ANALYSIS OF PLATES AND ARBITRARY SHELLS BY THE USE OF THE SEMILOOF ELEMENT

December, 1976
ANALYSIS OF PLATES AND ARBITRARY SHELLS BY THE USE OF THE SEMILOOF ELEMENT


December, 1976
ANALYSIS OF PLATES AND ARBITRARY SHELLS BY THE USE OF THE SEMILOOF ELEMENT

by


A thesis submitted to the University of Wales in candidature for the degree of Doctor of Philosophy

December, 1976

Department of Civil Engineering,
University College of Swansea,
SWANSEA

C/PH/42/76
DECLARATION

This is to certify that neither this thesis nor any part of it has been presented or is being concurrently submitted in candidature for any degree at any other University.

----------------------------
Candidate
CERTIFICATE OF RESEARCH

This is to certify that except where specific reference to other investigators is made, the work described in this thesis is the result of the investigation of the candidate.

---------------------
Candidate

---------------------
Director of Studies
ACKNOWLEDGEMENTS

I wish to express my gratitude to Professor V.S.S. Sá of the University of Opprto for his encouragement and support at all stages of this work.

In the University of Wales I was fortunate to work under Dr. D.R.J. Owen. His valuable guidance and encouragement are gratefully acknowledged.

I am also grateful to Professor O.C. Zienkiewicz for making the facilities of his department available for the completion of this research.

To Professor B.M. Irons of the University of Calgary, from whom I learned so much, my sincere acknowledgements are extended.

Thanks are also due to Dr. R.D. Wood and to my colleagues A.Z. Ijam, D. Shantaram, P.C. Jain, G.N. Pande, F. Albuquerque and F. Pimenta for many helpful discussions.

I am grateful to Mrs. N. Williams for typing most of the manuscript, to Mrs. J. Davies for having typed all my quarterly reports to INVOTAN, and to Mrs. A. McGail for preparing some of the drawings.

I thank also my wife and colleague, Lúcia, for many useful discussions and for her patience, encouragement and support at all stages of this work.

Finally I wish to acknowledge the financial support provided by commission INVOTAN of Junta Nacional de Investigacao Cientifica e Tecnologica and by Faculdade de Engenharia do Porto.
SUMMARY

The work reported in this thesis includes several topics in the field of finite element analysis of arbitrary plates and shells.

The semiloof shell element is used throughout and, based on the semiloof shell theory, a plate element is derived and used in various applications throughout the thesis. Detailed matrix formulations of these elements are presented and their performance tested for a number of linear problems.

A finite element eigenvalue solution is developed and various structural eigenvalue problems are solved using both plate and shell elements. These include free vibration analysis, stability analysis and analysis of the vibrations of initially stressed shells. Results and comparison with other solutions are presented for a wide range of problems.

Using an incremental and iterative method a computer program is developed for the geometrically non-linear analysis of shells. Results are presented using the semiloof shell element and comparisons are made with other solutions.

Both semiloof plate and shell elements are used for materially non-linear analysis, comparisons being made with other solutions where available. The computer programs developed use the initial stress method as the method of solution.

Finally several types of analysis are performed on a large reinforced thin shell. These include linear and geometrically non-linear analysis as well as analysis of stability and free vibrations.
The influence on each analysis of changes in boundary conditions is studied and the various results compared and criticised.

For the various types of analysis considered throughout this work several convergence studies are presented.
CHAPTER I  INTRODUCTION

CHAPTER II  THE SEMILOOF PLATE ELEMENT

1. Introduction
2. Initial (non constrained) version of the element
   2.1 Nodal configuration
   2.2 Displacement field
   2.3 Definition of the rotations
   2.4 Second derivatives
3. Final (constrained) version of the element
   3.1 Definition and formulation of the shear constraints
   3.2 Constraining of the element
   3.3 Element matrices
4. Patch test behaviour
   4.1 Square plate patch-tests
   4.2 Circular plate patch-tests
5. An assessment of the element performance
   5.1 Square plates
   5.2 Circular plates
   5.3 Variable thickness plates
6. Conclusions
   Bibliography

CHAPTER III  THE SEMILOOF SHELL ELEMENT

1. Introduction
2. Degrees of Freedom
3. In-Plane Behaviour
   3.1 Definition of Local Axes
   3.2 Displacements in Global Axes
   3.3 In-plane Displacements
   3.4 In-plane Derivatives
4. Out-of-plane Behaviour
   4.1 Out-of-plane Displacements
   4.2 Out-of-plane Derivatives
      a. Preliminary Considerations
      b. Determination of $(\partial U/\partial Z)^L$ and $(\partial V/\partial Z)^L$
c. Determination of $(\partial U/\partial Z)^N$ and $(\partial V/\partial Z)^N$

d. Final Expressions for $\partial U/\partial Z$ and $\partial V/\partial Z$

4.3 Second Derivatives

5. Element Constraining
   5.1 Constraining Equations
   5.2 Application of Constraints

6. Element Matrices

7. Results

8. Conclusions

Bibliography

CHAPTER IV

EIGENVALUE ANALYSIS

1. Introduction

2. The Eigenvalue Problem
   2.1 Formulation and Properties
   2.2 Method of Solution

3. Finite Element Eigenvalue Solution

4. Linear Stability Analysis
   4.1 Non Linear Strains
   4.2 Geometric Matrix
   4.3 Stability Analysis
   4.4 Numerical Results

5. Free Vibration Analysis
   5.1 Equations of Motion
   5.2 Mass Matrices
   5.3 Numerical Results

6. Vibration of Initially Stressed Shells
   6.1 Problem Formulation
   6.2 Numerical Results

7. Conclusions

Bibliography

CHAPTER V

GEOMETRICALLY NONLINEAR ANALYSIS

1. Introduction

2. Methods of Analysis

3. The Incremental Method
   3.1 General
   3.2 Stiffness Matrices
   3.3 Method of Solution
   3.4 Convergence Criteria

4. Results
5. Conclusions

Appendix

(Incremental Equilibrium Equations)

Bibliography

CHAPTER VI

ELASTO PLASTIC ANALYSIS

1. Introduction

2. Yield Conditions
   2.1 General
   2.2 Tresca's and Von Mises' criteria
   2.3 Yield Surfaces for Shells
   2.4 Strain Hardening

3. Plasticity Relations
   3.1 General
   3.2 Fundamental Assumptions
   3.3 Stress-strain Relations

4. Finite Element Formulations
   4.1 General
   4.2 The Initial Stress Method

5. Results

6. Conclusions

Bibliography

CHAPTER VII

APPLICATION OF THE PREVIOUS PROGRAMS TO A TYPICAL INDUSTRIAL PROBLEM

1. Introduction

2. Problem Definition
   2.1 Structure
   2.2 Load
   2.3 Boundary Conditions
   2.4 Finite Element Discretization

3. Linear Analysis
   3.1 Model I
   3.2 Model II
   3.3 Additional Models of Analysis

4. Elastic Stability Analysis
   4.1 Model II
   4.2 Model I

5. Vibration Analysis
6. Geometrically Nonlinear Analysis
   6.1 Model I
   6.2 Model II
7. Conclusions
   Bibliography

CHAPTER VIII   GENERAL CONCLUSIONS
**BASIC NOTATIONS**

\[
\begin{align*}
A & \quad \text{area integrating factor} \\
[A] & \quad \text{symmetric matrix (n\times n)} \\
a & \quad \text{vector containing derivatives of } F \\
\dot{a} & \quad \text{" } \text{" } \text{" } \text{" } \text{" } Q \\
[B] & \quad \text{matrix containing derivatives of shape functions} \\
& \quad \text{also (Chapter IV) symmetric positive definite matrix (n\times n)} \\
b & \quad \text{vector of body forces} \\
[C] & \quad \text{matrix of constraints} \\
& \quad \text{also (Chapter II) matrix with numerical values for the} \\
& \quad \text{shape functions} \\
C_{ijkl} & \quad \text{elasticity tensor} \\
[D] & \quad \text{elasticity matrix} \\
D_{ep} & \quad \text{elasto-plastic matrix} \\
\{d\} & \quad \text{vector of displacements for the current point } P(x,y,z) \\
d\lambda & \quad \text{scalar factor} \\
\text{de}_{ij} & \quad \text{increment of strain} \\
\text{de}^p_{ij} & \quad \text{plastic increment of strain} \\
\text{de}^e_{ij} & \quad \text{elastic increment of strain} \\
\{\delta^e\} & \quad \text{vector with nodal displacements for the element} \\
\lambda & \quad \text{eigenvalue} \\
L^j & \quad \text{shape function for loof or centre node } j \\
[L] & \quad \text{matrix of shape functions } L^j \\
[M] & \quad \text{diagonal matrix with mass densities} \\
M_x, M_y, M_{xy} & \quad \text{unit bending moments} \\
m_x, m_y, m_{xy} & \quad \text{nondimensional bending moments} \\
N^i & \quad \text{shape function for corner, mid-side or centre node } i \\
\{N\} & \quad \text{vector of shape functions } N^i \\
[N] & \quad \text{matrix } " \text{" } \text{" } \text{" } \text{" }
\end{align*}
\]
\[ N_A \] vector of shape functions (constrained)

\[ N_x, N_y, N_{xy} \] unit membrane forces

\[ n_x, n_y, n_{xy} \] nondimensional membrane forces

\[ P(x, y, z) \] current point with coordinates \((x, y, z)\)

\[ P_r(\lambda) \] term of a sturm sequence

\[ \{p^e\} \] vector of element nodal parameters

\[ \{p^e_A\} \] vector of retained nodal parameters

\[ \{p^e_B\} \] " " eliminated " "

\[ Q \] plastic potential

\[ Q_x, Q_m, Q_{tm} \] quadratic forms of \(n_x, m_x\) etc.

\[ \hat{\mathbf{r}}^j \] vector normal to the element side of a loof node \(j\)

\[ [R] \] matrix with components of vectors \(\hat{\mathbf{r}}^j\)

\[ E \] Young's modulus

\[ E_{ij} \] Green's strain tensor

\[ \varepsilon_x, \varepsilon_y, \varepsilon_{xy} \] strains in local axes (superscripts \(b\) and \(m\) refer to bending and membrane respectively)

\[ [\varepsilon], \varepsilon \] Green's strains

\[ [F] \] matrix of shape functions for semiloof shell element (constrained)

\[ \{r^e\} \] vector of element nodal forces

\[ F(\sigma_1, \sigma_2, \sigma_3) \] yield surface

\[ [G] \] matrix containing first derivatives of displacements

\[ [H] \] matrix of shape functions for semiloof plate element (constrained) also (Chapter III) matrix of direction cosines

\[ [I] \] identity matrix

\[ I_j \] side integrating factor

\[ J_1, J_2, J_3 \] functional determinants

\[ [J] \] Jacobian matrix

\[ [K_p] \] classical stiffness matrix
\([K_G]\) geometric stiffness matrix
\([K_D]\) initial displacement matrix
\([K_M]\) mass matrix
\(K_{ij}\) element of the stiffness matrix
\(\kappa\) hardening parameter
\(S_{ij}\) strain tensor
\(s_{ij}\) deviatoric stresses
\(\hat{S}^j\) vector along the side at loof node \(j\)
\([S]\) matrix with components of vectors \(\hat{S}^j\)
also (Chapter III) matrix of shape functions
\(\hat{T}^j\) thickness vector at loof node \(j\)
\(\hat{T}\) thickness vector at the current point \(P(x,y,z)\)
\([T]\) matrix of shape functions (constrained)
also (Chapter IV) matrix of stresses
\(T,t\) shell or plate thickness \((t\) also time in Chapter IV\)
\(u,v,w\) displacements referred to global axes \((0,x,y,z)\)
\(U,V,W\) displacements referred to local axes \((P,X,Y,Z)\)
\(\hat{X},\hat{Y},\hat{Z}\) local unit vectors
\(\omega\) normal displacement, plate
\(\theta^j_{XZ},\theta^j_{YZ}\) rotations at loof or centre node \(j\)
\(\{\theta^e_{XZ},\theta^e_{YZ}\}\) vectors of element nodal rotations
\((\xi,\eta)\) Gauss parameters for a surface
\([\sigma],\sigma\) stresses
CHAPTER I

INTRODUCTION

In many fields of science and engineering, various mechanical, thermal and electrical phenomena can be easily described in terms of differential equations, the solution of which allows an interpretation of the phenomena or model being considered.

Taking a general differential equation represented by

$$ Au = f(P) $$  \hspace{1cm} (1)

where $A$ is a positive definite (differential) operator defined on a certain field $D_A$ whose elements satisfy the boundary conditions of the problem,

$u$ is an element of $D_A$,

$f(P)$ is a known function with finite norm,

the solution of this equation involves finding a function $u(P) \in D_A$ which transforms (1) into an identity.

It is possible that the above equation might have been easily generated, but it is also very natural that its solution be extremely difficult if the physical domain of occurrence of the phenomena that it represents has a complicated shape. Such is the case of the differential equations that represent the Mathematical Theory of Elasticity. They have been known for a considerable number of decades (T-1), but their solution, even now, is not possible but for a reduced number of problems.

The above problem can however be reformulated in a different, but parallel way, by making use of the Minimal Functional Theorem.
This states (M-1):

If equation (1) has a solution, this gives to the following quadratic functional:

\[ F(u) = \int_{\Omega} \left[ u(P) A_u - 2u(P) f(P) \right] d\Omega \quad (2) \]

the least value of the values given by all the possible functions from \( D_A \), and conversely:

If there is in \( D_A \) a function which gives a minimal value to the functional (2), this function is the solution of equation (1).

For many problems of mechanics, the magnitude of the functional (2) is proportional to the potential energy of the system under consideration, the Minimum Functional Theorem being then equivalent to the principle of minimum potential energy.

From the numerical point of view a solution for the first problem (equation (1)) can be constructed in the form of an orthogonal series as follows:

\[ u_0(P) = \sum_{n=1}^{\infty} a_n \psi_n(P) \quad (3) \]

where

\[ a_n = \int_{\Omega} \psi_n A u_0 \, d\Omega \quad (4) \]

In these expressions \( \psi_n(P) \) represents a sequence of functions that is complete in the energy, i.e., any function with finite norm can be approximated within an arbitrary degree of precision by linear combinations of a finite number of functions \( \psi_n(P) \).

In order to define an approximate solution for the second problem (equation (2)), let us consider a functional \( \phi(u) \) whose values are positive bounded below. In these circumstances there exists an
exact lower bound, \( d \), for \( \phi(u) \):

\[
d = \inf \phi(u)
\]

A sequence \( (u_n)_{n \in \mathbb{N}} \) of functions belonging to the field of definition of \( \phi(u) \) is a minimizing sequence if

\[
\lim_{n \to \infty} \phi(u_n) = d
\]

We can now enunciate the following theorem of great relevance to a number of approximate methods (M-1):

If equation (1) has a solution, any sequence which is minimizing for functional (2) converges in energy to this solution.

This means that to get an approximate solution for the second problem it suffices to construct a minimizing sequence for functional (2) that represents with accuracy enough the problems to be solved.

The Ritz method constitutes such a method of constructing a minimizing sequence. For the application of this method a sequence of functions (coordinate functions or trial functions) belonging to \( D_A \) is chosen:

\[
\psi_1(P), \psi_2(P), \ldots, \psi_n(P),
\]

This sequence must obey the following conditions:

1. It must be complete in energy.
2. For any value of \( n \) the function \( \psi_j(P) \), \( j = 1, n \) must be linearly independent.

Having selected the sequence of functions, the application of the Ritz method follows now the following steps (M-1):

\( a \)

The unknown function \( u_n(P) \) is expressed in terms of the coordinate functions and some arbitrary parameters \( a_j \) by:
u_n(P) = \sum_{j=1}^{n} a_j \psi_j(P) \quad (5)

(b) Substituting \( u_n(P) \) for \( u(P) \) in the functional (2), this becomes a function of the parameters \( a_j \) with the following form:
\[
F(u_n) = \sum_{j,k=1}^{n} \left( \psi_k A \psi_j \, d\Omega \cdot a_j a_k \right) - 2 \sum_{j=1}^{n} \left( \int_{\Omega} \psi_j f \, d\Omega \cdot a_j \right)
\]
\( (6) \)

(c) As \( A \) is a positive definite operator, the vanishing of the first derivatives of this function gives a necessary and sufficient condition for a minimum to be obtained. We obtain therefore the following system of equations:
\[
\frac{\partial F(u_n)}{\partial a_i} = 0, \quad i=1,2,...,n
\]

(d) The solution of the previous system of equations yields the values of the coefficients \( a_i \), \( i=1,n \), that, substituted in (5) gives an approximate solution \( u_n(P) \).

The current literature shows other similar methods of obtaining a minimizing sequence (M-1, Z-1,2). Particularly important is the Galerkin method that is a generalization of the Ritz method for the case when the operator \( A \) is positive definite. These two methods yield therefore similar equations but when the operator \( A \) is only positive definite, the Ritz method is not applicable; the Galerkin method however can still be used.
The Galerkin-Ritz method is therefore of relatively simple application. The real difficulty lies in obtaining an appropriate set of trial functions. Theoretically it is always possible to find a complete set of trial functions, but its actual determination can be a very difficult task indeed.

This can be achieved in the finite element method by taking a different point of view in the selection of the trial functions.

In the finite element method the unknown function $u$ is approximated, not by trial functions defined over the whole domain, but by piecewise functions defined over a finite number of subdomains. Over these subdomains (finite elements) the trial functions (or shape functions) are uniquely described in terms of the values of the function at certain selected points (nodal points). One advantage of such an approximation is that each finite element can be isolated from the all domain and the corresponding function approximated in terms of its values at the nodal points. Within each finite element the trial functions can be given simple polynomial forms and the boundary conditions can also be easily imposed locally.

The accuracy of the process is now increased not by considering more and more complex trial functions, but simply keeping the same type of polynomials and increasing the number of elements.

Having described the finite element method a problem still remains: to know the kind of continuity that must exist between the individually defined fields in such a way that the piecewise finite element approximation converges to the right solution. We note that the different finite elements are compatibilized by the specification
of continuity conditions at coincident nodes of adjacent elements (reduced continuity conditions (0-1)). At these points, the values of the unknown function or functions and their principal derivatives must be the same. However, the imposition of these conditions does not imply that complete continuity exists along the element boundaries. This leads to the definition of conforming and nonconforming elements.

A type of finite element is said to be conforming if the continuity requirements along the element boundaries are met. When some discontinuity exists, the elements are designated as nonconforming.

We note now that the finite element method can be considered equivalent to the Galerkin-Ritz method when conformity is achieved (0-1).

Still a problem remains; that of satisfying the completeness requirement. Reference (0-1) demonstrates that completeness is achieved if the following conditions are met:

1. The general analytical expression for \( u_i \) within each element, is given as a polynomial with a number of arbitrary coefficients equal to the number of unit modes corresponding to the element.

2. If \( (P_i-1) \) is the maximum order of the principal derivatives for \( u_i \), the polynomial expression must contain a complete polynomial of the \( P_i^{th} \) degree, with all the terms affected by independent coefficients.

3. The terms of higher degree can vanish whatever the values taken by those coefficients.

Conformity and completeness constitute in the finite element method sufficient conditions for convergence to the right solution to be
achieved. Convergence can still be observed in certain cases (0-2) for nonconforming elements if completeness is achieved.

The requirements for convergence can easily be met for certain problems that do not involve high derivatives in its formulation. Such is the case for instance of the classical theory of elasticity for two and three dimensional media. More difficult however is to achieve conformity for plate and shell problems. Taking e.g., the case of the theory of thin plates, reference (Z-1), demonstrates that it is impossible to specify simple polynomial expressions for shape functions ensuring full compatibility, when only the normal displacement and its slopes are prescribed at the nodes.

Conformity has however been achieved (I-1, B-1, H-1, G-1) but this does not necessarily mean that a good performance is obtained. Furthermore, other difficulties can arise when imposing boundary conditions.

For these reasons, non-conforming, but convergent, elements are commonly used in the finite element practice (C-1,B-2). For the formulation and use of these elements the Ritz code is relaxed and appeal is made to the patch-test.

The patch test (I-2,3) establishes the convergence of a fine mesh by simulating what happens in the limit when the mesh is "infinitesimally" fine. It states:

If in a patch of elements the external nodes are given displacements corresponding to a state of constant stress, the internal nodes must take the values corresponding to the imposed state of constant stress.

Therefore, however much non-conforming an element may be, if it passes the patch-test it should be used with confidence as, in
the limit, convergence is always achieved.

Although initially used as a device to predict convergence, reference (0-3) states that the patch-test means much more than that and that it can play a very important role in the Mechanics of Discrete Media replacing the concept of convergence.

The semiloof element is a non-conforming element and therefore owes its existence to the patch-test. It is at the same time a very unorthodox element (I-4).

From the user point of view the element has the following degrees of freedom: 3 displacements at each corner and midside node; 2 rotations normal to the side at two points along each side (gauss points). This gives a total of 32 degrees of freedom for a quadrilateral and 24 for a triangle.

It is interesting to note that the finite element method has usually been a method of implementing infinitesimal theories, and the avenue usually considered by the various researchers when formulating shell elements is to use, for this formulation, one of the well known shell theories. This does not happen with the semiloof element that is a shell theory in itself. This theory is constructed in vector notation and the strains inside the element are defined with basis on the nodal parameters.

Another particularity of the semiloof element is the existence of two sets of shape functions, one for the displacements \((N^i_i, i=1,8)\) and the other for the rotations \((L^j_j, j=1,8)\). There are still two more shape functions corresponding to a centre node, \(N^9\) and \(L^9\), that, incidentally, are different from each other. It should also be noted that the rotations
are not derivatives of the displacements. (As pointed out in reference (0–1) one of the difficulties of achieving conformity for plates and shells results from the fact that rotations are usually regarded as derivatives of the transverse displacements).

As said before, the quadrilateral semiloof shell element has 32 degrees of freedom. Initially however, the element has more degrees of freedom, some of these being eliminated by imposing some shear constraints at discrete points of the element and globally on the element.

In view of the generality of the formulation the element can model shells of any shape with or without branches or sharp corners. In this respect a positive advantage is that the rotations are considered along the sides, eliminating therefore the problem of singularities at the corners.

The semiloof shell element has been tested for a number of simple linear problems (A–1, M–4). However, in view of the complexity and unorthodoxy of its formulation, a number of questions should still be put.

We start by noting that most of the problems reported refer to plates, and in view of the good results obtained it seems natural that the semiloof theory could be used, with advantage, for plate situations if the necessary modifications are introduced. At the same time, its performance in complex shell structures should be tested as problems of numerical instability and misbehaviour could arise.

In the field of eigenvalue analysis, some types of finite elements give spurious eigenvalues that are useless as approximations. This is the case of some cubic Hermitian elements as reported in
reference (S-1). This aspect should therefore be considered for
the semiloof element, and its performance in eigenvalue analysis
studied.

A similar problem can be put in relation to non-linear analysis,
particularly the problem of checking if some of the known bounds of
elasto-plastic analysis are respected. At the same time, convenient
methods of analysis should be chosen, not only for elasto-plastic analysis
but also for geometrically nonlinear analysis. The element performance
for these types of analysis should be examined.

For all the types of analysis previously referred, comparisons
should be made with other finite element solutions in order to check
the merits or dismerits of the element.

The work reported here constitutes an attempt to get answers to
the problems just enunciated. For this purpose a number of finite
element computer programs have been written and are described along the
thesis. This task has been made more difficult in view of the fact
of the semiloof element having different degrees of freedom for each
node. This made impossible the use of some existing non-linear and
eigenvalue routines at the Centre for Numerical Methods in Engineering
and therefore new programs have been developed (M-5, O-4).

A brief description of the work carried out in this research
will now be presented.

Chapter II presents a plate version of the semiloof element.
The same out-of-plane degrees of freedom as the shell element are used,
but, as the in-plane degrees of freedom are not considered, there is
a reduction in the number of nodal parameters from 32 to 16.
At the same time the definition of the bending strains is simplified because of the fact that some terms of the second derivatives become redundant and are consequently dropped from the formulation. The geometrical corrections needed for curved shell are unnecessary for plates and therefore are also eliminated. Apart from the reduction of the number of nodal parameters, these simplifications result as well in a considerable saving of computer time.

The patch test behaviour of the element is shown for various states of constant straining and rigid body motions. The performance of the element in more realistic problems is tested and comparisons are made with other known elements in problems involving both straight and curved boundaries.

In this chapter an historical introduction to the semiloof element is also presented.

In Chapter III a detailed matrix formulation of the semiloof shell element is presented. This formulation constitutes the background for the next chapters by laying the principles for the determination of various finite element matrices consequently used.

A typical shell problem is solved and, in order to test the element performance, the results given by the semiloof element are compared with the ones given by other elements. Further applications of the semiloof element in the field of linear analysis are reported in Chapter VII.

Chapter IV deals with various types of eigenvalue analysis. The eigenvalue problem is first formulated and some relevant properties are reviewed. A method of solution and the particularities of its
application in the finite element method context are presented and
justified. Based on this method a very versatile and general finite
element program is developed. This program can perform a simple
linear analysis and various types of eigenvalue analysis; both plate
and shell elements were considered in this program.

The instability of shells is formulated as an eigenvalue problem.
This shows the necessity of generating the usually called "geometric
matrix". This matrix depends on the state of stress in the element
and its generation is presented for the shell element. Some shell
stability problems are presented.

The equations of motion for systems involving distributed
mass and elasticity are then considered; its particularization for freely
oscillating structures leads to another type of eigenvalue problem.
This requires the generation of a "mass matrix", the formulation of which
is presented for both plate and shell elements. A number of vibration
problems involving both plate and shell structures are solved, the
corresponding eigenvalues and eigenvectors being compared with other
known solutions.

Finally, using the geometric and mass matrices (and also the
classical stiffness matrix), the vibration analysis of initially stressed
shells is formulated as an eigenvalue problem. The problem of a
cantilever plate subject to a membrane state of stress is solved,
showing the variation of the first three frequencies of vibration
with the level of the stresses.

The geometric matrix developed in this chapter is also used
in Chapters V and VI in which non-linear (geometric and material)
analysis are considered.
CHAPTER I - INTRODUCTION

Geometric non-linearity is the type of non-linearity that results from the fact of not being linear the relationship between the displacements undergone by a structure and the corresponding strains. Chapter V deals with this type of non-linearity.

Only the semiloof shell element is considered in this chapter once that, even for plates, this type of analysis requires the inclusion of the in-plane deformation that the plate element does not provide.

A review of various approaches and methods of analysis is presented with particular reference to the incremental method. The finite element program developed in this chapter uses an incremental and iterative process, i.e., the load is applied in increments and iterations are performed for each increment until the unbalanced forces become negligible.

This method seems to be particularly attractive for the semiloof shell element in view of the particularities of its formulation, namely the fact of the semiloof being a general shell element, with shape functions and stresses referred to local axes generated at the element surface.

A number of plate and shell problems are solved with this program and comparisons are made with other known solutions.

Chapter VI deals with the materially non-linear analysis of plates and shells. This type of analysis results from the fact of considering to be non-linear the relationship between stresses and the corresponding strains.
The yielding conditions for solids in general and shell structures in particular are briefly reviewed. The fundamental assumptions of the Theory of Plasticity are also considered, as well as the stress strain relations and the different finite element formulations.

The initial stress method is considered in detail once that this is the method used in the finite element program developed in this chapter. This program includes both plate and shell elements and, for this last case, there is the possibility of not considering the geometric matrix, to consider it as constant for each increment, or to update it for each iteration performed.

A number of plate and shell problems are solved and the results obtained compared with other theoretical or numerical results.

In Chapter VII the semiloof shell element, and other programs developed in this work and already referred to, are used to perform a series of numerical analysis on a large reinforced shell (storage tank) submitted to wind loading and loading resulting from its self-weight.

The analyses performed assume that the constituent material operates in the elastic range at all times; an assumption which is borne out by the numerical results obtained.

The behaviour of the tank under wind loading will be significantly influenced by the boundary conditions assumed between the base of the tank and the foundation. In practice, storage tanks are constructed so that they merely rest on a rolled base of graded material and therefore there is no resistance offered by the base to uplift of the tank in any region. Consequently two types of analysis are performed;
(i) Firstly, vertical deflection of the tank base is restrained at all points. Such an analysis indicates that tensile reactions exist over a certain proportion of the base region. (ii) Secondly, vertical restraint is eliminated from all points which exhibit tensile reactions in the first analysis and this model simulates lift off of the tank which is a phenomenon known to occur under operating conditions.

In this investigation, four separate types of analysis are performed:

(i) The tank is firstly assumed to behave linearly elastic. Two types of meshes are used and the corresponding results compared in order to assess the errors introduced by the coarser mesh. This type of analysis is known to be only an approximation, since non-linear effects must clearly influence the behaviour of such a slender structure. However, since linear analyses can be performed at a fraction of the cost of a non-linear solution linear computations are beneficial for examining the influences of various factors (such as boundary conditions) on the behaviour of the tank.

(ii) Elastic stability analysis. In this the buckling load of the structure is computed by an eigenvalue process. The buckling load of a thin shell may or may not be representative of the true collapse loading depending on the stress distribution within the structure. If the distribution of stress is relatively uniform the elastic buckling load will give an upper bound to the true collapse load. (reference (B-3)). However if there exists a possibility for transfer of load from one part of the shell to another after initial buckling, the actual collapse load may exceed this buckling value by an order of magnitude. For example, the collapse load of stiffened plate systems may be 10-20 times their elastic buckling load.
(iii) In such cases the collapse behaviour of a shell structure can only be numerically simulated by a large-deflection analysis. The method used for this analysis is described in detail in Chapter V.

(iv) Finally a natural frequency analysis is performed.

In this study the modes of free vibration of the structure and the corresponding natural frequencies are determined by means of an eigenvalue process.

Finally Chapter VIII makes a general balance of the thesis, presents the conclusions obtained, and gives recommendations for improvement and extension of this work.

As a final point we note that it is generally assumed along this thesis that the reader is familiar with the basic principles and techniques involved in the application of matrix methods to the linear and non-linear analysis of structures (generation of individual matrices, assembly of these matrices, etc.) as they are not explicitly considered. Detailed formulations can be seen in references (C-2, D-1, G-1, H-2, L-1, M-2,3, P-1, S-1, Z-1).
CHAPTER I - INTRODUCTION

BIBLIOGRAPHY

A-1 Albuquerque, F.
' A beam element for use with the semiloof shell element'

B-1 Bogner, F.K., Fox, R.L. and Schmit, L.A.
'The generation of inter-element-compatible stiffness and
mass matrices by the use of interpolation formulas'
Conference on Matrix Methods in Structural Mechanics, Ohio,
1966.

B-2 Baldwin, J.T., Razzaque, A. and Irons, B.
'Shape function subroutine for an isoparametric thin plate
element'
University of Wales, 1974.

B-3 Brush, D.O. and Almroth, B.O.
'Buckling of bars, plates and shells'

C-1 Clough, R.W. and Tocher, J.L.
'Finite element stiffness matrices for analysis of plate
bending'
Conference on Matrix Methods in Structural Mechanics, Ohio,
1966.

C-2 Cook, R.D.
'Concepts and applications of finite element analysis'

D-1 Desai, C.S. and Abel, J.F.
'Introduction to the finite element method (A numerical
method for engineering analysis)'

G-1 Gopalacharyulu, S.
'A higher order conforming rectangular plate element'

G-2 Gallagher, R.H.
'Finite element fundamentals'

H-1 Harvey, J.W. and Kelsey, S.
'Triangular plate bending elements with enforced compatibility'

H-2 Hinton, E. and Owen, D.R.J.
'Finite Element Programming'
CHAPTER I - INTRODUCTION

I-1  Irons, B.M.
     'A conforming quartic triangular element for plate bending'

I-2  Irons, B.M. and Razaque, A.
     'Experience with the patch-test for convergence of finite elements'
     In: The Mathematical Foundations of the Finite Element Method
     with Applications to Partial Differential Equations

I-3  Irons, B.M.
     'The patch test for engineers'
     University of Wales, March 1974.

I-4  Irons, B.M.
     'The semiloof shell element'
     In: Finite Elements for Thin Shells and Curved Members
     (Chapter 11) Ed. R.H. Gallagher, D.G. Ashwell,

L-1  Livesley, R.K.
     'Matrix methods of structural analysis'

M-1  Mikhlin, S.G.
     'Variational methods in mathematical physics'

M-2  Majid, K.I.
     'Non-linear structures'

M-3  Massonet, C., Deprez, G., Maquoi, R., Muller, R. and Fonder, G.
     'Calcul des structures sur ordinateur'
     Eyrolles Editeur, Masson et Cie Editeurs, 1972.

M-4  Martins, R.A.F.
     'Finite element eigenvalue solution employing the semiloof
     shell element'

M-5  Martins, R.A.F. and Owen, D.R.J.
     'Structural instability and natural vibrations analysis of
     thin arbitrary shells by the use of the semiloof element'

O-1  Oliveira, E.R.A.
     'Theoretical foundations of the finite element method'
     Laboratorio Nacional de Engenharia Civil, Lisbon, Memoria
     No.384.

O-2  Oliveira, E.R.A.
     'Completeness and convergence in the finite element method'
     Laboratorio Nacional de Engenharia Civil, Lisbon, Memoria
     No.373.
CHAPTER I - INTRODUCTION

O-3 Oliviera, E.R.A.
'Results on the convergence of the finite element method in structural and non-structural cases' Proceedings of the 1974 International Conference on Finite Element Methods in Engineering, held at the University of New South Wales, Australia, Editors V.A. Pulmano and A.P. Kabaila.

O-4 Owen, D.R.J. and Martins, R.A.F.
'Finite element analysis of an oil storage tank' University of Wales, Swansea, 1976

P-1 Przemieniecki, J.S.

S-1 Strang, G. and Fix, G.F.

T-1 Timoshenko, S.P. and Goodier

Z-1 Zienkiewicz, O.C.

Z-2 Zienkiewicz, O.C.
Book on the Finite Element Method (to be published).
CHAPTER II - THE SEMILOOF PLATE ELEMENT

1. INTRODUCTION

(a) The semiloof shell element (I-1) is the latest element in an evolution process extending for a considerable number of years. This process began with the line isoparametric elements of reference (E-1). In references (A-1,2) membrane and membrane-stack elements are developed, these allowing the representation of bending behaviour. They are derived from brick elements, the nodes along the thickness ($\zeta$ direction) being attached to a rigid straight line member subject to 3 deflections and 2 rotations. Each surface $\zeta = \text{constant}$ is treated as a membrane in plane stress and a factor 1.2 is introduced to simulate a parabolic shear distribution between membranes. This model proved remarkably efficient and has been widely used.

A further improvement in this model was achieved (Z-1) using a reduced integration technique. The 2x2 Gauss integration rule used not only made the element more economical but also made its use possible for the analysis of thin shells. The good performance of this element with reduced integration led to the first delinquent element. This element (I-2,B-1) uses a discrete Kirchhoff shear assumption (W-1) to constrain to zero the lateral shear at the 2x2 Gauss integrating points. These 8 constraints are then employed to eliminate the same number of variables (2 at each midside node). The semiloof shell element uses a similar process for eliminating unwanted variables but the constraining equations are now more sophisticated, some resulting from non-discrete constraints.

(b) Although the formulation of the semiloof shell element is complex, its performance (A-3, L-1, M-1,2) appears to justify this drawback.
In the formulation of an element with good performance, a judicious choice of the degrees of freedom must certainly be important. In reference (L-2) two methods are presented for the calculation of the stiffness matrix for a high performance element.

The first method is well established and consists of assembling a number of "small" elements by the usual finite element procedure in order to form a "big" element. Before the assembly of this element in a larger structure the equations of internal equilibrium are solved and the order of the stiffness matrix reduced. By a convenient choice of the displacements which are to be eliminated, the minimum energy can be closely approached.

In the second method a "big" element is formulated directly. The requirements of complete compatibility at the boundary of the element are dropped, and the continuity is only satisfied at a set of collocation points. Reference (L-2) uses the three points for a Gauss-Chebyshev integration rule as collocation points along each side of a rectangle. In the semiloof element continuity of normal slopes is imposed at the two Gauss points along each side. It should be noticed that none of these rules includes the extremes of the integration interval as integrating points, thus avoiding singularities at the corners of the element.

Excluding these 8 rotations along the sides, the semiloof shell element has the displacements along three orthogonal directions of the corner and midside nodes as degrees of freedom. The total number of degrees of freedom of the element is then 32.
(c) A great number of shell structures appearing in practice are reinforced with beams. These reinforcements can, in principle, be simulated by using real shell elements (D-1). This kind of reinforcement however as well as not being very versatile is also uneconomical because it unnecessarily increases the number of equations to be solved. A beam element to be utilised in conjunction with the semiloof element was developed in reference (A-3). This semiloof beam element has the same degrees of freedom as "one side" of the shell element, plus 3 orthogonal couples at each extremity, the total number of degrees of freedom then being 17. The couples at the extremities of the beam must be connected to other beams (or earthed) since for the shell element only displacements exist as degrees of freedom at the corner nodes.

(d) The semiloof element is applicable to general thin shell structures, and, as such, thin plates are amenable to solution by it. In reference (A-3) some plate problems are solved which show that very accurate results can be obtained with this element. Also reference (M-1) gives further evidence of this fact and in (M-2) it is shown that the element yields very accurate results for plate instability and vibration problems. Here, even with all the degrees of freedom that a shell necessitates, the results, in economical terms, compare very favourably with other plate elements.

This encouraged us to derive from the semiloof shell element a plate element. This semiloof plate element has the advantage of having only one half of the degrees of freedom needed for the shell. This is not however the only advantage. In fact the formulation of the strains
for the general shell is complex owing to the fact that the elements are, in principle, doubly curved. For the plate this formulation is simplified; the shape functions being easily presented in compact matrix expressions. The computing time for the calculation of the strains is therefore greatly reduced. Also some artificial doctoring of the geometry needed for shells (I-1) is unnecessary for plates, which simplifies the formulation even further.

(e) In this chapter the formulation of the semiloof plate element is presented, together with its patch-test performance. Its performance in more practical problems is also assessed by application to a number of plate problems with various shapes, loads and boundary conditions. Included in these are some plates with curved boundaries as these are sometimes a cause of dissatisfaction with some other elements which can only approximate this type of boundary by straight lines (C-1). Problems involving elements with variable thickness are also considered. The results of the problems are shown, it is hoped, with the necessary details in order to make possible a good assessment of the element performance. Unless otherwise stated the analytical solutions referred in this chapter are always from reference (T-1).

(f) It should be noted that the computer program developed includes, in fact, two elements, one triangle and one quadrilateral. However the formulation given here is always referred to the quadrilateral. The formulation for the triangle is, in principle, parallel, being different only in specific points. As will be seen later, the performance of the quadrilateral element is usually superior to that of the triangle.
2. INITIAL (NON-CONSTRAINED) VERSION OF THE ELEMENT

2.1 Nodal Configuration

(a) In Figure 1 the initial nodal configuration of the semiloof plate element is shown. The element is referred to a system of axes (0, x, y) and three types of nodes are considered:

- corner and midside nodes at which the displacements normal to the plane (x,y), \( \omega^i \) (i=1,8) are taken as parameters.

- loof nodes, located at the Gaussian quadrature position for the two point integration rule along the element sides. The two nodes along a side are therefore positioned at a distance \( \frac{x}{2\sqrt{3}} \) from the middle of the side.

The nodal parameters at these nodes are chosen to be the two rotations \( \theta^j_{XZ} \) (j = i,8) and \( \theta^j_{YZ} \) (j = 1,8) which are respectively normal and parallel to the element edge.

- central node at which the nodal parameters are the normal displacement \( \omega^9 \) and the two rotations, about the isoparametric curvilinear coordinates \( \xi \) and \( \eta \) respectively \( \theta^9_{XZ} \) and \( \theta^9_{YZ} \).

(b) The total number of degrees of freedom on this non constrained version of the element is then 27, and, for convenience in the following theory, they will be arranged in three vectors, one containing the normal displacements,

\[
\{\delta^6\} = \{\omega^1, \omega^2, \ldots, \omega^9\}^T
\]  

(1)

the other two containing the rotations,
\[
\{\theta_e^{xz}\} = \{\theta_{xZ}^1, \theta_{xZ}^2, \ldots, \theta_{xZ}^9\}^T \tag{2}
\]
\[
\{\theta_e^{yz}\} = \{\theta_{yZ}^1, \theta_{yZ}^2, \ldots, \theta_{yZ}^9\}^T \tag{3}
\]

Quantities referred to the corner, midside and central (displacement) nodes and to the loof and central (rotations) nodes will usually be indexed with respectively i and j.

As it will be seen later in this chapter, some of these variables will be eliminated, the final version of the element having only 16 degrees of freedom.

2.2 Displacement Field

(a) The coordinates for the current point \(P(x,y)\) of an element are assumed to be calculated by interpolation, using the coordinates of the 8 corner and midside nodes and a convenient set of shape functions. These nodal coordinates are assembled in the following two vectors:

\[
\{x^e\} = \{x^1, x^2, \ldots, x^8\}^T \tag{4}
\]
\[
\{y^e\} = \{y^1, y^2, \ldots, y^8\}^T \tag{5}
\]

and the interpolation functions, one for each node, are, in terms of the non-dimensional coordinates \((\xi, \eta)\) defined by:

\[
N^1 = \frac{1}{4} (1-\xi) (1-\eta) (-\xi-\eta-1)
\]
\[
N^2 = \frac{1}{2} (1-\xi^2) (1-\eta)
\]
\[
N^3 = \frac{1}{4} (1+\xi) (1-\eta) (\xi-\eta-1)
\]
\[
N^4 = \frac{1}{2} (1+\xi) (1-\eta^2)
\]
\[
N^5 = \frac{1}{4} (1+\xi) (1+\eta) (\xi+\eta-1)
\]
\[
N^6 = \frac{1}{2} (1-\xi^2) (1+\eta)
\]
\[
N^7 = \frac{1}{4} (1-\xi) (1+\eta) (-\xi+\eta-1)
\]
\[
N^8 = \frac{1}{2} (1-\xi) (1-\eta^2) \tag{6}
\]
CHAPTER II - THE SEMILOOF PLATE ELEMENT

Two typical shape functions of this set of 8, the ones for nodes 1 and 6, are represented in Figure 2 (a and b).

If for each point \( P(x,y) \) a vector is created defined by

\[
\{ \tilde{N}(\xi,\eta) \} = \{ N^1(\xi,\eta), N^2(\xi,\eta), \ldots, N^8(\xi,\eta) \}
\]  

(7)

the coordinates of that point are given by

\[
x(\xi,\eta) = \{ \tilde{N}(\xi,\eta) \} \{ x^e \}
\]  

(8)

\[
y(\xi,\eta) = \{ \tilde{N}(\xi,\eta) \} \{ y^e \}
\]  

(9)

or, expanding:

\[
x(\xi,\eta) = a_1 + a_2 \xi + a_3 \eta + a_4 \xi \eta + \ldots
\]

\[
y(\xi,\eta) = b_1 + b_2 \xi + b_3 \eta + b_4 \xi \eta + \ldots
\]

(10)

(b) The normal displacement at any point \( P(x,y) \) of an element is assumed to be expressed in terms of the displacement vector \( \{ \delta^e \} \) as defined in (1) and interpolation functions \( (N^i(x,y))_{i=1,9} \). Of these shape functions the first 8 are the ones defined in (6). The ninth one is defined by

\[
N^9 = (1-\xi^2)(1-\eta^2)
\]  

(6a)

and is represented in Figure 2c. Considering

\[
\{ N(\xi,\eta) \} = \{ \tilde{N}(\xi,\eta) \} \{ N^9(\xi,\eta) \}
\]  

(11)

the normal displacement, \( \omega \), is defined by

\[
\omega = \{ N \} \{ \delta^e \}
\]  

(12)

(c) The consideration of the shape function \( N^9 \) associated with the displacement at the centre of the element to define the displacement field, \( \omega(x,y) \) deserves explanation (I-1): it is dictated by the necessity of the element to pass a constant strain patch test. In fact, this patch-test demands that any parabolic displacement field:
\[ \omega = c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 xy + c_6 y^2 \]  \hspace{1cm} (13)

is capable of being exactly reproduced by the element. Substituting (8) and (9) in (13), the following expression is obtained:

\[ \omega = d_1 + d_2 \xi + d_3 \eta + d_4 \xi^2 \eta^2 + \ldots \]  \hspace{1cm} (14)

This implies that the functional basis defining the normal displacement field must, in particular, include the term \((\xi \eta)^2\). This however, cannot be found in the 8 shape functions (6) and hence the necessity of \(N^9\).

2.3 Definition of the Rotations

(a) The rotations of the loof and central nodes, as defined in section 1 must somehow have their orientation defined in the plane \((0, x, y)\). This orientation can be established (Figure 3) by considering a set of unit axes \((\mathbf{e}^j, \mathbf{x}^j, \mathbf{y}^j)\) orientated according to the directions of the rotations.

Considering, for example, the first loof node, the orientation of the rotation along the side, \(\theta^1_{YZ}\), can be defined by calculating a vector tangent to the side at that point. In fact side \(AB\) (Figure 3) can be considered as a curve defined by means of a parameter \(\xi\), and a vector tangent to such a curve is given by \((S-1)\)

\[ \hat{\mathbf{y}} = \frac{\partial \mathbf{y}}{\partial \xi} = \begin{bmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \end{bmatrix} \]  \hspace{1cm} (15)

or, using expressions (8) and (9)

\[ \hat{\mathbf{y}} = \begin{bmatrix} \frac{\partial \mathbf{N}}{\partial \xi} \{x^e\} \\ \frac{\partial \mathbf{N}}{\partial \xi} \{y^e\} \end{bmatrix} \]  \hspace{1cm} (16)
This vector can now be normalised by setting
\[ \hat{Y} = \left\{ \frac{\dot{Y}}{S} \right\} = \left\{ \begin{array}{c} Y_x \\ Y_y \end{array} \right\} \]  
(17)

where the scalar \( S \) is defined by:
\[ S = \left( \frac{\partial Y_x}{\partial \xi} \right)^2 + \left( \frac{\partial Y_y}{\partial \xi} \right)^2 \]  
(18)
\[ S = \left( \left( \frac{\partial \bar{N}_y}{\partial \xi} \{x^e\} \right)^2 + \left( \frac{\partial \bar{N}_y}{\partial \xi} \{y^e\} \right)^2 \right)^{\frac{1}{2}} \]  
(19)

A unit vector normal to \( \hat{Y} \) can now be easily calculated by imposing that the scalar product of these two vectors is zero. The vector obtained defines the orientation of the rotation normal to side AB and will be represented by
\[ \hat{X} = \{x\} = \left\{ \begin{array}{c} x^e \\ y^e \end{array} \right\} \]  
(20)

In a similar way unit local axes can be constructed at the other points.

(b) The rotations at the current point \( P(x,y) \) of the element are defined by using the local rotations defined in (2) and (3) and the corresponding shape functions for loof and central nodes. The principle behind the idea of a "big" element as considered in (L-2) is the imposition of conformity at certain points (collocation points) at the boundary of the element. These boundary points for the semiloof element are the two loof nodes along each side. The ninth node (at the centre), at which two orthogonal rotations are also included as parameters, is a mathematical necessity as it will be shown in the following (I-3).
It is proposed to start searching the shape functions for the 8 noded element of Figure 4. The functional basis for these shape functions must be a polynomial with 8 terms, the first six being the complete second order polynomial:

$$f(x, y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2$$  \hspace{1cm} (21)

As it is not possible to have a complete polynomial with 8 terms the choice of the remaining two terms will be made, in order to preserve symmetry, according to the two following rules:

- The basis must stay unaltered when $x$ and $y$ are exchanged, i.e.

  $$f(x, y) = f(y, x)$$  \hspace{1cm} (22)

- When the sign of $x$ or $y$ is reversed, the basis $f(x, y)$ plus $f(y, x)$ must be the same as the basis $f(x, y)$ plus $f(y, -x)$ for example.

These two rules are obeyed by adding to function (21) either the terms $x^2y$, $xy^2$ or $x^3y^3$. For this case the first two terms will be considered and the functional basis for the intended element will be:

$$f(x, y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^2 y + a_8 xy^2$$  \hspace{1cm} (23)

The shape function coefficients are now easily determined by constructing and inverting the (8x8) matrix $[C]$ defined by (Z-2, page 106):
\[
[C] = \begin{bmatrix}
1 & x^1 & y^1 & (x^1)^2 & (x^1y^1)^2 & \ldots & x^1(y^1)^2 \\
1 & x^2 & y^2 & (x^2)^2 & (x^2y^2)^2 & \ldots & x^2(y^2)^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x^8 & y^8 & (x^8)^2 & (x^8y^8)^2 & \ldots & x^8(y^8)^2 
\end{bmatrix}
\]  

(24)

where \((x^i, y^i)\) are the coordinates of node \(i\). Considering an element centred at the origin of the coordinates (Figure 4) the matrix \( [C] \) would be for this case:

\[
[C] = \begin{bmatrix}
1 & -\frac{1}{\sqrt{3}} & -1 & \frac{1}{3} & \frac{1}{3} & 1 & \frac{1}{3} & -\frac{1}{\sqrt{3}} \\
1 & \frac{1}{\sqrt{3}} & -1 & \frac{1}{3} & \frac{1}{3} & 1 & -\frac{1}{3} & \frac{1}{\sqrt{3}} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & -1 & -\frac{1}{\sqrt{3}} & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{\sqrt{3}} 
\end{bmatrix}
\]  

(25)

In this matrix column (1) is a linear combination of columns (4) and (6), more exactly:

\[
(\text{column } 4 + \text{column } 6) \times 3/4 = \text{column } 1
\]

(26)

which implies that matrix \([C]\) as defined in (25) is not invertible.

This example leads to the consideration of neutral functions for a particular functional basis \(f(x,y)\) (I-3). These are functions that take the value zero at every node of the element and are contained in the basis \(f(x,y)\). This is the case of the circle with radius 4/3, centred in the centre of the element and passing through the 8 loof nodes. The equation of this circle is, in fact, implicit in equation (26):

\[
x^2 + y^2 = 4/3
\]

(26a)
This difficulty can be overcome by adding to the element one (or more) extra nodes and to the basis (23) one (or more) extra terms in such a way that the following third rule is obeyed:

- the functional basis cannot contain any neutral function.

In the case of the semiloof element (I-4) a ninth node is considered at the centre of the element and we seek now an extra term to extend the functional basis (23) to 9 terms. Starting with the cubic terms:

\[ x^3, x^2y, xy^2, y^3 \]  \hspace{1cm} (27)

it can be seen that no single cubic term or combination of three terms can be made to obey rules 1 and 2. Also \((x^3 + xy^2)\) and \((x^2y + y^3)\) are disqualified because of the following neutral functions:

\[ x^3 + xy^2 - \frac{4}{3}x \quad \quad x^2y + y^3 - \frac{4}{3}y. \]  \hspace{1cm} (28)

By a similar argument some quartic terms are disqualified, and the final choice for the missing term is

\[ T_9 = x^3y - xy^3 \]  \hspace{1cm} (29)

The final functional basis for the element will then be

\[ f(x,y) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^2y + a_8xy^2 + a_9(x^3y - xy^3) \]  \hspace{1cm} (30)

A matrix \([C]\) similar to (24) can now be constructed and inverted and the shape functions \((L_j^3)_{j=1,9}\) calculated. A typical shape function for the loof nodes is shown on Figure 5.

It should be noted that \(L^9\) as defined here and \(N^9\) as defined in (6a) are not the same function.
2.4 Second Derivatives

(a) The strains in the plate will be calculated using, as a starting point, the rotations at the loof and central nodes. In order to define these strains the derivatives

\[ \frac{\partial U}{\partial Z} ; \frac{\partial V}{\partial Z} \]

will be first calculated. Here U and V represent displacements along a set of axes X and Y at a particular point P(x,y), and Z represents the direction normal to the mid-surface of the plate.

(b) As a preliminary exercise for this calculation, the beam of Figure 6 is considered. This beam is in the plane (X,Z) and is actuated by two rotations, \( \theta^1_{XZ} \) and \( \theta^2_{XZ} \), one at each extremity. The deformation of the beam under these rotations is shown in Figure 6.b. For a particular point, P, along the beam, the relative displacement between the upper and lower surface can be interpolated using the values at the extremities 1 and 2 which are respectively:

\[ \delta U_1 = t_1 \theta^1_{XZ} ; \quad \delta U_2 = t_2 \theta^2_{XZ} \]

Now using the shape functions of Figure 6.c the relative displacement between upper and lower surface is for point P,

\[ \delta U_p = t_1 \theta^1_{XZ} L^1_p + t_2 \theta^2_{XZ} L^2_p \]

where \( L^1_p \) and \( L^2_p \) are the values of shape functions \( L^1 \) and \( L^2 \) for point P.
CHAPTER II - THE SEMILOOF PLATE ELEMENT

The derivative $\partial U/\partial Z$ at this point can now be approximated by

$$
\left( \frac{\partial U}{\partial Z} \right)_p = \frac{\delta U}{t_p} = (t_1 \theta_{XZ}^1 \frac{L_1}{t_p} + t_2 \theta_{XZ}^2 \frac{L_2}{t_p}) \frac{1}{t_p}
$$

where $t_p$ is the beam thickness for point $P$.

(c) A similar procedure can now be adopted for the plate. First, for each loof node and for the central node, the following vectors are defined:

$$
\hat{R}_j = t_j \hat{x}_j \quad \hat{S}_j = t_j \hat{y}_j
$$

It should be noticed that both $\{\theta_{XZ}^e\}$ and $\{\theta_{XZ}^o\}$ contribute to the definition of the derivatives (31). The contribution of the first vector can be expressed as follows:

$$
\left( \frac{\partial U}{\partial Z} \right)_{XZ} = \sum_{j=1}^{9} \left\{ R_x^j \hat{x}_j \right\} \left\{ X_x \right\} \frac{L_j}{t} \theta_{XZ}^j
$$

$$
\left( \frac{\partial V}{\partial Z} \right)_{XZ} = \sum_{j=1}^{9} \left\{ R_y^j \hat{y}_j \right\} \left\{ Y_x \right\} \frac{L_j}{t} \theta_{XZ}^j
$$

where $t$ is the thickness for the point at which these expressions are calculated.

Similar expressions are obtained for the contribution of the second vector:

$$
\left( \frac{\partial U}{\partial Z} \right)_{YZ} = \sum_{j=1}^{9} \left\{ S_x^j \hat{x}_j \right\} \left\{ X_x \right\} \frac{L_j}{t} \theta_{YZ}^j
$$

$$
\left( \frac{\partial V}{\partial Z} \right)_{YZ} = \sum_{j=1}^{9} \left\{ S_y^j \hat{y}_j \right\} \left\{ X_y \right\} \frac{L_j}{t} \theta_{YZ}^j
$$
In order to write \( \partial U/\partial Z \) and \( \partial V/\partial Z \) in matrix form, matrices \([L], [R], \) and \([S] \) are defined as follows:

\[
[L] = \begin{bmatrix}
L^1 & 0 & 0 & \ldots & 0 \\
0 & L^2 & 0 & \ldots & 0 \\
0 & 0 & L^3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & L^9
\end{bmatrix}
\]  
\hspace{1cm} (40)

\[
[R] = \begin{bmatrix}
R^1_x & R^2_x & R^3_x & \ldots & R^9_x \\
R^1_y & R^2_y & R^3_y & \ldots & R^9_y
\end{bmatrix}
\]  
\hspace{1cm} (41)

\[
[S] = \begin{bmatrix}
S^1_x & S^2_x & S^3_x & \ldots & S^9_x \\
S^1_y & S^2_y & S^3_y & \ldots & S^9_y
\end{bmatrix}
\]  
\hspace{1cm} (42)

Noting that

\[
\frac{\partial U}{\partial Z} = \begin{bmatrix} \frac{\partial U}{\partial Z} \end{bmatrix}_{xz} + \begin{bmatrix} \frac{\partial U}{\partial Z} \end{bmatrix}_{yz}
\]  
\hspace{1cm} (43)

\[
\frac{\partial V}{\partial Z} = \begin{bmatrix} \frac{\partial V}{\partial Z} \end{bmatrix}_{xz} + \begin{bmatrix} \frac{\partial V}{\partial Z} \end{bmatrix}_{yz}
\]  
\hspace{1cm} (44)

and recalling (2) and (3) the following expressions are obtained:

\[
\frac{\partial U}{\partial Z} = \frac{1}{t} \left\{ (x)^T [K] [L] \{e^e_{xz}\} + (x)^T [S] [L] \{e^e_{yz}\} \right\}
\]  
\hspace{1cm} (45)

\[
\frac{\partial V}{\partial Z} = \frac{1}{t} \left\{ (y)^T [K] [L] \{e^e_{xz}\} + (y)^T [S] [L] \{e^e_{yz}\} \right\}
\]  
\hspace{1cm} (46)
(d) The following second derivatives of the displacements $U$ and $V$
can now be easily calculated:

\[
\frac{\partial^2 U}{\partial x \partial z} = \frac{1}{c} \left( \{x\}^T [R] \frac{\partial L}{\partial x} \{\theta^e_{xz}\} + \{x\}^T [S] \frac{\partial L}{\partial x} \{\theta^e_{yz}\} \right) \tag{47}
\]

\[
\frac{\partial^2 U}{\partial y \partial z} = \frac{1}{c} \left( \{x\}^T [R] \frac{\partial L}{\partial y} \{\theta^e_{xz}\} + \{x\}^T [S] \frac{\partial L}{\partial y} \{\theta^e_{yz}\} \right) \tag{48}
\]

\[
\frac{\partial^2 V}{\partial x \partial z} = \frac{1}{c} \left( \{y\}^T [R] \frac{\partial L}{\partial x} \{\theta^e_{xz}\} + \{y\}^T [S] \frac{\partial L}{\partial x} \{\theta^e_{yz}\} \right) \tag{49}
\]

\[
\frac{\partial^2 V}{\partial y \partial z} = \frac{1}{c} \left( \{y\}^T [R] \frac{\partial L}{\partial y} \{\theta^e_{xz}\} + \{y\}^T [S] \frac{\partial L}{\partial y} \{\theta^e_{yz}\} \right) \tag{50}
\]

These derivatives will be used in the next section for the
definition of the strains and stresses.
3. FINAL (CONSTRAINED) VERSION OF THE ELEMENT

The most critical part in the formulation of finite elements with discrete shear constraints is the definition and application of these constraints. A great amount of intuition associated with a process of "trial and error" appears to be the only way. Reference (I-5) gives an account of the struggle to define the appropriate shear constraints for the semiloof shell element. A number of different possibilities are enumerated and discussed and finally a set of eleven constraints are chosen.

The semiloof plate element uses the same constraints as the shell element although the resulting equations are different. The variables to be eliminated from the initial version of the element are the eight rotations along the side at each loof node, $\theta_{YZ}^j$ (j=1,8), the normal displacement at the central node $\omega^9$ and the two rotations $\theta_{XZ}^9$ and $\theta_{YZ}^9$ at the same point. The resulting element, Figure 7, has therefore 16 degrees of freedom, 8 rotations normal to the sides at the loof nodes and 8 normal displacements at the corner and mid-side nodes.

3.1 Definition and Formulation of the Shear Constraints

(a) The rotation $\theta_{YZ}^j$ at each loof node is eliminated by setting to zero the shear strain $\gamma_{XZ}$ at the same node. From here a set of 8 constraining equations (one for each loof node) are derived.

For the formulation of these equations the usual definition of shear strain is recalled (T-2):  

$$\gamma_{YZ} = \frac{\partial W}{\partial Y} + \frac{\partial V}{\partial Z}$$  (51)
The two terms on the right hand side of this expression are easily calculated using expressions (12) and (46), the following constraining equation being obtained:

\[
\left[ \frac{3N}{2Y} \right] \{\delta^e\} + \frac{1}{t} \left[ \{Y\}^T [R] [L] \{\theta^e_{xz}\} + \{Y\}^T [S] [L] \{\theta^e_{yz}\} \right] = 0 \quad (52)
\]

where the shape functions and shape functions derivatives are calculated for the corresponding loof node. It should be noticed that this expression represents 8 different equations, one for each loof node.

(b) For the elimination of the two rotations at the centre the following shear vector is defined:

\[
\begin{align*}
\vec{\gamma} &= \hat{x} \gamma_{xz} + \hat{y} \gamma_{yz} \\
\end{align*}
\]

The two constraining equations for these two variables are as follows:

\[
\begin{align*}
\int_{\Omega} \hat{R}^9 \vec{\gamma} \, d\Omega &= 0 \quad (54) \\
\int_{\Omega} \hat{S}^9 \vec{\gamma} \, d\Omega &= 0 \quad (55)
\end{align*}
\]

where \(\Omega\) represents the element area and \(d\Omega\) its differential.

Using (53) the integral equation (54) becomes:

\[
\int_{\Omega} \left( \{R^9 \times R^9\}_{X} \begin{bmatrix} X \\ X \end{bmatrix} \gamma_{xz} + \{R^9 \times R^9\}_{Y} \begin{bmatrix} Y \\ Y \end{bmatrix} \gamma_{yz} \right) \, d\Omega = 0 \quad (56)
\]

This integral will be evaluated numerically using a 2x2 point gauss rule.

If the values of \(X\), \(Y\), \(\gamma_{xz}\) and \(\gamma_{yz}\) are labelled with \(K\) for each of the 4 integrating points, equation (56) becomes:

\[
\sum_{K=1}^{4} \begin{bmatrix} \{R^9 \times R^9\}_{X} & \{R^9 \times R^9\}_{Y} \end{bmatrix}_{K} \begin{bmatrix} X \\ X \\ Y \\ Y \end{bmatrix}_{K} \gamma_{xz}^{K} + \begin{bmatrix} \{R^9 \times R^9\}_{X} & \{R^9 \times R^9\}_{Y} \end{bmatrix}_{K} \begin{bmatrix} Y \\ Y \\ X \\ X \end{bmatrix}_{K} \gamma_{yz}^{K} \Delta K \, t_{K} = 0 \quad (56a)
\]
where $t_K$ is the thickness for the integrating point $K$ and $A_K$ is the area integrating factor defined by

$$A_K = \frac{\partial x}{\partial \zeta} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \zeta}$$

(57)

The calculation of $\gamma_{xz}^K$ and $\gamma_{yz}^K$ in (56a) is made as in (51) for $\gamma_{yz}^K$.

For $\gamma_{xz}^K$ a similar expression can be derived:

$$\gamma_{xz}^K = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

(58)

or, using (12) and (46)

$$\gamma_{xz}^K = \left\{ \frac{\partial n}{\partial x} \right\} \{s^e\} + \frac{1}{t} \left\{ (x)^T \left[ R \right] L \right\} \{s^e_{xz}\} + \left\{ (x)^T \left[ S \right] L \right\} \{s^e_{yz}\}$$

(59)

For the tenth constraining equation, (55), an expression parallel to (56a) is obtained:

$$\sum_{K=1}^{4} \left\{ \left[ s^9 \right] \{x\} \right\} \gamma_{xz}^K + \left\{ \left[ s^9 \right] \{y\} \right\} \gamma_{yz}^K \left[ A_K \right] \{t_K\} = 0$$

(60)

(c) Finally, the normal displacement at the centre is eliminated by setting to zero the integral of the shear strain, $\gamma_{xz}$, along the element boundary, B:

$$\int_B \gamma_{xz} \, dB = 0$$

(61)

This integral is evaluated numerically using a 2 point gauss rule along each side. This implies the necessity of evaluating $\gamma_{xz}$ at each loof node. If these discrete values are denoted by $\gamma_{xz}^j$, (61) becomes:

$$\sum_{j=1}^{8} \gamma_{xz}^j \cdot I_j \cdot t_j = 0$$

(61.a)
where the integrating factor along the side $I_j$ is defined by

$$I_j^i = \left[ \begin{array}{c} y_x \\ y_y \end{array} \right]_j \begin{bmatrix} y_x \\ y_y \end{bmatrix}_{i,j}^i$$

(62)

(d) The 8 equations (52) and equations (56a), (60) and (61a) can be assembled in a single system of equations that in matrix form can be expressed as

$$[C] \{P^e\} = \{0\}$$

(63)

where $\{P^e\}$ represents a vector of the 27 nodal parameters of the element and $[C]$ is an (11x27) matrix containing shape functions and their derivatives as defined in sections (a), (b) and (c). Vector $\{P^e\}$ can be partitioned into two vectors, one, $\{P^e_A\}$ corresponding to the 16 degrees of freedom to be maintained, the other, $\{P^e_B\}$ corresponding to the 11 to be eliminated. Accordingly the matrix of constraints $[C]$ is also partitioned into $[C_A]$ and $[C_B]$ with dimensions respectively (11x16) and (11x11). Equation (2.63) then takes the form:

$$[C_A \mid C_B] \begin{bmatrix} P^e_A \\ \vdots \\ P^e_B \end{bmatrix} = \{0\}$$

(63a)

3.2 Constraining of the Element

(a) The normal displacement $\omega(x,y)$ and the derivatives $\partial^2 U/\partial x^2 Z$, $\partial^2 U/\partial y^2 Z$, $\partial^2 V/\partial x^2 Z$ and $\partial^2 V/\partial y^2 Z$ were previously defined and are expressed in terms of the element parameters in equations (12) and (47) to (50). These can be assembled in a single expression:
\[
\left\{ \omega \frac{\partial^2 U}{\partial x \partial z} \frac{\partial^2 U}{\partial y \partial z} \frac{\partial^2 V}{\partial x \partial z} \frac{\partial^2 V}{\partial y \partial z} \right\}^T = \{G\}^T = [S] \{p^e\} \tag{64}
\]

where \([S]\) is a (5x27) matrix containing shape functions and their derivatives. In a similar manner to section (3d) the nodal variables to be retained can be "separated" from the ones to be eliminated, equation (64) then becoming

\[
\{G\}^T = \begin{bmatrix} S_A & S_B \end{bmatrix} \begin{bmatrix} p^e_A \\ \hline p^e_B \end{bmatrix} \tag{65}
\]

where the dimensions of submatrices \([S_A]\) and \([S_B]\) are respectively (5x16) and (5x11).

(b) Equation (65) will now be constrained to 16 unknowns, or, in other words, \([p^e_B]\) will be eliminated from (65) by the use of equation (63a). This equation can be written as follows:

\[
[C_A] \{p^e_A\} + [C_B] \{p^e_B\} = \{0\} \tag{66}
\]

or

\[
\{p^e_B\} = -[C_B]^{-1} [C_A] \{p^e_A\} \tag{66a}
\]

Also (65) can be rewritten as

\[
\{G\}^T = \begin{bmatrix} S_A & S_B \end{bmatrix} \begin{bmatrix} p^e_A \\ \hline p^e_B \end{bmatrix} \tag{67}
\]

and substituting (66a) into this expression results in

\[
\{G\}^T = \begin{bmatrix} S_A \end{bmatrix} \{p^e_A\} - \begin{bmatrix} S_B \end{bmatrix} [C_B]^{-1} [C_A] \{p^e_A\} \tag{67a}
\]

or, finally

\[
\left\{ \omega \frac{\partial^2 U}{\partial x \partial z} \frac{\partial^2 U}{\partial y \partial z} \frac{\partial^2 V}{\partial x \partial z} \frac{\partial^2 V}{\partial y \partial z} \right\}^T = \begin{bmatrix} S_A \end{bmatrix} - \begin{bmatrix} S_B \end{bmatrix} [C_B]^{-1} [C_A] \{p^e_A\} = [H] \{p^e_A\} \tag{68}
\]

where \([H]\) is a (5x16) matrix.
CHAPTER II - THE SEMILOOF PLATE ELEMENT

The value of $\omega$ defined here will be used in this chapter for the definition of pressure loads, and in Chapter IV for the establishment of the element mass matrix. The second derivatives contained in (68) will be used for the definition of the strains and the stiffness matrix.

3.3 Element Matrices

The local strains are now defined in terms of the second derivatives contained in (68), as

$$
\begin{bmatrix}
\varepsilon_X \\
\varepsilon_Y \\
\varepsilon_{XZ}
\end{bmatrix} =
\begin{bmatrix}
-\frac{\partial^2 U}{\partial X \partial Z} \\
-\frac{\partial^2 V}{\partial Y \partial Z} \\
\frac{\partial^2 U}{\partial Y \partial Z} + \frac{\partial^2 V}{\partial X \partial Z}
\end{bmatrix} = [B] \{p_A^e\}
$$

(69)

where $[B]$ is a (3x16) matrix formed using the last 4 rows of matrix $[H]$. It is worth noting that, unlike in the classical plate theory, no second derivatives appear in this $[B]$ matrix.

The calculation of the stiffness matrix, $[K^e]$, follows the usual procedure in finite element formulation of plate bending problems (Z-2, Chapt. 10) and is given by

$$
[K^e] = \int_V [B]^T [D] [B] \, dV
$$

(70)

where $V$ represents the volume of the element and the elasticity matrix, $[D]$ is

$$
[D] = \frac{Et^3}{12(1-\nu^2)}
$$

$$
\begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & (1-\nu)/2
\end{bmatrix}
$$

(71)
t, \nu and E being respectively the thickness, Poisson's ratio and Young's modulus.

The nodal forces due to distributed loads, \{F^e\}, are

\[
\{F^e\} = \int_A \{N_A\} p \, dA
\]

(72)

where \(A\) is the element area, \(\{N_A\}\) is a vector composed of the first line of matrix \([H]\) and \(p = p(x,y)\) is the distributed load function.

Once the element parameters are known the element stresses, \{\sigma\} can be calculated by

\[
\{\sigma\} = [D] [B] \{F^e\}_A
\]

(73)

Perhaps it is unnecessary to say that expressions (70) and (72) are numerically integrated. The following 5 point rule has been used in the examples shown in this chapter:

\[
\int_{-1}^{1} \int_{-1}^{1} \phi(\xi,\eta) \, d\xi d\eta = 0.2 \phi(0,0) + 0.95 \sum_1^{4} \phi(\pm0.592348878 \pm 0.592348878)
\]

(74)

The stresses \{\sigma\} shown later in this chapter are calculated at the 2x2 Gauss quadrature points.

4. PATCH TEST BEHAVIOUR OF THE ELEMENT

The performance of an element in practice is usually assessed by the solution of a number of problems whose analytical solution is known. A refinement of the mesh is then shown to yield better results, the exact solution being possibly obtained when the size of the element converges to zero.
In this chapter we start by considering this last situation, that is the limit situation when the elements become infinitesimal. In this case the elements are in a state of constant strain, so we merely have to consider such a state and study the behaviour of a patch of elements under these conditions. If the constant strain state cannot be reproduced the element should be treated with caution.

The problems solved fall into two categories: problems including only elements with straight edges and problems which include elements with curved edges. This division was done in order to isolate the difficulty of approximating exactly a curved boundary.

4.1 Square Plate Patch Tests

A square plate with side length 2.0 and thickness 0.05 was the basis for a number of tests, including the state of constant strain of Figure 8 and the rigid body motion of Figure 9; different types of meshes were used. The conditions and results of these tests are now presented.

(a) State of constant strain using the one element mesh of Figure 10.a. The boundary conditions are as shown on Table 1 and the principal stresses given by the program for each of the 4 Gauss points are:

300.00000 on direction (1.00000, 0.00000)
0.00000 on direction (0.00000, 1.00000)

<table>
<thead>
<tr>
<th>Node</th>
<th>Variable Prescribed Node Value</th>
<th>Variable</th>
<th>Prescribed Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.00000</td>
<td>4</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0.00000</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2.00000</td>
<td>8</td>
</tr>
</tbody>
</table>

(see note next page)

TABLE 1
(b) Rigid body motion using the mesh of Figure 10a.

The boundary conditions are included in Table 2 and all the stresses are 0.00000.

<table>
<thead>
<tr>
<th>Node</th>
<th>Variable</th>
<th>Prescribed Value</th>
<th>Node</th>
<th>Variable</th>
<th>Prescribed Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.10000</td>
<td>2</td>
<td>2</td>
<td>0.02500</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.12500</td>
<td>2</td>
<td>3</td>
<td>0.02500</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.15000</td>
<td>4</td>
<td>2</td>
<td>-0.02500</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.17500</td>
<td>4</td>
<td>3</td>
<td>-0.02500</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0.20000</td>
<td>6</td>
<td>2</td>
<td>-0.02500</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>0.17500</td>
<td>6</td>
<td>3</td>
<td>-0.02500</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0.15000</td>
<td>8</td>
<td>2</td>
<td>-0.02500</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>0.12500</td>
<td>8</td>
<td>3</td>
<td>-0.02500</td>
</tr>
</tbody>
</table>

**TABLE 2**

(c) State of constant strain using the 9 elements mesh of Figure 10b.

The boundary conditions are shown in Table 3 and the stresses at each of the 36 Gauss points are 300.00000 along direction (1.00000, 0.00000)

0.00000 along direction (0.00000, 1.00000)

<table>
<thead>
<tr>
<th>Node</th>
<th>Variable</th>
<th>Prescribed Value</th>
<th>Node</th>
<th>Variable</th>
<th>Prescribed Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.00000</td>
<td>8</td>
<td>3</td>
<td>2.00000</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>0.00000</td>
<td>14</td>
<td>2</td>
<td>2.00000</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>2.00000</td>
<td>14</td>
<td>3</td>
<td>2.00000</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>2.00000</td>
<td>16</td>
<td>2</td>
<td>-2.00000</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>2.00000</td>
<td>16</td>
<td>3</td>
<td>-2.00000</td>
</tr>
</tbody>
</table>

Note: Variable 1 refers to the normal displacement, variables 2 and 3 (mid side nodes only) refer to the 2 rotations along the side.
(d) Rigid body motion using the 9 elements mesh of Figure 10b. The boundary conditions for this test are shown on Table 4. Under these boundary conditions the internal nodes must take values according to the movement imposed and the stresses must be zero. The values given by the program for the stresses are (0.00000) and the displacements of the internal nodes are shown on Table 5.

<table>
<thead>
<tr>
<th>Node</th>
<th>Variable</th>
<th>Prescribed Value</th>
<th>Node</th>
<th>Variable</th>
<th>Prescribed Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.10000</td>
<td>2</td>
<td>2</td>
<td>0.02500</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.11000</td>
<td>2</td>
<td>3</td>
<td>0.02500</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.12000</td>
<td>4</td>
<td>2</td>
<td>0.02500</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.13500</td>
<td>4</td>
<td>3</td>
<td>0.02500</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0.15000</td>
<td>14</td>
<td>2</td>
<td>0.02500</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>0.16000</td>
<td>14</td>
<td>3</td>
<td>0.02500</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0.17000</td>
<td>16</td>
<td>2</td>
<td>-0.02500</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>0.18500</td>
<td>16</td>
<td>3</td>
<td>-0.02500</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>0.20000</td>
<td>6</td>
<td>2</td>
<td>-0.02500</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>0.19250</td>
<td>6</td>
<td>3</td>
<td>-0.02500</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>0.18500</td>
<td>8</td>
<td>2</td>
<td>-0.02500</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>0.16750</td>
<td>8</td>
<td>3</td>
<td>-0.02500</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>0.15000</td>
<td>10</td>
<td>2</td>
<td>-0.02500</td>
</tr>
<tr>
<td>14</td>
<td>1</td>
<td>0.14000</td>
<td>10</td>
<td>3</td>
<td>-0.02500</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
<td>0.13000</td>
<td>12</td>
<td>2</td>
<td>-0.02500</td>
</tr>
<tr>
<td>16</td>
<td>1</td>
<td>0.11500</td>
<td>12</td>
<td>3</td>
<td>-0.02500</td>
</tr>
</tbody>
</table>

**TABLE 4**

<table>
<thead>
<tr>
<th>Node</th>
<th>Value of the Displacement</th>
<th>Node</th>
<th>Value of the Displacement</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Theoretical</td>
<td></td>
<td>Theoretical</td>
</tr>
<tr>
<td></td>
<td>F.E.M.</td>
<td></td>
<td>F.E.M.</td>
</tr>
<tr>
<td>17</td>
<td>0.12750</td>
<td>27</td>
<td>0.14250</td>
</tr>
<tr>
<td>18</td>
<td>0.13500</td>
<td>28</td>
<td>0.13625</td>
</tr>
<tr>
<td>19</td>
<td>0.13250</td>
<td>29</td>
<td>0.14875</td>
</tr>
<tr>
<td>20</td>
<td>0.15250</td>
<td>30</td>
<td>0.16750</td>
</tr>
<tr>
<td>21</td>
<td>0.16625</td>
<td>31</td>
<td>0.16500</td>
</tr>
<tr>
<td>22</td>
<td>0.16250</td>
<td>32</td>
<td>0.16520</td>
</tr>
<tr>
<td>23</td>
<td>0.15500</td>
<td>33</td>
<td>0.15250</td>
</tr>
<tr>
<td>24</td>
<td>0.14750</td>
<td>34</td>
<td>0.16250</td>
</tr>
<tr>
<td>25</td>
<td>0.14125</td>
<td>35</td>
<td>0.16875</td>
</tr>
<tr>
<td>26</td>
<td>0.14500</td>
<td>36</td>
<td>0.17625</td>
</tr>
</tbody>
</table>

**TABLE 5**
(e) State of constant strain using the 2 element mesh of Fig.11a. The boundary conditions are shown on Table 6 and stresses at each of the 6 points were calculated to be

300.00000 along direction (1.00000, 0.00000)
0.00000 along direction (0.00000, 1.00000)

<table>
<thead>
<tr>
<th>Node</th>
<th>Variable</th>
<th>Prescribed Value</th>
<th>Node</th>
<th>Variable</th>
<th>Prescribed Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.00000</td>
<td>4</td>
<td>3</td>
<td>2.00000</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0.00000</td>
<td>8</td>
<td>2</td>
<td>-2.00000</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2.00000</td>
<td>8</td>
<td>3</td>
<td>-2.00000</td>
</tr>
</tbody>
</table>

TABLE 6

(f) Rigid body motion using the 2 element mesh of Figure 11a. All the calculated stresses have the value (0.00000) and the normal displacement of point 9, theoretically 0.15, is reproduced exactly: 0.150000000. The boundary conditions are indicated in Table 7.

<table>
<thead>
<tr>
<th>Node</th>
<th>Variable</th>
<th>Prescribed Value</th>
<th>Node</th>
<th>Variable</th>
<th>Prescribed Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.10000</td>
<td>2</td>
<td>2</td>
<td>0.02500</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.12500</td>
<td>2</td>
<td>3</td>
<td>0.02500</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.15000</td>
<td>4</td>
<td>2</td>
<td>-0.02500</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.17500</td>
<td>4</td>
<td>3</td>
<td>-0.02500</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0.20000</td>
<td>6</td>
<td>2</td>
<td>-0.02500</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>0.17500</td>
<td>6</td>
<td>3</td>
<td>-0.02500</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0.15000</td>
<td>8</td>
<td>2</td>
<td>-0.02500</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>0.12500</td>
<td>8</td>
<td>3</td>
<td>-0.02500</td>
</tr>
</tbody>
</table>

TABLE 7
(g) State of constant strain using the 8 element mesh of Figure 11b.

The principal stresses calculated are

300.00000 along direction (1.00000, 0.00000)

0.00000 along direction (0.00000, 1.00000)

The boundary conditions are shown in Table 8.

<table>
<thead>
<tr>
<th>Node</th>
<th>Variable</th>
<th>Prescribed Value</th>
<th>Node</th>
<th>Variable</th>
<th>Prescribed Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.00000</td>
<td>4</td>
<td>3</td>
<td>2.00000</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0.00000</td>
<td>8</td>
<td>2</td>
<td>-2.00000</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2.00000</td>
<td>8</td>
<td>3</td>
<td>-2.00000</td>
</tr>
</tbody>
</table>

TABLE 8

(h) Rigid body motion using the 8 element mesh of Figure 11b.

The boundary conditions for this test are shown in Table 9. All the stresses calculated are zero (0.00000) and all the internal nodes displacements are reproduced exactly (Table 10).

<table>
<thead>
<tr>
<th>Node</th>
<th>Variable</th>
<th>Prescribed Value</th>
<th>Node</th>
<th>Variable</th>
<th>Prescribed Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.10000</td>
<td>2</td>
<td>2</td>
<td>0.02500</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.12500</td>
<td>2</td>
<td>3</td>
<td>0.02500</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.15000</td>
<td>4</td>
<td>2</td>
<td>-0.02500</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.17500</td>
<td>4</td>
<td>3</td>
<td>-0.02500</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0.20000</td>
<td>6</td>
<td>2</td>
<td>-0.02500</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>0.17500</td>
<td>6</td>
<td>3</td>
<td>-0.02500</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0.15000</td>
<td>8</td>
<td>2</td>
<td>-0.02500</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>0.12500</td>
<td>8</td>
<td>3</td>
<td>-0.02500</td>
</tr>
</tbody>
</table>

TABLE 9
CHAPTER II - THE SEMILOOF PLATE ELEMENT

<table>
<thead>
<tr>
<th>Node</th>
<th>Value of the Displacement Theoretical</th>
<th>F.E.M.</th>
<th>Node</th>
<th>Value of the Displacement Theoretical</th>
<th>F.E.M.</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>0.11625</td>
<td>0.1162500000</td>
<td>16</td>
<td>0.1325</td>
<td>0.132500000</td>
</tr>
<tr>
<td>10</td>
<td>0.124375</td>
<td>0.1243750000</td>
<td>17</td>
<td>0.140625</td>
<td>0.140625000</td>
</tr>
<tr>
<td>11</td>
<td>0.149375</td>
<td>0.1493750000</td>
<td>18</td>
<td>0.14875</td>
<td>0.148750000</td>
</tr>
<tr>
<td>12</td>
<td>0.174375</td>
<td>0.1743750000</td>
<td>19</td>
<td>0.1575</td>
<td>0.157500000</td>
</tr>
<tr>
<td>13</td>
<td>0.183125</td>
<td>0.1831250000</td>
<td>20</td>
<td>0.16625</td>
<td>0.166250000</td>
</tr>
<tr>
<td>14</td>
<td>0.158125</td>
<td>0.1581250000</td>
<td>21</td>
<td>0.149375</td>
<td>0.149375000</td>
</tr>
<tr>
<td>15</td>
<td>0.14125</td>
<td>0.1412500000</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE 10

As the previous results show, states of constant strain are reproduced with high accuracy for patches of arbitrary quadrilaterals and triangles.

4.2 Circular Plate Patch Tests

In this section results are given for a quarter of a circular plate in a state of constant strain. This is represented in Figure 12, where the plate dimensions and material properties are also given.

(a) State of constant strain using the 1 element mesh of Figure 13a. The boundary conditions are as shown in Table 11 and impose a constant stress equal to (300.0).

<table>
<thead>
<tr>
<th>Node</th>
<th>Variable</th>
<th>Prescribed Value</th>
<th>Node</th>
<th>Variable</th>
<th>Prescribed Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>0.00000</td>
<td>6</td>
<td>1</td>
<td>0.00000</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0.00000</td>
<td>6</td>
<td>2</td>
<td>2.00000</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.00000</td>
<td>6</td>
<td>3</td>
<td>2.00000</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.00000</td>
<td>7</td>
<td>1</td>
<td>0.00000</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>2.00000</td>
<td>8</td>
<td>2</td>
<td>0.00000</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>2.00000</td>
<td>8</td>
<td>3</td>
<td>0.00000</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0.00000</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE 11
The stresses are shown in Table 12, where the worst stress (298.09435) differs 0.635% from the exact value.

<table>
<thead>
<tr>
<th>Element</th>
<th>Point Coordinate x</th>
<th>y</th>
<th>Princ. Stress</th>
<th>Direction</th>
<th>Princ. Stress</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2123</td>
<td>0.2123</td>
<td>299.83569</td>
<td>0.70711</td>
<td>0.70711</td>
</tr>
<tr>
<td></td>
<td>0.7809</td>
<td>0.1877</td>
<td>300.33087</td>
<td>0.70436</td>
<td>0.70984</td>
</tr>
<tr>
<td></td>
<td>0.6588</td>
<td>0.6588</td>
<td>299.60313</td>
<td>0.70711</td>
<td>0.70711</td>
</tr>
<tr>
<td></td>
<td>0.1877</td>
<td>0.7809</td>
<td>300.33087</td>
<td>0.70984</td>
<td>0.70436</td>
</tr>
</tbody>
</table>

TABLE 12

(b) State of constant strain using the 9 elements mesh of Fig. 13b.

The boundary conditions are shown in Table 13 and the stresses in Table 14.

<table>
<thead>
<tr>
<th>Node</th>
<th>Variable</th>
<th>Prescribed Value</th>
<th>Node</th>
<th>Variable</th>
<th>Prescribed Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>22</td>
<td>2</td>
<td>0.00000</td>
<td>31</td>
<td>2</td>
<td>0.00000</td>
</tr>
<tr>
<td>22</td>
<td>3</td>
<td>0.00000</td>
<td>31</td>
<td>3</td>
<td>0.00000</td>
</tr>
<tr>
<td>23</td>
<td>2</td>
<td>0.00000</td>
<td>38</td>
<td>2</td>
<td>0.00000</td>
</tr>
<tr>
<td>23</td>
<td>3</td>
<td>0.00000</td>
<td>38</td>
<td>3</td>
<td>0.00000</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.00000</td>
<td>11</td>
<td>2</td>
<td>2.00000</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.00000</td>
<td>11</td>
<td>3</td>
<td>2.00000</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>0.00000</td>
<td>12</td>
<td>2</td>
<td>2.00000</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>0.00000</td>
<td>12</td>
<td>3</td>
<td>2.00000</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0.00000</td>
<td>13</td>
<td>2</td>
<td>2.00000</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>0.00000</td>
<td>13</td>
<td>3</td>
<td>2.00000</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>0.00000</td>
<td>14</td>
<td>2</td>
<td>2.00000</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>0.00000</td>
<td>14</td>
<td>3</td>
<td>2.00000</td>
</tr>
<tr>
<td>12</td>
<td>1</td>
<td>0.00000</td>
<td>15</td>
<td>2</td>
<td>2.00000</td>
</tr>
<tr>
<td>13</td>
<td>1</td>
<td>0.00000</td>
<td>15</td>
<td>3</td>
<td>2.00000</td>
</tr>
<tr>
<td>14</td>
<td>1</td>
<td>0.00000</td>
<td>16</td>
<td>2</td>
<td>2.00000</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
<td>0.00000</td>
<td>16</td>
<td>3</td>
<td>2.00000</td>
</tr>
<tr>
<td>16</td>
<td>1</td>
<td>0.00000</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE 13
In this table the worst stress has the value 300.24021 and corresponds to an error of 0.080%. This represents a considerable improvement compared with the maximum error for the test previously referred. It is worth noting that a five element mesh with the same number of nodes along the curved side of the plate (Figure 13c) does not yield better values than the one element mesh of Figure 13a; neither does a 16 element mesh with only 4 elements along the curved boundary (Figure 13d) yield better results than the mesh of Figure 13b. This appears to indicate that the bigger errors observed in the circular plate, when compared with the square plate, are originated by the difficulty of exactly representing the boundary.

<table>
<thead>
<tr>
<th>Element</th>
<th>Point Coordinate</th>
<th>Princ. Stress</th>
<th>Direction</th>
<th>Princ. Stress</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>x</td>
<td>y</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.0758</td>
<td>0.1924</td>
<td>299.99455</td>
<td>0.83150</td>
</tr>
<tr>
<td></td>
<td>0.2829</td>
<td>0.3715</td>
<td>300.02535</td>
<td>0.40328</td>
</tr>
<tr>
<td></td>
<td>0.1509</td>
<td>0.5676</td>
<td>299.97935</td>
<td>0.50580</td>
</tr>
<tr>
<td></td>
<td>0.0404</td>
<td>0.4985</td>
<td>299.93848</td>
<td>0.86345</td>
</tr>
<tr>
<td>2</td>
<td>0.1856</td>
<td>0.0736</td>
<td>299.98889</td>
<td>0.45412</td>
</tr>
<tr>
<td></td>
<td>0.4501</td>
<td>0.0553</td>
<td>299.93192</td>
<td>0.39651</td>
</tr>
<tr>
<td></td>
<td>0.5378</td>
<td>0.2064</td>
<td>300.10481</td>
<td>0.26149</td>
</tr>
<tr>
<td></td>
<td>0.3866</td>
<td>0.2747</td>
<td>300.01131</td>
<td>0.99997</td>
</tr>
<tr>
<td>3</td>
<td>0.6423</td>
<td>0.0500</td>
<td>300.05025</td>
<td>0.71292</td>
</tr>
<tr>
<td></td>
<td>0.9030</td>
<td>0.0540</td>
<td>299.97361</td>
<td>0.74988</td>
</tr>
<tr>
<td></td>
<td>0.8973</td>
<td>0.1999</td>
<td>299.93954</td>
<td>0.84565</td>
</tr>
<tr>
<td></td>
<td>0.6746</td>
<td>0.1864</td>
<td>300.02511</td>
<td>0.98380</td>
</tr>
<tr>
<td>4</td>
<td>0.6502</td>
<td>0.2723</td>
<td>300.03048</td>
<td>0.86933</td>
</tr>
<tr>
<td></td>
<td>0.8697</td>
<td>0.3013</td>
<td>299.96002</td>
<td>0.42964</td>
</tr>
<tr>
<td></td>
<td>0.8011</td>
<td>0.4295</td>
<td>299.95721</td>
<td>0.99829</td>
</tr>
<tr>
<td></td>
<td>0.5515</td>
<td>0.3700</td>
<td>300.03785</td>
<td>0.35426</td>
</tr>
<tr>
<td>5</td>
<td>0.5248</td>
<td>0.4503</td>
<td>300.09107</td>
<td>0.99934</td>
</tr>
<tr>
<td></td>
<td>0.7533</td>
<td>0.5213</td>
<td>300.02820</td>
<td>0.94853</td>
</tr>
<tr>
<td></td>
<td>0.6932</td>
<td>0.6412</td>
<td>299.90339</td>
<td>0.99234</td>
</tr>
<tr>
<td></td>
<td>0.5509</td>
<td>0.5712</td>
<td>299.95400</td>
<td>0.97240</td>
</tr>
<tr>
<td>6</td>
<td>0.2408</td>
<td>0.6073</td>
<td>299.90645</td>
<td>0.65012</td>
</tr>
<tr>
<td></td>
<td>0.3874</td>
<td>0.4734</td>
<td>299.92570</td>
<td>0.89277</td>
</tr>
<tr>
<td></td>
<td>0.4659</td>
<td>0.5727</td>
<td>300.19392</td>
<td>0.40271</td>
</tr>
<tr>
<td></td>
<td>0.3759</td>
<td>0.6466</td>
<td>299.94302</td>
<td>-0.27053</td>
</tr>
<tr>
<td>7</td>
<td>0.4511</td>
<td>0.6982</td>
<td>299.96892</td>
<td>0.98362</td>
</tr>
<tr>
<td></td>
<td>0.5310</td>
<td>0.6378</td>
<td>299.96245</td>
<td>0.52782</td>
</tr>
<tr>
<td></td>
<td>0.6304</td>
<td>0.7164</td>
<td>299.92210</td>
<td>-0.03529</td>
</tr>
<tr>
<td></td>
<td>0.5215</td>
<td>0.7998</td>
<td>300.04823</td>
<td>0.26103</td>
</tr>
<tr>
<td>8</td>
<td>0.2135</td>
<td>0.7184</td>
<td>299.96453</td>
<td>0.83008</td>
</tr>
<tr>
<td></td>
<td>0.3658</td>
<td>0.7199</td>
<td>300.01192</td>
<td>0.99128</td>
</tr>
<tr>
<td></td>
<td>0.4283</td>
<td>0.8642</td>
<td>300.05508</td>
<td>0.94927</td>
</tr>
<tr>
<td></td>
<td>0.2856</td>
<td>0.8880</td>
<td>300.06245</td>
<td>0.57326</td>
</tr>
<tr>
<td>9</td>
<td>0.0334</td>
<td>0.6925</td>
<td>300.09225</td>
<td>0.81800</td>
</tr>
<tr>
<td></td>
<td>0.1242</td>
<td>0.7111</td>
<td>299.97710</td>
<td>0.92420</td>
</tr>
<tr>
<td></td>
<td>0.1832</td>
<td>0.9071</td>
<td>300.03803</td>
<td>0.90649</td>
</tr>
<tr>
<td></td>
<td>0.0495</td>
<td>0.9165</td>
<td>300.03946</td>
<td>0.67169</td>
</tr>
</tbody>
</table>

TABLE 14
(c) State of constant strain using the 2 element mesh of Figure 14a. The stresses in this case are not as accurate as in case (a) which has approximately the same number of degrees of freedom. The maximum stress calculated was (308.50789) which corresponds to an error of 2.8% and the minimum stress was (281.86827) or, a 6.4% error. The boundary conditions are as in case (a).

(d) State of constant strain using the 14 element mesh of Figure 14b. The maximum stress is in this case (310.04916), which represents an error of 3.3% and the minimum (290.83060) (error 3%). There is therefore a significant increase of the accuracy between the two cases (c) and (d), although not as significant as between cases (a) and (b). The boundary conditions for this case are the same as in case (b).

5. AN ASSESSMENT OF THE ELEMENT PERFORMANCE

In practice states of constant strain are very rare and so the performance of a finite element must be assessed in more complicated and real displacement fields. It is well known for instance that the Morley constant strain element (M-3) passes the patch test but this does not mean that it is a good performer. In fact its performance is very poor when compared with some of the more recent elements. In this section some practical solutions obtained with the semiloof plate element are presented and the results compared with the theoretical solutions and with the solutions obtained from other elements.
5.1 Square Plates

Square plates are widely used in the literature for assessing the performance of finite elements and so due to the availability of results for comparisons, they are also considered here. Four problems were solved with meshes of 1, 4 and 16 elements. These are:

- simply supported square plate under uniform load
- clamped square plate under uniform load
- simply supported square plate under point load at the centre
- clamped square plate under a point load at the centre

The results obtained for the normal displacement at the centre of the plate are shown in Table 15 and in Figures 15, 16, 17, 18; comparisons with other elements are presented (T-3).

<table>
<thead>
<tr>
<th>Number Elements</th>
<th>S.S. Uniform Load</th>
<th>Clamped Unif. Load</th>
<th>S.S. Point Load</th>
<th>Clamped Point Load</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.08604</td>
<td>1.38952</td>
<td>1.31527</td>
<td>7.02689</td>
</tr>
<tr>
<td>4</td>
<td>4.04672</td>
<td>1.26418</td>
<td>1.21242</td>
<td>6.07251</td>
</tr>
<tr>
<td>16</td>
<td>4.05807</td>
<td>1.26252</td>
<td>1.7764</td>
<td>5.76935</td>
</tr>
<tr>
<td>Exact</td>
<td>4.06</td>
<td>1.26</td>
<td>1.16</td>
<td>5.60</td>
</tr>
</tbody>
</table>

**TABLE 15** - Displacements at the centre (multipliers as in Figures 15 to 18).

The results obtained with the semiloof plate element are remarkably good for the cases of uniform load. For the clamped plate under point loading, the results, as shown in Figure 18, do not compare as well with other elements but the accuracy obtained is still fairly high.

In Figures 19, 20, 21, 22, the results for the displacements and stresses along certain lines are plotted. The results for the 16
element cases are shown on full line and compared with the cases of 1 and 4 elements. It can be seen that very accurate results are obtained even for the 1 element case.

5.2 Circular Plates

The exact representation of curved boundaries is an important source of error in finite element approximations. If many elements are used along the boundaries a straight line approximation to the curved boundary can give reasonably accurate results, but, using high performance elements, only a small number of elements is needed, and the curved boundaries can then have a marked effect on the performance. Unfortunately most of the results reported in the literature refer to straight line boundaries, which makes comparisons difficult.

In this section a number of problems is solved involving a circular plate and the results compared with the ones given by the High Precision Element of reference (C-1). The results for comparison are taken from reference (C-1) which also develops a special purpose element based on reference (C-2) but with one of the sides being curved. In order to assess the degree of convergence each problem was solved with three types of meshes, Figure 23. The mesh of Figure 23c will be referred to in the tables as having 4 divisions (radial or angular divisions), the mesh of Figure 23b as having 2 divisions, and the one of Figure 23a as having 1 division. Reference (C-1) uses, for uniformly loaded plates, meshes whose general pattern is shown in Figure 23d and for point loaded plates meshes similar to the one of Figure 23c.
Both of these are referred to here as having 4 divisions. We note that meshes similar to the one of Figure 23d cannot be used for the semiloof square elements because they would be very ill-formed. At the same time the displacement field for the semiloof does not include a quartic polynomial in r, as the analytical solution demands for convergence to be expected just by circumferential refinement. The finite element of reference (C-2) has as degrees of freedom 18 generalized displacements which are the normal displacement and its first and second derivatives at each of three corners, and uses an incomplete quintic polynomial to represent the displacement field.

In the following, the displacements are compared at the centre of the plate. The bending moments, for the semiloof plate element are given for the Gauss points nearer to the centre or to the edge. These Gauss points are indicated by 1 and 2 in Figures 23a,b,c.

(a) Simply supported plate under uniform load.

The central deflections for the various cases are given in Table 16 where it can be seen that better results are obtained with the semiloof plate element. The bending moments are given in Table 17, where it can be seen that the semiloof, starting with a smaller error (2.9%) for the 1 division case, ends with a worse result (0.60) for the 4 divisions case.

<table>
<thead>
<tr>
<th>Divisions</th>
<th>Refer. C-1</th>
<th>Error %</th>
<th>S.L. Plate</th>
<th>Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.50082</td>
<td>2.05</td>
<td>6.30822</td>
<td>0.97</td>
</tr>
<tr>
<td>2</td>
<td>6.41040</td>
<td>0.63</td>
<td>6.36057</td>
<td>0.15</td>
</tr>
<tr>
<td>3</td>
<td>6.38880</td>
<td>0.29</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>6.38081</td>
<td>0.17</td>
<td>6.36634</td>
<td>0.06</td>
</tr>
<tr>
<td>5</td>
<td>6.37703</td>
<td>0.11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Exact</td>
<td></td>
<td></td>
<td>6.37019</td>
<td></td>
</tr>
</tbody>
</table>

TABLE 16 Central deflection ($10^2 wD/qa^b$) for a simply supported circular plate under uniform load.
CHAPTER II - THE SEMILOOF PLATE ELEMENT

<table>
<thead>
<tr>
<th>Number of Divisions</th>
<th>Central Moment</th>
<th>Moment at Gauss Point</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Refer. C-1</td>
<td>Exact</td>
</tr>
<tr>
<td>1</td>
<td>1.93567</td>
<td>2.0625</td>
</tr>
<tr>
<td>2</td>
<td>2.06582</td>
<td>&quot;</td>
</tr>
<tr>
<td>3</td>
<td>2.06606</td>
<td>&quot;</td>
</tr>
<tr>
<td>4</td>
<td>2.06987</td>
<td>&quot;</td>
</tr>
<tr>
<td>5</td>
<td>2.06412</td>
<td>&quot;</td>
</tr>
</tbody>
</table>

**TABLE 17** Bending moments ($10M/Qa^2$) for a simply supported circular plate under uniform load.

(b) Clamped plate under uniform load.

Results for deflections and bending moments are given in Tables 18 and 19. Better results are always yielded by the semiloof element. The difference is particularly large for the 1-division case, the results of the semiloof being approximately 17 times better.

<table>
<thead>
<tr>
<th>Divisions</th>
<th>Refer. C-1</th>
<th>Error (%)</th>
<th>S.L. Plate</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.359</td>
<td>51.0</td>
<td>1.51489</td>
<td>3.05</td>
</tr>
<tr>
<td>2</td>
<td>1.679</td>
<td>7.5</td>
<td>1.55419</td>
<td>0.53</td>
</tr>
<tr>
<td>3</td>
<td>1.609</td>
<td>3.0</td>
<td>1.55876</td>
<td>0.24</td>
</tr>
<tr>
<td>4</td>
<td>1.578</td>
<td>1.0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 18** Central deflection ($10^2wD/Qa^4$) for a clamped circular plate under uniform load.

(c) Simply supported plate under point load.

The values for the deflections are given in Table 20. The results of the High Precision Element are this time superior. As for the square plates the results for point load cases are worse than for problems with pressure loading.
TABLE 20 Central deflection ($10^2 \text{ wD/Pa}^2$) for a simply supported circular plate under point load.

(d) Clamped plate under point load.

Deflections and moments are given in Tables 21 and 22. The deflections for the High Performance Element are still better than the semiloof element, but incomparably better results are obtained with the semiloof element for the bending moments.

TABLE 21 Central deflection ($10^2 \text{ wD/Pa}^2$) for a clamped circular plate under point load

<table>
<thead>
<tr>
<th>Divisions</th>
<th>Refer. C-1</th>
<th>Error %</th>
<th>S.L. Plate</th>
<th>Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.49394</td>
<td>25.4</td>
<td>2.46342</td>
<td>23.8</td>
</tr>
<tr>
<td>2</td>
<td>2.05390</td>
<td>3.24</td>
<td>2.24352</td>
<td>12.8</td>
</tr>
<tr>
<td>3</td>
<td>2.01505</td>
<td>1.29</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2.00297</td>
<td>0.68</td>
<td>2.04522</td>
<td>2.81</td>
</tr>
<tr>
<td>5</td>
<td>1.99778</td>
<td>0.42</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE 22 Bending moments ($10 \text{ M/P}$) for a clamped circular plate under point load.

<table>
<thead>
<tr>
<th>Number of Divisions</th>
<th>Radial Edge Moment Refer. C-1</th>
<th>Exact</th>
<th>Error %</th>
<th>Radial Moment at Gauss Point 2 Semiloof Plate</th>
<th>Exact</th>
<th>Error %</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.3166</td>
<td>0.79577</td>
<td>191.1</td>
<td>0.58508</td>
<td>0.59753</td>
<td>2.1</td>
</tr>
<tr>
<td>2</td>
<td>1.1720</td>
<td>&quot;</td>
<td>47.3</td>
<td>0.69452</td>
<td>0.69557</td>
<td>0.15</td>
</tr>
<tr>
<td>3</td>
<td>1.0188</td>
<td>&quot;</td>
<td>28.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.9530</td>
<td>&quot;</td>
<td>19.8</td>
<td>0.74973</td>
<td>0.74529</td>
<td>0.60</td>
</tr>
<tr>
<td>5</td>
<td>0.9168</td>
<td>&quot;</td>
<td>15.2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
5.3 VARIABLE THICKNESS PLATES

The method of numerical integration used for the element matrices (section 3.3) allows an easy treatment of variable thickness plates. The accuracy of the results is here assessed through the consideration of two simply supported circular plates subject to uniform load. These plates have linearly varying thickness along the radius, the ratio between the bigger (centre) and smaller (edge) thickness being 1.5 and 2.33.

The solutions are compared with the numerical values given in (T-1) and are shown in Figures 24 and 25. In these figures the results obtained with a mesh of 16 elements (for a quarter of the plate) are drawn in a full line which passes through the 3 points for which numerical values are given in (T-1). The results for a 3 element mesh is also almost coincident with the 16 element line.

6. CONCLUSIONS

Based on the semiloof shell theory, a plate element has been derived, a full matrix formulation of the element being presented in this chapter.

Results for various patch-tests involving both straight and curved boundaries are given. The states of constant strain were exactly reproduced for the straight boundary cases and very nearly for the cases involving curved boundaries. In this case a mesh refinement along the curved boundary very much improves the results.

The element has been tested for other types of situation and several convergence studies, involving both constant and variable thickness
plates, are presented. These show a very high rate of convergence to the exact solution.

Compared with other elements, the semiloof plate element proved to be very competitive, both displacements and stresses being reproduced with high accuracy, particularly for the case of uniformly distributed loads.
BIBLIOGRAPHY

A-1 Ahmad,S.
'Curved finite elements in the analysis of solid, shell and plate structures'

A-2 Ahmad,S.
'Pseudo-isoparametric elements for shell and plate analysis'
J.B.C.S.A. Conference on Recent Advances in Stress Analysis,

A-3 Albuquerque,F.
'A beam element for use with the semiloof shell element'

B-1 Baldwin,J.T., Razzaque, A. and Irons,B.
'Shape function subroutine for an isoparametric thin plate element'
Civil Engineering Department, University of Wales, Swansea.

'Triangular elements in plate bending - conforming and non-
conforming solutions'
Conf. on Matrix Methods in Structural Mechanics, Ohio, 1965.

B-3 Bogner,F.K., Fox,R.L. and Schmidt,L.A.
'The generation of interelement compatible stiffness and mass
matrices by the use of interpolation formulas'

B-4 Bell,K.
'A refined triangular plate bending finite element'

C-1 Chernuka,M.W., Cowper,G.R., Lindberg,G.M. and Olson,M.D.
'Finite element analysis of plates with curved edges'
Int. J. for Numerical Methods in Engineering, Vol.4, 1972,
p. 49, 65.

C-2 Cowper,G.R., Koska,E., Lindberg,G.M. and Olson,M.D.
'A high precision triangular plate bending element'
Aeronautical Report LR-514 - National Research Council of Canada,
1968.

C-3 Clough,R.W. and Tocher,J.L.
'Finite element matrices for analysis of plate bending'

C-4 Connor,J.J. and Will,G.
'A triangular flat plate bending element'
TR-68-3 - Department of Civil Engineering, M.I.T., 1968.
D-1 Davies, J.D., Parekh, C.J. and Zienkiewicz, O.C.  
'Analysis of slabs with edge beams'  
University of Wales, Swansea, 1968.

E-1 Ergatoudis, J.G.  
'Isoparametric finite elements in two and three dimensional analysis'  

I-1 Irons, B.M.  
'The semiloof shell element'  

I-2 Irons, B.M.  
'A further modification of Ahmad's shell element'  

I-3 Irons, B.M.  
Comment on "A higher order conforming rectangular plate element" by S. Gopalacharyulu.  

I-4 Irons, B.M. and Razzaque, A.  

I-5 Irons, B.M.  
'Postscript to the semiloof shell element'  
University of Wales, Swansea.

L-1 Locke, R.  
'An analysis of an intersection problem using the semiloof shell element'  

L-2 Loof, H.W.  
'The economical computation of stiffness of large structural elements'  

M-1 Martins, R.A.F.  
'Finite element eigenvalue solution employing the semiloof shell element'  
M-2 Martins, R.A.F. and Owen, D.R.J.
'Structural instability and natural vibration analysis of thin arbitrary shells by use of the semiloof element'

M-3 Morley, L.S.D.
'A triangular equilibrium element with linearly varying bending moments for plate bending problems'

S-1 Souza, J.R.
'Analise infinitesimal I'
Universidade do Porta.

T-1 Timoshenko, S.P. and Woinowsky-Krieger, S.
'Theory of plates and shells'

T-2 Timoshenko, S.P. and Goodier
'Theory of Elasticity'

T-3 Tottenham, H. and Brebbia, C. (editors)
'Finite element techniques in structural mechanics'
Stress Analysis Publishers.

T-4 Tocher, J.L.
'Analysis of plate bending using triangular elements'

V-1 Veubeke, F.de and Sander, G.
'An equilibrium model for plate bending'

W-1 Wempner, G.A., Oden, T. and Kross, D.A.
'Finite element analysis of thin shells'

Z-1 Zienkiewicz, O.C., Taylor, R.L. and Too, J.M.
'Reduced integration technique in general analysis of plates and shells'

Z-2 Zienkiewicz, O.C.
'The finite element method in engineering science'
Fig. 1 - Initial Nodal Configuration

Fig. 2 - Typical Shape Functions, N₁
FIG. 3 - LOCAL AXES (p^i, X^j, Y^j)

FIG. 4 - LOOP NODES
**Fig. 5** - Shape function for loop node 1

**Fig. 6** - Beam actuated by rotations

**Fig. 7** - Final degrees of freedom
DIMENSIONS: SQUARE PLATE WITH SIDE LENGTH EQUAL 2.0 AND THICKNESS EQUAL 0.05
MATERIAL PROPERTIES: E = 3000.0, ν = 0.3
ROTATION EQUAL 2.0 ALONG SIDES AD AND BC

FIGURE 8 - STATE OF CONSTANT STRESS IMPOSED ON A SQUARE PLATE

DIMENSIONS: SQUARE PLATE WITH SIDE LENGTH EQUAL 2.0 AND THICKNESS EQUAL 0.05
MATERIAL PROPERTIES: E = 3000.0, ν = 0.3
DISPLACEMENTS: 0.15 AT POINT A, 0.20 AT POINT B, 0.15 AT POINT C.

FIGURE 9 - RIGID BODY MOTION IMPOSED ON A SQUARE PLATE
CHAPTER II - THE SEMILOOF PLATE ELEMENT

FIGURE 10

FIGURE 11
DIMENSIONS: ONE QUARTER OF A CIRCULAR PLATE WITH
RADIUS 1.0 AND THICKNESS 0.075
MATERIAL PROPERTIES: E = 3000.0, ν = 0.25
ROTATIONS: 0.0 ALONG AB AND AC; 2.0 ALONG BC.

FIGURE 12 - STATE OF CONSTANT STRESS IMPOSED ON
A CIRCULAR PLATE.

FIGURE 13
FIGURE 15  SIMPLE SUPPORTED SQUARE PLATE UNDER UNIFORMLY DISTRIBUTED LOAD

FIGURE 16  CLAMPED SQUARE PLATE UNDER UNIFORMLY DISTRIBUTED LOAD
CHAPTER II - THE SEMILOOF PLATE ELEMENT

FIGURE 17 - SIMPLE SUPPORTED SQUARE PLATE UNDER A CONCENTRATED LOAD AT THE CENTRE

FIGURE 18 - CLAMPED SQUARE PLATE UNDER A CONCENTRATED LOAD AT THE CENTRE
CHAPTER II - THE SEMILOOF PLATE ELEMENT

**Material Properties**

\[ \varepsilon = 3000.0 \; \; \; \gamma = 0.3 \]

\[ \text{THIK.} = 0.05 \; \; \; \text{SIDE} = 4.0 \]

\[ \text{UNIF. LOAD, } Q = 1.0 \]

**Normal Displacements Along AB**

**Principal Stresses Along AC**

*In the direction normal to AC*

Error for points A and C less than 0.3%

**Principal Stresses Along AC**

*In direction AC*

Error for points A and C less than 0.3%

*Fig. 19 - Simply supported square plate uniformly loaded*
FIG 20 - CLAMPED SQUARE PLATE UNIFORMLY LOADED
FIG. 21 SIMPLY SUPPORTED SQUARE PLATE WITH A CONCENTRATED LOAD AT THE CENTRE
**Figure 22** Clamped square plate with a concentrated load at the centre.
Figure 23 - Meshes for circular plate
Fig. 24 - Simply sup. circ. plate with linearly varying thickness ($\frac{T_{max}}{T_{min}} = 2.33$) (Uniform load, $p=1$)

Error for the displacement at the center:
- 3 elem. case: 1%
- 16 elem. case: 0.4%

MAT. PROP.:
- $E = 3000$
- $\nu = 0.25$
- $T_{min} = 0.05$

MAX. CONST. THICK. (THEOR. VALUES)
- $0.1$ to $0.2$
- $0.3$ to $0.4$
- $0.5$ (radius) $0.8$ $0.9$

MIN. CONST. THICK. (THEOR. VALUES)
- $0.1$ to $0.2$
- $0.3$ to $0.4$
- $0.5$ (radius) $0.8$ $0.9$

Radial Stress
- 350
- 400
- 450

Tang. Stress
- 350
- 400
- 450

$\bigcirc \rightarrow$ Refer. T-1
$\rightarrow$ SemiLoof (16 elem.)
$\square \rightarrow$ SemiLoof (3 elem.)
FIG. 25 - SIMPLY SUPPORTED CIRC. PLATE WITH LINEARLY VARYING THICKNESS ($\frac{T_{\text{max}}}{T_{\text{min}}} = 1.5$)

(ERROR FOR THE DISPLACEMENT AT THE CENTER:
3 ELEM. CASE: 1%  | 16 ELEM. CASE: 1%)
CHAPTER III - THE SEMILOOF SHELL ELEMENT

1. INTRODUCTION

An historical introduction to the semiloof element was presented in the last chapter and will not be elaborated further.

This chapter presents a matrix formulation of the semiloof shell element based on references (I-1) to (I-5) and in our own experience of work with the element in various branches of structural mechanics. This chapter constitutes therefore the background to the following chapters and establishes the principles for the derivation of some matrices to be used later; giving at the same time a more detailed formulation of the element than that presented in the above references. The semiloof element, we think, will be increasingly used in the near future and a matrix formulation of the element similar to the one presented here may also be welcomed by potential users.

Section 7 of the chapter shows a comparison, for a linear problem, between the semiloof element and other elements presented in the literature. Further linear applications of the element will be presented in Chapter VII.

2. DEGREES OF FREEDOM

Similar to the plate element presented previously, the semiloof shell element has some "initial" degrees of freedom which are eliminated at the element level by conveniently specifying a number of constraints. Figure 1 shows a quadrilateral semiloof shell element referred to a system of axes \((\mathbf{0}, x, y, z)\). This figure shows the initial degrees of freedom and nodal configuration of the element.
Three types of nodes are shown in this figure: the ones represented with a ring which will be called "corner and midside nodes", the ones represented with a cross, "loof nodes" and the one represented with a ring and a cross, "centre node".

At the corner and midside nodes the displacements $u^i$, $v^i$ and $w^i$ along respectively the axes $x$, $y$ and $z$ are taken as parameters.

The parameters at the loof nodes are two rotations: one perpendicular to the side, say $\theta^j_{xz}$, and the other along the side, say $\theta^j_{yz}$.

Finally the central node combines these two types of parameters: three displacements, $u^9$, $v^9$ and $w^9$, and two rotations $\theta^9_{xz}$ and $\theta^9_{yz}$ along the $\xi$ and $\eta$ axis.

For convenience at this stage, the nodal parameters will be arranged as follows:

<table>
<thead>
<tr>
<th>${\delta^e}$</th>
<th>displ. along x at corner node 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u^1$</td>
<td>&quot; &quot; &quot; y &quot; &quot; &quot; 1</td>
</tr>
<tr>
<td>$v^1$</td>
<td>&quot; &quot; &quot; z &quot; &quot; &quot; 1</td>
</tr>
<tr>
<td>$w^1$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$u^2$</th>
<th>displ. along x at midside node 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v^2$</td>
<td>&quot; &quot; &quot; y &quot; &quot; &quot; 2</td>
</tr>
<tr>
<td>$w^2$</td>
<td>&quot; &quot; &quot; z &quot; &quot; &quot; 2</td>
</tr>
<tr>
<td>$\vdots$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$u^9$</th>
<th>displ. along x at central node 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v^9$</td>
<td>&quot; &quot; &quot; y &quot; &quot; &quot; 9</td>
</tr>
<tr>
<td>$w^9$</td>
<td>&quot; &quot; &quot; z &quot; &quot; &quot; 9</td>
</tr>
</tbody>
</table>

(1a)
\[
\{\theta^e_{XZ}\} = \begin{cases} 
\theta^1_{XZ} & \text{rotation normal to the side at loof node 1} \\
\theta^2_{XZ} & \text{" " " " " " " " 2} \\
\vdots & \vdots \\
\theta^9_{XZ} & \text{rotation along } \xi \text{ at the central node 9} 
\end{cases}
\] (1b)

\[
\{\theta^e_{YZ}\} = \begin{cases} 
\theta^1_{YZ} & \text{rotation along the side at loof node 1} \\
\theta^2_{YZ} & \text{" " " " " " " " 2} \\
\vdots & \vdots \\
\theta^9_{YZ} & \text{rotation along } \eta \text{ at the central node} 
\end{cases}
\] (1c)

After eliminating all the degrees of freedom at the centre node and all the local rotations along the element sides of the element will have a total of 32 degrees of freedom. In the next two sections the element will be formulated with the total number of degrees of freedom included. The constraining of the element will be presented in section 5.

3. IN-PLANE BEHAVIOUR

Consider a general element of a general structure, referred to a system of global coordinates \((0,x,y,z)\), and a general point of the element, \(P\), with coordinates \((x,y,z)\), Figure 2.

To define the in-plane behaviour of the element, we must be able to define for each point \(P(x,y,z)\) the displacements contained in a plane tangent to the surface of the element at the point considered.

This implies the necessity of creating at each point \(P(x,y,z)\) of the element (or at specific points if a system of numerical integration is used) a set of local axes with one axis normal to the
surface and the other two contained in the plane tangent to the element (see Figure 2).

If the surface of the element is considered to be defined in parametric form by means of the two Gauss parameters \((\xi, \eta)\), Figure 2, the referred set of local axes can be generated without difficulty.

3.1 Definition of the Local Axes

In order to obtain a parametric representation of the surface, the coordinates of the general point \(P(x,y,z)\) will be interpolated through the shape functions \((N_i^i)_{i=1,8}\) corresponding to nodes 1 to 8 of Figure 1, and defined by expression (6) of the previous chapter.

These shape functions are functions of \(\xi\) and \(\eta\) and the coordinates \((x,y,z)\) of the point \(P\) will be defined by

\[
\begin{align*}
  x &= x^1 N^1 + x^2 N^2 + \ldots + x^8 N^8 = \sum_{i=1}^{8} x^i N^i \\
  y &= y^1 N^1 + y^2 N^2 + \ldots + y^8 N^8 = \sum_{i=1}^{8} y^i N^i \\
  z &= z^1 N^1 + z^2 N^2 + \ldots + z^8 N^8 = \sum_{i=1}^{8} z^i N^i
\end{align*}
\]  

(2)

where \((x^i,y^i,z^i)\) are the coordinates of the node \(i\).

The following local derivatives can now be easily calculated:

\[
\begin{align*}
  \frac{\partial x}{\partial \xi} &= \sum_{i=1}^{8} x^i \frac{\partial N^i}{\partial \xi} & \frac{\partial y}{\partial \eta} &= \sum_{i=1}^{8} y^i \frac{\partial N^i}{\partial \eta} \\
  \frac{\partial y}{\partial \xi} &= \sum_{i=1}^{8} y^i \frac{\partial N^i}{\partial \xi} & \frac{\partial z}{\partial \eta} &= \sum_{i=1}^{8} z^i \frac{\partial N^i}{\partial \eta} \\
  \frac{\partial z}{\partial \xi} &= \sum_{i=1}^{8} z^i \frac{\partial N^i}{\partial \xi} & \frac{\partial x}{\partial \eta} &= \sum_{i=1}^{8} x^i \frac{\partial N^i}{\partial \eta}
\end{align*}
\]  

(3-5)
The components of a vector $\hat{Z}$ normal to the surface at the point $P(x,y,z)$ are given by reference (S-2).

$$\hat{Z} = \frac{\partial P}{\partial \xi} \wedge \frac{\partial P}{\partial \eta}$$

or, expanding (a)

$$\hat{Z} = \begin{pmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \\ \frac{\partial z}{\partial \xi} \end{pmatrix} \wedge \begin{pmatrix} \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \eta} \\ \frac{\partial z}{\partial \eta} \end{pmatrix} = \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{8} x_i \frac{\partial N_i}{\partial \xi} \\ \sum_{i=1}^{8} y_i \frac{\partial N_i}{\partial \xi} \\ \sum_{i=1}^{8} z_i \frac{\partial N_i}{\partial \xi} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{8} x_i \frac{\partial N_i}{\partial \eta} \\ \sum_{i=1}^{8} y_i \frac{\partial N_i}{\partial \eta} \\ \sum_{i=1}^{8} z_i \frac{\partial N_i}{\partial \eta} \end{pmatrix} (6a)$$

Normalizing the vector we will get a vector $\hat{z}$ in the direction of $\hat{Z}$ and defined by

$$\hat{z} = \frac{\hat{Z}}{\sqrt{J_1^2 + J_2^2 + J_3^2}} = [Z] = \begin{pmatrix} Z_x \\ Z_y \\ Z_z \end{pmatrix}$$

where the functional determinants $J_1$, $J_2$ and $J_3$ are given by:

$$J_1 = \frac{\partial (y,z)}{\partial (\xi,\eta)} = \begin{vmatrix} 8 & \sum_{i=1}^{8} y_i \frac{\partial N_i}{\partial \xi} & \sum_{i=1}^{8} y_i \frac{\partial N_i}{\partial \eta} \\ 8 & \sum_{i=1}^{8} z_i \frac{\partial N_i}{\partial \xi} & \sum_{i=1}^{8} z_i \frac{\partial N_i}{\partial \eta} \end{vmatrix}$$

(a) The determinant of expression (6a) is obviously a "dummy determinant" assumed to be expanded according to Laplace's rule along the first row.
\[ J_2 = \frac{\partial (z, x)}{\partial (\xi, \eta)} = \begin{vmatrix} \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} & \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} \\ \sum_{i=1}^{8} x_i \frac{\partial N_i}{\partial \xi} & \sum_{i=1}^{8} y_i \frac{\partial N_i}{\partial \eta} \end{vmatrix} \] (9)

\[ J_3 = \frac{\partial (x, y)}{\partial (\xi, \eta)} = \begin{vmatrix} \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} & \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} \\ \sum_{i=1}^{8} y_i \frac{\partial N_i}{\partial \xi} & \sum_{i=1}^{8} z_i \frac{\partial N_i}{\partial \eta} \end{vmatrix} \] (10)

We note that expressions (6a), (8), (9) and (10) are easily calculated for each point under consideration.

Having determined the unit vector \( \hat{z} \) normal to the surface of the element we move now to the determination of a unit vector \( \hat{x} \) along a line with one of the parameters kept as constant, say \( \eta = \) constant.

To determine this vector we note that the line \( \eta = \) constant is a space curve defined by means of the parameter \( \xi \). The components of a vector tangent to a space curve defined by a parameter \( \xi \) are given by reference (S-2).

\[ \hat{x} = \frac{\partial \vec{p}}{\partial \xi} = \begin{pmatrix} \frac{\partial x}{\partial \xi} \\ \frac{\partial y}{\partial \xi} \\ \frac{\partial z}{\partial \xi} \end{pmatrix} = \begin{pmatrix} 8 \sum_{i=1}^{8} x_i \frac{\partial N_i}{\partial \xi} \\ 8 \sum_{i=1}^{8} y_i \frac{\partial N_i}{\partial \xi} \\ 8 \sum_{i=1}^{8} z_i \frac{\partial N_i}{\partial \xi} \end{pmatrix} \] (11)

This vector can now be normalized, as follows:

\[ \hat{x} = \frac{\hat{x}}{\vec{S}} = \{x\} = \begin{pmatrix} x_x \\ x_y \\ x_z \end{pmatrix} \] (12)
where $S$ is the scalar defined by

$$S = \sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2}$$ (13)

or

$$S = \sqrt{\left(\sum_{i=1}^{8} x_i \frac{\partial N_i^1}{\partial \xi}\right)^2 + \left(\sum_{i=1}^{8} y_i \frac{\partial N_i^1}{\partial \xi}\right)^2 + \left(\sum_{i=1}^{8} z_i \frac{\partial N_i^1}{\partial \xi}\right)^2}$$ (14)

The last vector needed to complete the local orthonormal set of axes, $\hat{Y}$, is now easily calculated as the vectorial product of the other two, $\hat{X}$ and $\hat{Z}$, that is:

$$\hat{Y} = \hat{X} \wedge \hat{Z} = \begin{vmatrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{vmatrix} = \begin{vmatrix} i & j & k \\ X_x & X_y & X_z \\ Y_x & Y_y & Y_z \end{vmatrix}$$ (15)

or

$$\hat{Y} = \{Y\} = \begin{Bmatrix} Y_x \\ Y_y \\ Y_z \end{Bmatrix}$$

The set of axes $(P, X, Y, Z)$ is shown in Figure 2.

3.2 Displacements in Global Axes

The displacements, referred to the global axis $(O,x,y,z)$, of each point $P(x,y,z)$ of the element will be denoted respectively by $u$, $v$ and $w$, and assembled together in the vector $\{d\}$, as follows:

$$\{d\} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$ (16)
These displacements will be interpolated inside the element using the nine shape functions \( (N_i)_{i=1,9} \) defined in the previous chapter by the following expressions:

\[
\begin{align*}
    u &= u^1 N^1 + u^2 N^2 + \ldots + u^9 N^9 \\
    v &= v^1 N^1 + v^2 N^2 + \ldots + v^9 N^9 \\
    w &= w^1 N^1 + w^2 N^2 + \ldots + w^9 N^9
\end{align*}
\]  

(17)

where, as seen in section 2, \( u^i, v^i, w^i \) are the displacements along the global axes \( x, y \) and \( z \), respectively, for the node \( i \).

If we define a matrix of shape functions \([N]\) by means of the following expression:

\[
[N] = \begin{bmatrix}
    N^1 & 0 & 0 & | & N^2 & 0 & 0 & | & \ldots & | & N^9 & 0 & 0 \\
    0 & N^1 & 0 & | & 0 & N^2 & 0 & | & \ldots & | & 0 & N^9 & 0 \\
    0 & 0 & N^1 & | & 0 & 0 & N^2 & | & \ldots & | & 0 & 0 & N^9
\end{bmatrix}
\]  

(18)

we can write the displacements in global axes in a simple matrix equation:

\[
\{d\} = [N] \{\delta^e\}
\]  

(19)

\( \{\delta^e\} \) as defined in section 2.

3.3 In-Plane Displacements

The displacements in local axes (in-plane displacements) will be denoted by \( U \) and \( V \) respectively along the axes \( \hat{x} \) and \( \hat{y} \), and can be obtained as the projections of \( \{u,v,w\} \) on \( \hat{x} \) and \( \hat{y} \) respectively, i.e.,

\[
U = \{x\}^T \{d\}
\]  

(20)

\[
V = \{y\}^T \{d\}
\]  

(21)
These expressions can be rewritten using expression (19) as follows:

\[ U = \{x\}^T \{N\} \{\delta_e\} \]  
\[ V = \{y\}^T \{N\} \{\delta_e\} \]  

(22)

(23)

3.4 In-Plane Derivatives

In order to define the membrane strains, section 6, we must be able to determine the derivatives with respect to \(X\) and \(Y\) of the in-plane displacements \(U\) and \(V\).

This implies the necessity of determining the derivatives of the shape functions with respect to \(X\) and \(Y\).

Referring to the appendix of reference (I-1) we can write:

\[
\begin{bmatrix}
\frac{\partial N_i}{\partial X} \\
\frac{\partial N_i}{\partial Y}
\end{bmatrix}
= \begin{bmatrix}
\hat{X} \cdot \hat{\xi} & \hat{X} \cdot \hat{\eta} \\
\hat{Y} \cdot \hat{\xi} & \hat{Y} \cdot \hat{\eta}
\end{bmatrix} - \begin{bmatrix}
\frac{\partial N}{\partial \xi} \\
\frac{\partial N}{\partial \eta}
\end{bmatrix}
\]

(24)

Now using expression (11) to determine \(\hat{\xi}\) and a similar expression to determine \(\hat{\eta}\) this becomes:

\[
\begin{bmatrix}
\frac{\partial N_i}{\partial X} \\
\frac{\partial N_i}{\partial Y}
\end{bmatrix}
= \begin{bmatrix}
\{X, X, X\} \{\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \xi} \frac{\partial z}{\partial \xi}\}^T & \{X, X, X\} \{\frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \eta} \frac{\partial z}{\partial \eta}\}^T \\
\{Y, Y, Y\} \{\frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \xi} \frac{\partial z}{\partial \xi}\}^T & \{Y, Y, Y\} \{\frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \eta} \frac{\partial z}{\partial \eta}\}^T
\end{bmatrix}
\left(\begin{bmatrix}
\frac{\partial N_i}{\partial \xi} \\
\frac{\partial N_i}{\partial \eta}
\end{bmatrix}
\right)
\]

(25)

In this expression the derivatives with respect to \(\xi\) and \(\eta\) of the point coordinates are easily determined using expressions (3), (4) and (5).
Now we define the matrices \( \frac{\partial N}{\partial x} \) and \( \frac{\partial N}{\partial y} \) as follows:

\[
\begin{bmatrix}
\frac{\partial N_1}{\partial x} & 0 & 0 & \frac{\partial N_2}{\partial x} & 0 & 0 & \cdots & \frac{\partial N_9}{\partial x} & 0 & 0 \\
0 & \frac{\partial N_1}{\partial x} & 0 & 0 & \frac{\partial N_2}{\partial x} & 0 & \cdots & 0 & \frac{\partial N_9}{\partial x} & 0 \\
0 & 0 & \frac{\partial N_1}{\partial x} & 0 & 0 & \frac{\partial N_2}{\partial x} & \cdots & 0 & 0 & \frac{\partial N_9}{\partial x}
\end{bmatrix}
\] (25a)

\[
\begin{bmatrix}
\frac{\partial N_1}{\partial y} & 0 & 0 & \frac{\partial N_2}{\partial y} & 0 & 0 & \cdots & \frac{\partial N_9}{\partial y} & 0 & 0 \\
0 & \frac{\partial N_1}{\partial y} & 0 & 0 & \frac{\partial N_2}{\partial y} & 0 & \cdots & 0 & \frac{\partial N_9}{\partial y} & 0 \\
0 & 0 & \frac{\partial N_1}{\partial y} & 0 & 0 & \frac{\partial N_2}{\partial y} & \cdots & 0 & 0 & \frac{\partial N_9}{\partial y}
\end{bmatrix}
\] (25b)

whose elements are easily calculated by means of the matrix equation (25).

The derivatives of the in-plane displacements are now calculated using expressions (22) and (23) and are given by:

\[
\frac{\partial U}{\partial x} = (x)^T \left[ \frac{\partial N}{\partial x} \right] \{\delta^e\}
\] (26)

\[
\frac{\partial U}{\partial y} = (x)^T \left[ \frac{\partial N}{\partial y} \right] \{\delta^e\}
\] (27)

\[
\frac{\partial V}{\partial x} = (y)^T \left[ \frac{\partial N}{\partial x} \right] \{\delta^e\}
\] (28)

\[
\frac{\partial V}{\partial y} = (y)^T \left[ \frac{\partial N}{\partial y} \right] \{\delta^e\}
\] (29)
4. OUT-OF-PLANE BEHAVIOUR

We are interested now in determining for each point \( P(x,y,z) \) of the element, the displacement, \( W \), along the normal to the element surface, i.e. the displacement along the local axis \( \hat{Z} \). This displacement is used, together with the displacements \( U \) and \( V \) already defined in section 3 to determine the mass matrix, used in problems of vibrations, and for the consideration of distributed loads.

The derivatives with respect to \( Z \) of the displacements \( U \) and \( V \) will also be determined as they will be used for the determination of the geometric matrix.

4.1 Out-of-Plane Displacements

The determination of the out-of-plane displacement \( W \) is done following the principle explained in 3.3: we project \( \{u,v,w\} \) on \( \hat{Z} \), that is

\[
W = \{Z\}^T \{d\}
\]  
(30)

or, using (19)

\[
W = \{Z\}^T \{N\} \{\varepsilon^e\}
\]  
(31)

4.2 Out-of-Plane Derivatives

(a) Preliminary considerations

As we shall see in the chapter concerned with instability analysis, for the determination of matrices involved, the following derivatives of the displacements are needed:

\[
\frac{\partial u}{\partial Z}
\]  
(32)

\[
\frac{\partial v}{\partial Z}
\]  
(33)
The values of these derivatives at a particular point \( P(x,y,z) \) of the element can be divided into two parts as follows:

\[
\frac{\partial u}{\partial z} = \left( \frac{\partial u}{\partial z} \right)^L + \left( \frac{\partial u}{\partial z} \right)^N \tag{34}
\]

\[
\frac{\partial v}{\partial z} = \left( \frac{\partial v}{\partial z} \right)^L + \left( \frac{\partial v}{\partial z} \right)^N \tag{35}
\]

where the terms with superscript "L" represent the effect of the rotations at the loof and central node, and the terms with superscript "N" represent the effect of the displacements at the corner midside nodes and central node.

As for the case of the plate element, the definition of the orientation of the loof rotations is made by considering at each loof node a set of local axes \((P^i, X^i, Y^i, Z^i)\) as shown in Figure 1.

The determination of these sets of local axes is easily accomplished as in (3.1) and we take then for granted that unit local axes represented by:

\[
\hat{x}^j = \{x^j\} = \begin{bmatrix} x^j_x \\ x^j_y \\ x^j_z \end{bmatrix} \tag{36}
\]

\[
\hat{y}^j = \{y^j\} = \begin{bmatrix} y^j_x \\ y^j_y \\ y^j_z \end{bmatrix} \tag{37}
\]

\[
\hat{z}^j = \{z^j\} = \begin{bmatrix} z^j_x \\ z^j_y \\ z^j_z \end{bmatrix} \tag{38}
\]
can be formed at each loof node (or centre node) j.

We note now that, unlike the case of the plate element, where the thickness at any point of the element can be considered as a scalar, it is necessary for the shell to generate a thickness vector.

We therefore generate at each loof node a vector that we shall call the "non corrected thickness vector", $\hat{T}^j$, defined by

$$\hat{T}^j = T^j \hat{z}^j$$  \hspace{1cm} (39)

and alternatively represented by

$$\hat{T}^j = \{T^j\} = \begin{bmatrix} T^j_x \\ T^j_y \\ T^j_z \end{bmatrix}$$

(40)

where $T^j$ is the scalar thickness at node j.

This vector has, obviously, the direction and orientation of the unit vector $\hat{z}^j$ and its name suggests that it must be corrected.

This correction aims at making zero the component of the thickness vector along the element boundary. For details see reference (I-1).

The "corrected thickness vector" at node j will from now on be called simply the "thickness vector" and will be represented by

$$T^j = \{T^j\} = \begin{bmatrix} T^j_x \\ T^j_y \\ T^j_z \end{bmatrix}$$

(41)
We now generate
\[
\hat{\mathbf{R}}^j = \mathbf{T}^j \mathbf{R}^j = \{R^j\} = \begin{bmatrix} R^j_x \\ R^j_y \\ R^j_z \end{bmatrix}
\]
which defines the rotation shared by two neighbouring elements and
\[
\hat{\mathbf{s}}^j = \mathbf{T}^j \mathbf{s}^j = \{s^j\} = \begin{bmatrix} s^j_x \\ s^j_y \\ s^j_z \end{bmatrix}
\]
which defines the slope along the edge of the element.

(b) Determination of \((\partial U/\partial Z)^I\) and \((\partial V/\partial Z)^I\)

We assume that at each point \(P(x,y,z)\) of the element the rotations \(\theta_{xZ}\) and \(\theta_{yZ}\) can be interpolated by means of the loof and central node shape functions, \((L^j)_{j=1,9}\), and the corresponding nodal values \(\theta_{xZ}^j\) and \(\theta_{yZ}^j\).

We note that these shape functions have been defined in the previous chapter.

Once again we are faced with the necessity of generating local axes \((P,X,Y,Z)\) at any point \(P(x,y,z)\), and once these are generated we can write the contribution of \(\theta_{xZ}^j\) to \((\partial U/\partial Z)^I\) and \((\partial V/\partial Z)^I\) as follows:
\[
\left(\frac{\partial U}{\partial Z}\right)_{xZ}^I = \sum_{j=1}^9 \left( \begin{array}{c} x_x^j \\ x_y^j \\ x_z^j \end{array} \right) L^j \frac{1}{T} \theta_{xZ}^j
\]
\[
\frac{\partial V}{\partial Z}_{XZ} = \sum_{j=1}^{9} \left\{ R^j_x R^j_y R^j_z \right\} \begin{bmatrix} Y_x \\ Y_y \\ Y_z \end{bmatrix} L^j \frac{1}{T} \theta^j_{XZ}
\]

where \( T \) is the shell thickness at point \( P(x,y,z) \).

The contributions of \( \theta^j_{YZ} \) to \( \frac{\partial U}{\partial Z}^L \) and \( \frac{\partial V}{\partial Z}^L \) can similarly be written as:

\[
\frac{\partial U}{\partial Z}_{YZ} = \sum_{j=1}^{9} \left\{ s^j_x s^j_y s^j_z \right\} \begin{bmatrix} X_x \\ X_y \\ X_z \end{bmatrix} L^j \frac{1}{T} \theta^j_{YZ}
\]

\[
\frac{\partial V}{\partial Z}_{YZ} = \sum_{j=1}^{9} \left\{ s^j_x s^j_y s^j_z \right\} \begin{bmatrix} Y_x \\ Y_y \\ Y_z \end{bmatrix} L^j \frac{1}{T} \theta^j_{YZ}
\]

In order to write \( \frac{\partial U}{\partial Z}^L \) and \( \frac{\partial V}{\partial Z}^L \) in matrix form we define matrices \([L]\), \([R]\) and \([S]\) as follows:

\[
[L] = \begin{bmatrix}
L_1 & 0 & 0 & \ldots & 0 \\
0 & L_2 & 0 & \ldots & 0 \\
0 & 0 & L_3 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & L_9 \\
\end{bmatrix}
\]

\[
[R] = \begin{bmatrix}
R^1_x & R^2_x & R^3_x & \ldots & R^9_x \\
R^1_y & R^2_y & R^3_y & \ldots & R^9_y \\
R^1_z & R^2_z & R^3_z & \ldots & R^9_z \\
\end{bmatrix}
\]

(45)

(46)

(47)

(48)

(49)
\[
[S] = \begin{bmatrix}
S_1^x & S_2^x & S_3^x & \cdots & S_9^x \\
S_1^y & S_2^y & S_3^y & \cdots & S_9^y \\
S_1^z & S_2^z & S_3^z & \cdots & S_9^z \\
\end{bmatrix}
\]
\[(50)\]

Now we can write
\[
\left[\frac{\partial U}{\partial Z}\right]^L = \left[\frac{\partial U}{\partial Z}\right]_{XZ}^L + \left[\frac{\partial U}{\partial Z}\right]_{YZ}^L
\]
\[(51)\]

and, if we recall the definition of \(\theta_{XZ}^e\) and \(\theta_{YZ}^e\) from section 2,
\[
\left[\frac{\partial U}{\partial Z}\right]_{XZ}^L = \frac{1}{T} \{x\}^T [R] [L] \{\theta_{XZ}^e\} + \frac{1}{T} \{x\}^T [S] [L] \{\theta_{YZ}^e\}
\]
\[(52)\]

Similarly we have \(\left[\frac{\partial V}{\partial Z}\right]_{XZ}^L = \left[\frac{\partial V}{\partial Z}\right]_{YZ}^L \)
\[(53)\]

or \(\left[\frac{\partial V}{\partial Z}\right]_{XZ}^L = \frac{1}{T} \{y\}^T [R] [L] \{\theta_{XZ}^e\} + \frac{1}{T} \{y\}^T [S] [L] \{\theta_{YZ}^e\}\)
\[(54)\]

(c) Determination of \(\frac{\partial U}{\partial Z}\) and \(\frac{\partial V}{\partial Z}\)

We assume that the vector thickness \(T\) at any point \(P(x,y,z)\) of the element can be interpolated by means of the shape functions \(L_j\), that is
\[
\hat{T} = \{T\} = \begin{bmatrix}
T_x \\
T_y \\
T_z \\
\end{bmatrix} = \begin{bmatrix}
\sum_{j=1}^{9} T_{x}^j L_j \\
\sum_{j=1}^{9} T_{y}^j L_j \\
\sum_{j=1}^{9} T_{z}^j L_j \\
\end{bmatrix}
\]
\[(55)\]
This vector \( \mathbf{T} \), defined at each point \( P(x,y,z) \) of the element, is not necessarily normal to the midsurface of the element, what implies the supplementary contributions \( \left( \frac{\partial U}{\partial Z} \right)^N \) to \( \frac{\partial U}{\partial Z} \) and \( \left( \frac{\partial V}{\partial Z} \right)^N \) to \( \frac{\partial V}{\partial Z} \) that we are interested in determining.

In Figure 3 we represent the plane XZ for a particular point \( P \), and the thickness vector (assumed to be in this plane) is also shown. The thickness vector is not normal to the midsurface which means that we are comparing point \( B \) with point \( A \) that is not orthogonally above, and that \( U \) varies with \( Z \).

This variation can be approximated as follows:

\[
\left( \frac{\partial U}{\partial Z} \right)^N_{XZ} = \frac{U_B - U_A}{T} \quad (56)
\]

where the index \( XZ \) means that we are confining ourselves to the plane \( XZ \).

Now we note that

\( (U_B - U_A) = \) (variation of \( U \) per unit of \( X \)) \* (length of variation along \( X \)),

or, in mathematical terms:

\[
(U_B - U_A) = \frac{\partial U}{\partial X} T_X \quad (57)
\]

where, as shown in Figure 3, \( T_X \) is the component of the vector \( \mathbf{T} \) along the local axis \( X \).

Equation \( (56) \) can then be rewritten as follows:

\[
\left( \frac{\partial U}{\partial X} \right)^N_{XZ} = \frac{\partial U}{\partial X} T_X \frac{1}{T} \quad (58)
\]
This equation leads us to the conclusion that we must be able to determine the components of the vector \( \mathbf{T} \) on the set of axes \((P,X,Y,Z)\).

This is easily accomplished by projecting \( \mathbf{T} \) along the axes \((X,Y,Z)\); we get then for \( \mathbf{T}_X \):

\[
\mathbf{T}_X = \{T_x \ T_y \ T_z\} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}
\]

The other components are similarly calculated:

\[
\mathbf{T}_Y = \{T_x \ T_y \ T_z\} \begin{bmatrix} Y \\ X \\ Z \end{bmatrix} \quad \text{(59)}
\]

\[
\mathbf{T}_Z = \{T_x \ T_y \ T_z\} \begin{bmatrix} Z \\ X \\ Y \end{bmatrix} \quad \text{(60)}
\]

The derivative with respect to \(X\) of the displacement \(U\), contained in expression (58) is calculated using equation (26) and we finally get:

\[
\left( \frac{\partial U}{\partial X} \right)^N_{XZ} = -\frac{1}{T} \mathbf{T}_X (X)^T \left[ \frac{\partial N}{\partial X} \right] \{\delta^e\} \quad \text{(62)}
\]

We note now that in Figure 3 we confined ourselves to the plane \(XZ\). Let us suppose now that \( \mathbf{T} \) has a component along the \(Y\) axis as shown in Figure 4. Then we get another contribution to \( \left( \frac{\partial U}{\partial Z} \right)^N_{YX} \), which, because it refers to the plane \(YZ\) will be labelled "YZ", i.e.
\[
\left( \frac{\partial u}{\partial z} \right)^N_{YZ} = -\frac{1}{T} T_x \{x\}^T \left[ \frac{\partial N}{\partial y} \right] \{\delta^e\}
\]

(63)

There is however a supplementary contribution to \( \left( \frac{\partial u}{\partial z} \right)^N \) that we will label with the indice "w", and which results from the fact that the displacement \( w \) along \( Z \) at the lower surface has an \( X \) component in the upper surface.

In fact, the normal direction to the upper surface becomes:

\[
\left[ -\frac{\partial T_z}{\partial x}, -\frac{\partial T_z}{\partial y}, 1 \right]
\]

(64)

and the variation of \( U \) across the thickness of the shell will be

\[
\delta U = \frac{\partial T_z}{\partial x} w
\]

(65)

or, approximately

\[
\frac{\partial U}{\partial z} = \frac{1}{T} \frac{\partial T_z}{\partial x} w
\]

(66)

Using now expression (31) we can write

\[
\left( \frac{\partial u}{\partial z} \right)^N_w = \frac{1}{T} \frac{\partial T_z}{\partial x} \{z\}^T [N] \{\delta^e\}
\]

(67)

In this expression we still have to calculate \( \frac{\partial T_z}{\partial x} \). This is however easily done if we use expressions (60), (55) and (24).

In fact, from (55) and (60) we have

\[
T_z = Z_x \left( \sum_{j=1}^{9} T_x^j L^j \right) + Z_y \left( \sum_{j=1}^{9} T_y^j L^j \right) + Z_z \left( \sum_{j=1}^{9} T_z^j L^j \right)
\]

(68)

or

\[
T_z = \sum_{j=1}^{9} \left( Z_x T_x^j + Z_y T_y^j + Z_z T_z^j \right) L^j
\]

(68a)
CHAPTER III - THE SEMILOOF SHELL ELEMENT

The derivative with respect to $X$ of this function is easily calculated:

$$
\frac{\partial T}{\partial X} = \sum_{j=1}^{9} \left( Z_x T^j_x + Z_y T^j_y + Z_z T^j_z \right) \frac{\partial L^j}{\partial X}
$$

(69)

where $\frac{\partial L^j}{\partial X}$ is calculated by an expression similar to (24) where the "$N^i, N^i$" are changed to "$L^j, L^j$".

In equation (67) everything is now known and the final expression for $\left[ \frac{\partial U}{\partial Z} \right]^N$ is:

$$
\left[ \frac{\partial U}{\partial Z} \right]^N = \left[ \frac{\partial U}{\partial Z} \right]_{XZ}^N + \left[ \frac{\partial U}{\partial Z} \right]_{YZ}^N + \left[ \frac{\partial U}{\partial Z} \right]_w^N
$$

(70)

or

$$
\left[ \frac{\partial U}{\partial Z} \right]^N = \frac{1}{T} \left\{ -T_X \{X\}^T \left[ \frac{\partial N}{\partial X} \right] - T_Y \{Y\}^T \left[ \frac{\partial N}{\partial Y} \right] + \frac{\partial T}{\partial X} \{Z\}^T \{N\} \right\}\{\delta^e\}
$$

(71)

A similar expression is obtained for $\left[ \frac{\partial V}{\partial Z} \right]^N$:

$$
\left[ \frac{\partial V}{\partial Z} \right]^N = \frac{1}{T} \left\{ -T_X \{X\}^T \left[ \frac{\partial N}{\partial X} \right] - T_Y \{Y\}^T \left[ \frac{\partial N}{\partial Y} \right] + \frac{\partial T}{\partial X} \{Z\}^T \{N\} \right\}\{\delta^e\}
$$

(72)

(d) Final expressions for $\partial U/\partial Z$ and $\partial V/\partial Z$

Using (34), (52) and (71) we can finally write:

$$
\frac{\partial U}{\partial Z} = \frac{1}{T} \{X\}^T \left[ R \right] \left[ L \right] \{\theta^e_{XZ}\} + \frac{1}{T} \{X\}^T \left[ S \right] \left[ L \right] \{\theta^e_{YZ}\} +
$$

$$
+ \frac{1}{T} \left\{ -T_X \{X\}^T \left[ \frac{\partial N}{\partial X} \right] - T_Y \{X\}^T \left[ \frac{\partial N}{\partial Y} \right] + \frac{\partial T}{\partial X} \{Z\}^T \{N\} \right\}\{\delta^e\}
$$

(73)
Similarly, using (35), (54) and (72)

\[
\frac{\partial y}{\partial z} = \frac{1}{T} \{Y\}^T \{R\} \{L\} \{\theta^e_{xz}\} + \frac{1}{T} \{S\} \{L\} \{\theta^e_{yz}\} + \\
+ \frac{1}{T} \left\{ - T_X \{y\}^T \left[ \begin{array}{c} \frac{\partial N}{\partial X} \\ \frac{\partial N}{\partial Y} \\ \frac{\partial T_z}{\partial Y} \end{array} \right] - T_Y \{y\}^T \left[ \begin{array}{c} \frac{\partial N}{\partial Y} \\ \frac{\partial N}{\partial Y} \\ \frac{\partial T_z}{\partial Y} \end{array} \right] \right\} \{\delta^e\} 
\]

(74)

4.3 Second Order Derivatives

In order to define the bending strains in the element we must be able to determine the following second derivatives of the displacements:

\[
\frac{\partial^2 U}{\partial X^2} \quad \frac{\partial^2 U}{\partial Y^2} \\
\frac{\partial^2 V}{\partial X^2} \quad \frac{\partial^2 V}{\partial Y^2} 
\]

(75)

These are easily determined using the results of the previous section, specifically equations (73) and (74).

In fact from (73) we get:

\[
\frac{\partial^2 U}{\partial X^2} = \frac{1}{T} \{x\}^T \{R\} \left[ \begin{array}{c} \frac{\partial L}{\partial X} \{\theta^e_{xz}\} + \frac{1}{T} \{x\}^T \{S\} \left[ \begin{array}{c} \frac{\partial L}{\partial X} \{\theta^e_{yz}\} + \\
+ \frac{1}{T} \left\{ - \frac{\partial T_x}{\partial X} \{x\}^T \left[ \begin{array}{c} \frac{\partial N}{\partial X} \\ \frac{\partial N}{\partial Y} \\ \frac{\partial T_z}{\partial Y} \end{array} \right] - \frac{\partial T_x}{\partial X} \{x\}^T \left[ \begin{array}{c} \frac{\partial N}{\partial Y} \\ \frac{\partial N}{\partial Y} \\ \frac{\partial T_z}{\partial Y} \end{array} \right] \right\} \{\delta^e\} 
\right\} \{\delta^e\} 
\]

(76)

\[
\frac{\partial^2 U}{\partial Y^2} = \frac{1}{T} \{x\}^T \{R\} \left[ \begin{array}{c} \frac{\partial L}{\partial Y} \{\theta^e_{xz}\} + \frac{1}{T} \{x\}^T \{S\} \left[ \begin{array}{c} \frac{\partial L}{\partial Y} \{\theta^e_{yz}\} + \\
+ \frac{1}{T} \left\{ - \frac{\partial T_y}{\partial Y} \{x\}^T \left[ \begin{array}{c} \frac{\partial N}{\partial X} \\ \frac{\partial N}{\partial Y} \\ \frac{\partial T_z}{\partial Y} \end{array} \right] - \frac{\partial T_y}{\partial Y} \{x\}^T \left[ \begin{array}{c} \frac{\partial N}{\partial Y} \\ \frac{\partial N}{\partial Y} \\ \frac{\partial T_z}{\partial Y} \end{array} \right] \right\} \{\delta^e\} 
\right\} \{\delta^e\} 
\]

(77)
And from (74):

\[
\frac{\partial^2 Y}{\partial x \partial z} = \frac{1}{T} \{Y\}^T \left[ \frac{\partial L}{\partial x} \right] \{\theta^e_{xz}\} + \frac{1}{T} \{Y\}^T \left[ \frac{\partial L}{\partial x} \right] \{\theta^e_{yz}\} + \\
+ \frac{1}{T} \left\{ - \frac{\partial T}{\partial x} \{Y\}^T \left[ \frac{\partial \alpha}{\partial x} \right] - \frac{\partial T}{\partial x} \{Y\}^T \left[ \frac{\partial \alpha}{\partial y} \right] + \frac{\partial T}{\partial y} \{Z\}^T \left[ \frac{\partial \alpha}{\partial x} \right] \right\} \{\delta^e\} \tag{78}
\]

Finally, also from (74):

\[
\frac{\partial^2 Y}{\partial y \partial z} = \frac{1}{T} \{Y\}^T \left[ \frac{\partial L}{\partial y} \right] \{\theta^e_{xz}\} + \frac{1}{T} \{Y\}^T \left[ \frac{\partial L}{\partial y} \right] \{\theta^e_{yz}\} + \\
+ \frac{1}{T} \left\{ - \frac{\partial T}{\partial y} \{Y\}^T \left[ \frac{\partial \alpha}{\partial x} \right] - \frac{\partial T}{\partial y} \{Y\}^T \left[ \frac{\partial \alpha}{\partial y} \right] + \frac{\partial T}{\partial y} \{Z\}^T \left[ \frac{\partial \alpha}{\partial y} \right] \right\} \{\delta^e\} \tag{79}
\]

Of these equations, only the determination of the derivatives with respect to X and Y of \(T_X\) and \(T_Y\) has not been discussed yet.

For the determination of these quantities we recall equations (55) and (59) and (60), from which we conclude:

\[
T_X = \left( \sum_{j=1}^{9} T^j_x L^j \right) X_x + \left( \sum_{j=1}^{9} T^j_y L^j \right) X_y + \left( \sum_{j=1}^{9} T^j_z L^j \right) X_z \tag{80}
\]

\[
T_Y = \left( \sum_{j=1}^{9} T^j_x L^j \right) Y_x + \left( \sum_{j=1}^{9} T^j_y L^j \right) Y_y + \left( \sum_{j=1}^{9} T^j_z L^j \right) Y_z \tag{81}
\]

These equations can be modified further:

\[
T_X = \sum_{j=1}^{9} \left( T^j_x X_x + T^j_y X_y + T^j_z X_z \right) L^j \tag{82}
\]

\[
T_Y = \sum_{j=1}^{9} \left( T^j_x Y_x + T^j_y Y_y + T^j_z Y_z \right) L^j \tag{83}
\]
Now the derivatives with respect to $X$ and $Y$ of $T_X$ are given by

$$\frac{\partial T_X}{\partial X} = \sum_{j=1}^{9} (T^j_x x^j_x + T^j_y y^j_x + T^j_z z^j_z) \frac{\partial L^j}{\partial X},$$  \hspace{1cm} (84)

$$\frac{\partial T_X}{\partial Y} = \sum_{j=1}^{9} (T^j_x x^j_y + T^j_y y^j_y + T^j_z z^j_z) \frac{\partial L^j}{\partial Y},$$  \hspace{1cm} (85)

similar expressions being obtained for $\frac{\partial T_Y}{\partial X}$ and $\frac{\partial T_Y}{\partial Y}$.

The derivatives with respect to $X$ and $Y$ of the shape functions contained in (84) and (85) are calculated using an equation similar to (24) where the "N^i N^i" are transformed to "L^i L^i".

5. ELEMENT CONSTRAINING

The degrees of freedom corresponding to the displacements of the central node are combined to create a deflection normal to the element at its centre. Of the remaining 43 degrees of freedom a further 11 variables are eliminated to give a final total of 32 degrees of freedom for the element, which are:

24 displacement components with respect to the global axes at the corner and midside nodes

8 rotations, $\theta^j_{xz}$, normal to the element edge at each loop node.

Variable elimination is made by imposing eleven shear constraints on the behaviour of the element as for the plate element (section 3 of last chapter). The formulation of the constraints is similar to that done for the plate element, except that different constraining equations are now obtained.
The shear strain $\gamma_{YZ}$ at each loof node is given by

$$
\gamma_{YZ} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}
$$

(16)

The first term on the right hand side of this expression is obtained by differentiating expression (31) and evaluating it at the particular loof node considered:

$$
\frac{\partial w}{\partial y} = \{z\}^T \left[ \frac{\partial N}{\partial y} \right] \{\delta^e\}
$$

(87)

The second term is obtained by evaluating expression (74) at the loof node. The resulting constraining equation will be:

$$
\frac{1}{T} \{y\}^T \left[ \begin{array}{c} R^x \\ R^y \\ R^z \end{array} \right] \left[ \begin{array}{c} \theta^e_x \\ \theta^e_y \\ \theta^e_z \end{array} \right] + \frac{\partial N}{\partial y} - \{T\} \left[ \begin{array}{c} \frac{\partial N}{\partial y} \end{array} \right] +

\frac{\partial T}{\partial y} \{z\}^T \left[ \begin{array}{c} \frac{\partial N}{\partial y} \end{array} \right] + \{T\} \left[ \begin{array}{c} \frac{\partial N}{\partial y} \end{array} \right] \{\delta^e\} = 0
$$

(88)

This equation, one for each loof node is used to eliminate the rotation $\theta_{YZ}$ at the loof node.

For the elimination of the two rotations at the centre, equations similar to equations (56a) and (60) of the previous chapter are obtained. These are respectively:

$$
\sum_{k=1}^{4} \left\{ \begin{array}{c} R^x \\ R^y \\ R^z \end{array} \right\}_k \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \gamma^k_{XZ} + \left\{ \begin{array}{c} R^x \\ R^y \\ R^z \end{array} \right\}_k \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \gamma^k_{YZ} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \gamma^k_{YZ} A_k T_k = 0
$$

(89)

and

$$
\sum_{k=1}^{4} \left\{ \begin{array}{c} S^x \\ S^y \\ S^z \end{array} \right\}_k \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \gamma^k_{XZ} + \left\{ \begin{array}{c} S^x \\ S^y \\ S^z \end{array} \right\}_k \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \gamma^k_{YZ} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \gamma^k_{YZ} A_k T_k = 0
$$

(90)
In these expressions $T_k$ is the scalar thickness at the integrating point $k$ and $A_k$ is the area integrating factor defined by

$$A_k = (A_x^2 + A_y^2 + A_z^2)^{\frac{1}{2}}$$

(91)

where:

$$\begin{vmatrix} A_x \\ A_y \\ A_z \end{vmatrix} = \{A_k\} = \tilde{A}_k = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x/\partial \xi & \partial y/\partial \xi & \partial z/\partial \xi \\ \partial x/\partial \eta & \partial y/\partial \eta & \partial z/\partial \eta \end{vmatrix}$$

(92)

This being a dummy determinant expanded along the first row according to Laplace's rule.

In equations (89) and (90) the only quantity whose determination has not yet been discussed is $\gamma_{XZ}^k$. It follows however the same lines as $\gamma_{YZ}$ defined by (86).

As for the elimination of the centre displacement, an expression similar to expression (61a) of the last chapter is used:

$$\sum_{j=1}^{8} \gamma_{XZ}^j I_j T_j = 0$$

(93)

$T_j$ being the element thickness at the loof node, $\gamma_{XZ}^j$ being the local shear as defined by an expression parallel to (88) and the side integrating factor $I_j$ being defined by

$$I_j = \left( \{Y_x Y_y Y_z\}_j \begin{vmatrix} Y_x \\ Y_y \\ Y_z \end{vmatrix} \right)^{\frac{1}{2}}$$

(94)
5.2 Application of Constraints

Apart from the definition of the constraining equations, the elimination of the unwanted variables in the semiloof shell element is made in a similar way to the process employed for the plate element. Following the definitions of section 3 of the last chapter, the quantities relevant for the determination of the classic stiffness matrix, geometric matrix, applied loads and mass matrix can be put together in a single matrix of shape functions, \([F]\), defined by:

\[
\left\{ U \ V \ W \ \frac{\partial U}{\partial X} \ \frac{\partial U}{\partial Y} \ \frac{\partial V}{\partial X} \ \frac{\partial V}{\partial Y} \ \frac{\partial U}{\partial Z} \ \frac{\partial V}{\partial Z} \ \frac{\partial^2 U}{\partial X \partial Z} \ \frac{\partial^2 V}{\partial Y \partial Z} \right\}^T = [F] \{ p_\text{e}^e \} = \left[ [S_A] - [S_B] [C_B]^{-1} [C_A] \right] \{ p_\text{e}^e \} \quad (95)
\]

6. ELEMENT MATRICES

The element stiffness matrix \([K_E^e]\) can now be easily determined. It is composed of two parts, one corresponding to the in-plane behaviour, the other to the bending effects. Defining the element strains as

\[
\left\{ \varepsilon^m \right\} = \left\{ \begin{array}{l} \varepsilon_X^m \\ \varepsilon_Y^m \\ \varepsilon_{XZ}^m \\ \varepsilon_X^b \\ \varepsilon_Y^b \\ \varepsilon_{XZ}^b \\ \end{array} \right\} = \left\{ \begin{array}{l} \frac{\partial U}{\partial X} \\ \frac{\partial V}{\partial Y} \\ \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} \\ - \frac{\partial^2 U}{\partial X \partial Z} \\ - \frac{\partial^2 V}{\partial Y \partial Z} \\ - \left( \frac{\partial^2 U}{\partial Y \partial Z} + \frac{\partial^2 V}{\partial X \partial Z} \right) \end{array} \right\} \quad (96)
\]
where the indexes _m_ and _b_ refer to "membrane" and "bending" respectively, these can be expressed in terms of the nodal parameters by making use of matrix \([F]\) as defined by equation (95).

Using the standard notation of (Z-L) we can write:

\[
\begin{bmatrix}
-\varepsilon^m \\
\varepsilon^b
\end{bmatrix} = \begin{bmatrix}
-B^m \\
-B^b
\end{bmatrix} \{P_A^e\} = [B] \{P_A^e\}
\]

the stiffness matrix being therefore defined by

\[
[K_e^e] = \int_V [B]^T [D] [B] \, dV
\]

(98)

The elasticity matrix \([D]\) includes now both a membrane and a bending part and is defined by

\[
[D] = \begin{bmatrix}
D^m & 0 \\
- & - \\
0 & D^b
\end{bmatrix}
\]

(99)

where \([D^b]\) is given by expression (71) of chapter II, and \([D^m]\) is defined by:

\[
[D^m] = \frac{E}{1-\nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & (1-\nu)/2
\end{bmatrix}
\]

(100)

The pressure load is defined, by using the shape functions corresponding to the displacement normal to the surface \((W)\), by

\[
\{F_p^e\} = \int_A \{N_p\} \, p \, dA
\]

(101)

\({F_p^e}\) being a vector of nodal forces, \({N_p}\) a vector equal to the third line of matrix \([F]\), \(A\) the area of the midsurface of the element, and \(p=p(\xi, \eta)\) the pressure load function.
The body forces along any of the axes $x, y$ or $z$ can also be easily taken into account. For this we define first at any point of the element (in fact at each integrating point) a matrix $[H]$ containing direction cosines of the unit vectors $(\hat{x}, \hat{y}, \hat{z})$ as previously defined:

$$[H] = \begin{bmatrix} X_x & Y_x & Z_x \\ X_y & Y_y & Z_y \\ X_z & Y_z & Z_z \end{bmatrix}$$  \hspace{1cm} (102)

If the body forces along the axes are defined by

$$\{b\} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$  \hspace{1cm} (103)

The corresponding nodal forces $\{F^e_b\}$ will be defined by

$$\{F^e_b\} = \int_V \{b\}^T [H] [T] \, dV$$  \hspace{1cm} (104)

$[T]$ being a $(3 \times 32)$ matrix equal to the first 3 rows of matrix $[F]$.

Integration of expressions (98), (101) and (104) was carried out numerically using either the 5 point rule defined by equation (74) of Chapter II or a Gauss rule with 2x2 points.
CHAPTER III - THE SEMILOOF SHELL ELEMENT

7. RESULTS

Only one application of the semiloof shell element to linear problems will be presented in this Chapter. Further applications are considered in Chapter VII. This is a cylindrical shell roof as shown in Figure 5. The structure is subject to a vertical load shown on the figure and is supposed to have the following material properties:

\[ E = 30 \times 10^6 \text{ psi} \]

\[ \nu = 0.0 \]

The dimensions and types of meshes used are shown in the figure.

This structure has been analysed by various researchers, and some results, together with the ones obtained using the semiloof element, are shown in Figure 6. This figure shows studies of convergence using the following elements:

(a) Flat 15 degrees of freedom element, Ref. C-1, S-1
(b) " 15 " " " " " Ref. S-1
(c) " 12 " " " " Ref. B-1
(d) Curved 27 " " " " Ref. B-2
(e) " 36 " " " " Ref. C-2
(f) " 36 " " " " Ref. B-2
(g) " 27 " " " " Ref. D-1
(h) Semiloof shell element

It can be seen that the semiloof element compares very well with the other elements showing a very high rate of convergence.
8. CONCLUSIONS

A matrix formulation of the semiloof shell element has been presented in this chapter, this constituting the basic background for the following chapters. The semiloof element was compared with other elements and proved to give very accurate results even with coarse meshes, the rate of convergence being considerably higher than for the other elements with which it was compared.
BIBLIOGRAPHY

B-1  Bäcklund, J., Wennerström, H.
    'Finite element analysis of elasto-plastic shells'
    Int. J. for Numerical Methods in Engineering, Vol.8, 1974,
    p.415, 424.

B-2  Bonnes, G., Dhat, G., Giroux, Y.M. and Robichaud, L.P.A.
    'Curved triangular elements for the analysis of shells'
    Proc. Second Conf. on Matrix Methods in Structural Mechanics,
    Ohio, 1968.

C-1  Clough, R.W., Johnson, C.P.
    'A finite element approximation for the analysis of thin shells'
    Int. J. Solids and Structures, 4, 1968, p.43, 60.

C-2  Cowper, G.R., Lindberg, G.M., and Olson, M.D.
    'A shallow shell finite element of triangular shape'

D-1  Dupuis, G.A., Hibbitt, H.D., McNamara, S.F. and Marcal, P.V.
    'Non-linear material and geometric behaviour of shell
    structures'
    Comp. and Structures, 1, 1972, p. 223, 239.

I-1  Irons, B.M.
    'The semiloof shell element'
    Chapter 11 of "Finite Elements for Thin Shells and Curved Members"
    Edit. by R.H. Gallagher and D.G. Ashwell (John Wiley & Sons).

I-2  Irons, B.M.
    'Un nouvel element des coques generales - semiloof'

I-3  Irons, B.M.
    'Postscript to the semiloof shell element'
    University of Wales, Swansea.

I-4  Irons, B.M.
    Comment on 'A higher order conforming rectangular plate element'
    by S. Gopalacharyulu, Int. J. for Numerical Methods in

I-5  Irons, B.M. and Razzaque, A.
    'Experience with the patch-test for convergence of finite
    elements'
    In: The Mathematical Foundations of the Finite Element Method with
    Applications to Partial Differential Equations, ed. A.K. Aziz
S-1 Strickland, G.E. and Loden, W.A.
'A doubly curved triangular shell element'
Proc. Second Conf. of Matrix Methods in Structural Mechanics,
Ohio, 1968.

S-2 Souza, J.R.
'Analise Infinitesimal I'
Universidade do Porto.

Z-1 Zienkiewicz, O.C.
'The finite element method in engineering science'
FIG. 1 - NODAL CONFIGURATION OF THE SEMILOOF ELEMENT.
Figure 2 - Local Axes
UNIFORM LOAD ALONG Z:
-90 lb/sq ft

(DIMENSIONS IN FEET)

FIGURE 5
CHAPTER IV - EIGENVALUE ANALYSIS

CHAPTER IV

EIGENVALUE ANALYSIS

1. INTRODUCTION

In this chapter the Semiloof plate and shell elements are used for the solution of eigenvalue problems, a finite element eigenvalue solution system being developed for this purpose. An initial and very restricted version of this program, applicable only for the determination of the first buckling load, and without the possibility of determining the corresponding buckling shape, was presented in reference (M-1). This program has now been extended (M-2) for the determination of any number of eigenvalues and eigenvectors in the analysis of both elastic stability and vibrations of stressed and unstressed plates and shells. The program can also undertake a simple linear analysis which makes it a very versatile finite element package.

The eigenvalue solution is based on a Sturm sequence property and is presented in detail in sections 2 and 3. Both the Semiloof shell and plate elements are included in the program and a number of problems are solved to test their performance in eigenvalue analysis.

We note that reference (M-1) presents a number of buckling problems using the Semiloof shell element and compares the solutions obtained with other existing solutions. Therefore only one buckling problem, with different loads and boundary conditions is presented in this chapter.
2. THE EIGENVALUE PROBLEM

2.1 Formulations and Properties

Consider the following system of $n$ linear equations

$$\left[ [A] - \lambda [I] \right] \{x\} = \{0\} \quad (1)$$

in which $[A]$ is a square matrix of order $n$
$[I]$ is an identity matrix of order $n$
$\{x\}$ is a vector of $n$ unknowns, and
$\lambda$ is a parameter.

This system of equations is homogenous, and therefore, whatever value $\lambda$ takes, it always admits the trivial solution

$$\{x\} = \{0\} \quad (2)$$

For the system of equations (1) to admit a non trivial solution it is necessary and sufficient that the rank of the matrix

$$\left[ [A] - \lambda [I] \right] \quad (3)$$

be less than the number of equations, i.e. it must be

$$\det \left[ [A] - \lambda [I] \right] = 0 \quad (4)$$

This represents a polynomial equation of order $n$:

$$a_0 + a_2 \lambda + \ldots + a_{n-1} \lambda^{n-1} + (-1)^n \lambda^n = 0 \quad (5)$$

with $n$ solutions, in principle complex, with multiplicities up to $n$, that is, with the possibility of all the roots being equal.
for which there are no positive pivots in the triangularized form of 
\[
[A] - (\text{WRIGHT})[B]
\] (this means that there are no eigenvalues larger than 
\text{WRIGHT}); then we find a number \text{WLEFT} for which there is one positive 
pivot (this implies that there is one eigenvalue larger than \text{WLEFT}. We 
can therefore conclude that the largest eigenvalue of (10) lies between 
\text{WLEFT} and \text{WRIGHT}. The accuracy of the determined eigenvalue at this 
stage will depend on the length of the interval (\text{WLEFT}, \text{WRIGHT}).

e) By interval halving a better approximation can be 
obtained but the accuracy obtainable by this method is limited by the 
stability of the process of elimination. For all problems solved to date 
a very good convergence was observed and no difficulty was experienced in 
obtaining the eigenvalues accurate up to 8 significant figures, simply by 
interval halving.

However, in order to increase the efficiency of the solution 
algorithm, a linear interpolation scheme can be adopted as soon as the 
eigenvalue is localized.

Obviously, the process can be applied to determine any number 
of positive or negative eigenvalues.

3. FINITE ELEMENT EIGENVALUE SOLUTION

Apart from the advantages referred to previously, the finite 
element eigenvalue solution presented here has the further advantage of 
being easily implemented once a solution for a static finite element 
problem is available. The present solution centres on a frontal equation 
solver originated by Irons (I-1). A brief description of the frontal 
equation solver will therefore be presented.
Equation 3 can now take the place previously occupied by equation 1 and the complete system of equations will become:

\[
\begin{bmatrix}
K_{11} & K_{12} & K_{14} & u_1 & f_1 \\
K_{23} & K_{22} & K_{24} & u_2 & f_2 \\
K_{33} & K_{32} & K_{34} & u_3 & f_3 \\
K_{43} & K_{42} & K_{44} & u_4 & f_4 \\
\end{bmatrix}
\begin{bmatrix}
K_{11} \\
K_{23} \\
K_{33} \\
K_{43} \\
\end{bmatrix}
\begin{bmatrix}
\bar{K}_{11} \\
\bar{K}_{21} + K_{22} \\
\bar{K}_{31} + K_{24} \\
\bar{K}_{41} + K_{42} \\
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
\end{bmatrix}
= \begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4 \\
\end{bmatrix}
\]

We now note that equation 3 can be removed but $K_{33}$ will not be a pivot once that the variable 3 is fixed. In fact the unknown in this equation will be $R_3$ calculated during the back-substitution process.

Therefore, keeping a record of all the fixed variables, the "pseudo-pivots" can be easily detected and discarded for the counting involved in the rule of section 2.2d.

After dealing with equation 3 the process of Gaussian elimination follows as previously described. Another point should however be noted: that in order to optimize the computer utilization the final ordering of the equations can be very different from that implied by the node numbering. This however does not alter the eigenvalues. In fact "permuting two rows of a matrix results in its determinant changing sign" (G-3), which implies that its eigenvalues do not suffer any modification.

As a final point we note that the elimination of one equation for the eigenvector calculation is easily achieved by fixing the corresponding variable.

Figure 2 shows a flow chart of the computer program. The exact definition of the stiffness matrices to be assembled for each case of application of the program (instability, vibrations and vibrations of
CHAPTER IV - EIGENVALUE ANALYSIS

loaded structures) are defined in the next 3 sections. Figure 3 shows in more detail the part of the program implied by the double arrow marked A in figure 2.

4. LINEAR STABILITY

In this section the linear stability analysis of a shell is formulated as an eigenvalue problem. No large deflection effects are taken into consideration and an elastic analysis with an arbitrary load is performed before the buckling analysis in order to define the state of stress in the structure. The buckling load is then determined as a factor which multiplied by the applied load gives the buckling load.

This type of approach requires the computation of a geometric or initial stresses matrix. This stiffness matrix will also be used in section 6 for the vibration analysis of shells subject to a system of initial stresses, and in chapter V for the geometrically nonlinear analysis of plates and shells.

4.1 Non Linear Strains

Consider a continuous body referred to a rectangular Cartesian coordinate system (O, x, y, z), and, in particular, two neighbouring points of this body defined by the following coordinates:

\[
P \rightarrow \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad Q \rightarrow \begin{bmatrix} X + dX \\ Y + dY \\ Z + dZ \end{bmatrix}
\]  \quad (18)

If the body undergoes a deformation, points P and Q will change their position. Representing by U, V and W the components of the displacement vector along the axes x, y and z respectively the coordinates of points P and Q will become:
\[ P^* \rightarrow \begin{cases} X + U \\ Y + V \\ Z + W \end{cases}, \quad Q^* \rightarrow \begin{cases} X + dX + U + dU \\ Y + dY + V + dV \\ Z + dZ + W + dW \end{cases} \]  

(19)

Assuming that \( U(X,Y,Z) \), \( V(X,Y,Z) \) and \( W(X,Y,Z) \) are uniform and continuous functions, by definition of differential of a function, we can write:

\[
\begin{align*}
    dU &= \frac{\partial U}{\partial X} \, dX + \frac{\partial U}{\partial Y} \, dY + \frac{\partial U}{\partial Z} \, dZ \\
    dV &= \frac{\partial V}{\partial X} \, dX + \frac{\partial V}{\partial Y} \, dY + \frac{\partial V}{\partial Z} \, dZ \\
    dW &= \frac{\partial W}{\partial X} \, dX + \frac{\partial W}{\partial Y} \, dY + \frac{\partial W}{\partial Z} \, dZ 
\end{align*}
\]

(20)

Defining the (3x3) matrix \([j]\) by:

\[
[j] = \begin{bmatrix}
    \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} & \frac{\partial U}{\partial Z} \\
    \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} & \frac{\partial V}{\partial Z} \\
    \frac{\partial W}{\partial X} & \frac{\partial W}{\partial Y} & \frac{\partial W}{\partial Z}
\end{bmatrix}
\]

(21)

expressions (20) can be written in a sole matrix equation as

\[
\begin{bmatrix}
    dU \\
    dV \\
    dW
\end{bmatrix} = [j] \begin{bmatrix}
    dX \\
    dY \\
    dZ
\end{bmatrix}
\]

(22)

The line segment \( PQ \) also changes during the deformation.

In fact, before deformation we had

\[
\begin{bmatrix}
    dX \\
    dY \\
    dZ
\end{bmatrix}
\]

(23)
becoming after deformation

\[
P*Q* \rightarrow \begin{bmatrix} \frac{dX}{dY} + [j] \frac{dX}{dZ} \\ \frac{dY}{dZ} \end{bmatrix}
\]

or

\[
P*Q* \rightarrow \begin{bmatrix} [I] + [j]^T \end{bmatrix} \begin{bmatrix} \frac{dX}{dY} \\ \frac{dY}{dZ} \end{bmatrix}
\]  

(24a)

The local deformation can be conveniently measured by considering the difference

\[
(P*Q*)^2 - (PQ)^2
\]  

(25)

The first term of this expression is given by:

\[
(P*Q*)^2 = (dX \ dY \ dZ) \begin{bmatrix} [I] + [j]^T \end{bmatrix} \begin{bmatrix} [I] + [j] \end{bmatrix} \begin{bmatrix} \frac{dX}{dY} \\ \frac{dY}{dZ} \end{bmatrix}
\]  

(25a)

and the second by

\[
(PQ)^2 = (dX \ dY \ dZ) \begin{bmatrix} \frac{dX}{dY} \\ \frac{dY}{dZ} \end{bmatrix}
\]  

(25b)

Expression (25) is then defined as

\[
(P*Q*)^2 - (PQ)^2 =
\]

\[
= (dX \ dY \ dZ) \begin{bmatrix} [I+j]^T \end{bmatrix} \begin{bmatrix} [I+j] - [I] \end{bmatrix} \begin{bmatrix} \frac{dX}{dY} \\ \frac{dY}{dZ} \end{bmatrix}
\]  

(26)
The \((3x3)\) matrix in this expression defines the Green's strains \([\varepsilon]\), which are

\[
2[\varepsilon] = \begin{bmatrix}
[I+j^T] & [I+j] - I
\end{bmatrix}
\]

or

\[
[\varepsilon] = \frac{1}{2}\begin{bmatrix}
[j] + [j]^T + [j]^T[j]
\end{bmatrix}
\]

\[
[\varepsilon] = \frac{1}{2}\begin{bmatrix}
[j] + [j]^T + \frac{1}{2}[j]^T[j]
\end{bmatrix}
\]

(27b)

The first term on the right hand represents the usual first order strains and the second term the second order strains.

4.2 Geometric Matrix

The use of the linear strains of expression \((27b)\) in a finite element context provides the classical stiffness equation

\[
[K_E] \{\delta\} = \{p\}
\]

(28)

where \([K_E]\) classical stiffness matrix

\(\{\delta\}\) vector of nodal parameters

\(\{p\}\) vector of nodal forces

If the second order strains are considered the strain energy will have an additional term, resulting in an additional matrix (geometric matrix) for the equilibrium equations. This additional term results from the work done by stresses acting on the second order strains.

For the determination of the geometric matrix the stresses are first determined by means of a linear elastic analysis. Once the stresses are calculated the second order energy will be given by

\[
U_2 = \frac{1}{2} \int_V [\sigma] : \begin{bmatrix}
[j]^T[j]
\end{bmatrix} dV
\]

(29)
where $\sigma$ is a matrix of stress components. Expanding this expressions
for a two-dimensional medium we obtain:

$$U_2 = \frac{1}{2} \int_V \left[ \begin{array}{cc} \sigma_{XX} & \sigma_{XY} \\ \sigma_{XY} & \sigma_{YY} \end{array} \right] \left[ \begin{array}{c} \frac{\partial U}{\partial x} \\ \frac{\partial U}{\partial y} \end{array} \right]^2 + \left[ \begin{array}{c} \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \end{array} \right]^2 + \left[ \begin{array}{c} \frac{\partial W}{\partial x} \\ \frac{\partial W}{\partial y} \end{array} \right]^2 + \left[ \begin{array}{c} \frac{\partial U}{\partial x} \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \frac{\partial V}{\partial y} + \frac{\partial W}{\partial x} \frac{\partial W}{\partial y} \end{array} \right] \right] dV$$

(30)

This expression can be rearranged as follows:

$$U_2 = \frac{1}{2} \int_V \left( \frac{\partial U}{\partial x} \right)^2 \sigma_{XX} + \left( \frac{\partial U}{\partial y} \right)^2 \sigma_{YY} + \left( \frac{\partial V}{\partial x} \right)^2 \sigma_{XY} + \left( \frac{\partial V}{\partial y} \right)^2 \sigma_{YY} dV +$$

$$+ \frac{1}{2} \int_V \left( \frac{\partial W}{\partial x} \right)^2 \sigma_{XX} + \left( \frac{\partial W}{\partial y} \right)^2 \sigma_{YY} dV +$$

$$+ \frac{1}{2} \int_V \left( \frac{\partial U}{\partial x} \frac{\partial V}{\partial y} + \frac{\partial V}{\partial x} \frac{\partial U}{\partial y} \right) \sigma_{XY} dV$$

(31)

or further:

$$U_2 = \frac{1}{2} \int_V \left( \frac{\partial U}{\partial x} \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \frac{\partial V}{\partial y} + \frac{\partial W}{\partial x} \frac{\partial W}{\partial y} \right) \left[ \begin{array}{cc} \sigma & (\text{zeros}) \\ (\text{zeros}) & \sigma \end{array} \right] \left[ \begin{array}{c} \frac{\partial U}{\partial x} \\ \frac{\partial U}{\partial y} \\ \frac{\partial V}{\partial x} \\ \frac{\partial V}{\partial y} \\ \frac{\partial W}{\partial x} \\ \frac{\partial W}{\partial y} \end{array} \right] dV$$

(32)
In order to derive the geometric matrix, the derivatives contained in this expression must now be expressed in terms of nodal parameters. This aspect was discussed in Chapter 3 where an expression for these derivatives was derived. This is implicit in the matrix equation (95) of that chapter and, defining \([G]\) as a submatrix of \([F]\) equal to lines 3 to 7 of this matrix the second order strain energy will be given by:

\[
U_2 = \frac{1}{2} \int_V \{p_e^e\}^T [G]^T [T] [G] \{p_e^e\} \, dv
\]  

(33)

where

\[
T = \begin{bmatrix}
    [\sigma] & \text{zeros} \\
    \text{zeros} & [\sigma]
\end{bmatrix}
\]  

(34)

The geometric matrix, \([K_G]\), is now easily calculated from (33) by making use of the principle of stationary energy and is

\[
\]  

(35)

4.3 Stability Analysis

The system of equations describing the behaviour of the structure, for a load less than the critical one, is obtained by performing the first variation of the total potential energy \(\pi\):

\[
\delta \pi = [K_E] \{p_A\} + [K_G] \{p_A\} - \{p\} = 0
\]  

(36)

For buckling to occur this system of equations must be indeterminate. As the system is non homogenous this only can happen if the determinant of the system becomes zero:
\[
\det \left[ \begin{bmatrix} K_E \end{bmatrix} + \begin{bmatrix} K_G \end{bmatrix} \right] = 0 \tag{37}
\]

In order to find for what load this happens the structure can be subjected to increasing loads, checking at the same time when equality (37) occurs.

Now we note that the stiffness matrix \([K_E]\) is independent of the load level and that \([K_G]\) is directly proportional to it once that the initial configuration of the structure is always taken as reference. Therefore, the effect of different levels of load on equation (36) can be accounted for by simply modifying the matrix \([K_G]\) by means of a parameter \(\lambda\), which is, in fact, equivalent to the formulation of the following eigenvalue problem:

\[
\begin{bmatrix} K_E \end{bmatrix} + \lambda \begin{bmatrix} K_G \end{bmatrix} \begin{bmatrix} P_A \end{bmatrix} = \{0\} \tag{38}
\]

As a final point we note that the restrictions on the form of matrices \([A]\) and \([B]\) as defined in equation (8) are satisfied by the geometric matrix \([K_G]\) and conventional stiffness matrix \([K_E]\) respectively. Also the eigenvalues of

\[
\begin{bmatrix} K_G \end{bmatrix} - \lambda \begin{bmatrix} K_E \end{bmatrix} \begin{bmatrix} P_e \end{bmatrix} = \{0\} \tag{39}
\]

are easily related to the eigenvalues of (38): they are symmetric and inverse.

4.4 Numerical Results

In this section a cylindrical shell with the following characteristics is analysed:

- shell radius: 76 m
- shell length: 23 m
shell thickness 0.02 m
Young's Modulus $200 \times 10^6$ KN/m$^2$
Poisson ratio 0.3

The mesh used is similar to the one shown on figure 3 of chapter 7 except that no stiffeners are considered.

Fixing all the three displacements at the upper and lower boundaries and considering the shell subject to an external pressure only an eigenvalue $\lambda_1 = 1.28$ was determined. The same eigenvalue was calculated by considering the shell self weight as loading as well as the external pressure. The corresponding eigenvalues are shown in figures 4 and 5 respectively. It can be seen that for the case in which the self weight of the shell is considered (figure 5) the upper part of the shell shows no buckling. This is natural since both extremities of the shell were considered fixed (pinned), which induces, owing to the self weight, traction stresses in this part of the shell.

Using the same mesh but allowing the upper boundary to be free to move in the longitudinal direction a slightly higher eigenvalue was obtained: $\lambda_1 = 1.30$.

Although no analytical solution is available for this problem some empirical formulae (B-1) give the first buckling mode at a pressure of 1.8 which compares reasonably well with the value obtained here. In Chapter 7 some more stability problems are considered.

5. FREE VIBRATION ANALYSIS

5.1 Equation of Motion

The main objective in the field of structural dynamic analysis is the determination, for a structure subject to time varying load, the
corresponding time versus displacement history. The equations of motion are a means of describing this displacement history.

The formulation of the equations of motion can be achieved in various ways (C-2):

- Direct formulation using d'Alembert principle
- Formulating using the principle of virtual work
- Formulation using Hamilton's principle

These three formulations are equivalent and lead to identical equations of motion.

For systems involving mass and elasticity terms distributed over complex shapes the principle of virtual work constitutes a very convenient means of deriving the equations of motion.

The principle of virtual work generalised to include dynamic conditions can be expressed as (P-2):

$$\delta S_1 = \delta Q - \int_V \rho \delta d^T \ddot{d} \, dV$$

(40)

where $\delta S_1$ is the increment of elastic strain
$\delta Q$ is the increment of virtual work of the applied forces
$\rho$ is the density of the body
and $d$ lists the relevant displacements

Denoting by $\varepsilon$ and $\sigma$ the relevant strains and stresses, by $\phi$ and $X$ the surface and body forces and by $P$ the directly applied loads, we have
\[ \delta S_i = \int_V \delta \epsilon^T \sigma \, dV \] (41)

\[ \delta Q = \int_S \delta d^T \phi \, dS + \int_V \delta d^T X \, dV + \delta p_A \, P \] (42)

where \( \delta p_A \) represent the virtual displacements of corresponding to forces \( P \).

Using the usual finite element interpolation process for the displacements:

\[ d = a p_A \] (43)

where \( a \) is a matrix (or vector) of shape functions. Expressing the relevant strains in terms of displacements by

\[ \delta \epsilon = B \delta p_A \] (44)

and making use of the generalised Hooke's law, (40) becomes:

\[ \int_V \rho \delta p_A^T a^T a p_A \, dV + \int_V \delta p_A^T B^T D B p_A \, dV = \]

\[ = \int_S \delta p_A^T a^T \phi \, dS + \int_V \delta p_A^T a^T X \, ds + \delta p_A^T P \] (45)

The equilibrium equations can therefore be written as

\[ K_M p_A + K_E p_A = P \] (46)

where \( K_E \) is the usual stiffness matrix

\( K_M \) is the mass matrix, defined by

\[ K_M = \int_V \rho \, a^T a \, dV \] (47)
and
\[ R = P - \int_S a^T \phi \, dS + \int_V a^T X \, dV \] (48)
represents the total nodal forces.
Equation (46) can be used to derive the equations for the free vibrations of structure.

For this case we have
\[ K_M \ddot{P}_A + K_E P_A = 0 \] (49)
and since the free vibrations are harmonic the displacements \( P_A \) can be written as
\[ P_A = q \, e^{i\omega t} \] (50)
where \( q \) is a vector of amplitudes of displacements
\( \lambda \) the circular frequency of oscillation and
\( t \) is the time

Substituting this value of \( P_A \) into equation (49) and cancelling the factor \( e^{i\omega t} \) we obtain
\[ \left( -\lambda^2 K_M + K_E \right) q = 0 \] (51)
which represent the equations of motion for a freely oscillating system.
This is equivalent to an eigenvalue problem similar to the one represented by (8). In matrix notation (51) becomes
\[ \left[ [K_E] - \lambda^2 [K_M] \right] \{q\} = \{0\} \] (52)
Comparing this equation with (8) and considering \( [K_E] \) as \([B]\) and \( [K_M] \) as \([A]\)
the eigenvalues of (52) are the inverse of the eigenvalues of (8).

5.2 Mass Matrices

In this section the derivation of the consistent mass matrices for the semiloof plate and shell elements is presented.

For the calculation of the mass matrix for the plate element the only relevant displacement is the displacement normal to the plate surface. This is defined in terms of nodal parameters in matrix equation (68) of chapter II. Using therefore a vector \( \{ N_A \} \) composed of the elements of the first row of matrix \( [H] \), the mass matrix for the semiloof plate element is defined by

\[
[K_M] = \int_V \rho \{ N_A \}^T \{ N_A \} \, dV
\]  

(53)

For the case of the shell element all three displacements \( U, V \) and \( W \) as defined in chapter III must be considered. Recalling matrix equation (95) of chapter III and defining a (3x32) matrix made up of the first 3 rows of matrix \( [F] \), the mass matrix for the Semiloof shell element is defined by:

\[
[K_M] = \int_V [T]^T [M] [T] \, dV
\]  

(54)

where

\[
[M] = \begin{bmatrix}
\rho & 0 & 0 \\
0 & \rho & 0 \\
0 & 0 & \rho
\end{bmatrix}
\]  

(55)

The integrals of equations (53) and (54) have of course been evaluated numerically. For the numerical results presented in the next section three integrating rules have alternatively been used:
- 4 and 9 point Gaussian integrating rules
- 5 point rule as defined by (74) of chapter II

5.3 Numerical Results

A number of vibration problems involving both plates and shells will be presented in this section. For the case of the plates, unless stated otherwise the following material properties are assumed:

Young's modulus, $E = 3000.0$

Poisson's ratio, $\nu = 0.3$

Thickness, $t = 0.05$

Mass density, $\rho = 1.0$

Also, unless stated otherwise, the frequencies of vibration are given in radians/sec and a 2x2 Gauss integrating rule is used.

a) Simply supported square plate

Table 1 shows the first 14 eigenvalues corresponding to the vibrations of a simply supported square plate with side length equal to 4.0'. The shell program has been used, and, as shown in table 1, one quarter of the plate was analysed, using three different types of boundary conditions. This table includes also the theoretical values (T-1) and even for relatively coarse meshes a good comparison with theory is evident. Figure 6 shows a comparison of the values obtained with the Semilooft shell element and other elements. The elements considered are listed below and are fully described in reference (D-1).

ACM - Non-conforming complete rectangle. Cubic model with 12 external degrees of freedom
BFS - Complete conforming rectangle. Cubic Hermitian polynomial model with 16 external degrees of freedom

P - Conforming but incomplete rectangle. Contains polynomial terms up to 6th order with 12 external degrees of freedom

LCCT - Model composed of 2 triangles and cubic variation.

With the exception of the BFS element the performance of the Semiloof element is superior to all the others considered. It should be noted that we are using a shell element to solve a plate problem. If only the plate bending degrees of freedom were employed the convergence curve of the Semiloof element would be translated to the left resulting in an even more favourable comparison. Figure 7 studies the convergence of the first 3 eigenvalues with increasing number of elements. Figures 8, 9 and 10 show respectively the first second and third eigenvectors obtained using meshes of 1, 4 and 16 elements. As it can be seen, even the coarser meshes yield quite accurate eigenvectors. Higher order vibrating modes are given in figures 11 to 14, and are generally in close agreement with the theoretical shape.

Table 2 gives the first 3 eigenvalues for the same problem but calculated using the plate element and 3 different integrating rules. It can be seen that the modifications of the integrating rule does not affect the results significantly.

b) Clamped square plate

This problem was solved using the plate program. The plate, with a side length of 2.0, was discretized into 16 elements. Table 3 shows the theoretical and computed values for the first seven frequencies of vibration. A close agreement can be observed between the calculated
and the theoretical values. We note that the program detected double
eigenvalues for the second, sixth and seventh mode. This is natural since,
unlike the case of the first, third, fourth and fifth modes which have two
lines of symmetry, the second and sixth modes have one line of symmetry only.
This implies the possibility of obtaining two equal modes of vibration
rotated by $90^\circ$ in relation to each other (the theoretical eigenvalue and
eigenvector for the seventh mode is not included in reference S-1). The
eigenvectors obtained for the second, fourth and sixth modes are shown in
figures 15 and 16.

c) Square plate with two adjacent edges clamped

The theoretical and computed eigenvalues corresponding to
the first five frequencies are given in table 4. A close agreement
between them can be observed. The computed eigenvalues for the sixth to
the tenth modes are also included in this table, but not the corresponding
theoretical values as these are not given in reference (S-1). Some
eigenvectors are given in figures 17 and 18.

This problem was solved using the plate program and the plate,
having a side length of 2.0, was discretized into 16 elements.

d) Circular plate with clamped edges

Results for the first 3 eigenvalues using meshes of 3, 16
and 27 elements are given in figure 19, where a convergence study is also
presented. The theoretical values shown are taken from reference (T-1).
Results obtained using a 1 element mesh are as follows:
\[ \lambda_1 = 9.6256 \]
\[ \lambda_2 = 21.8733 \]
\[ \lambda_3 = 50.8860 \]

and constitute a fairly good approximation for such a coarse mesh. Using the 3 elements mesh the following higher order eigenvalues were also calculated:

\[ \lambda_4 = 39.5608 \]
\[ \lambda_5 = 43.5581 \]
\[ \lambda_6 = 62.9133 \]
\[ \lambda_7 = 87.0804 \]
\[ \lambda_8 = 114.2412 \]
\[ \lambda_9 = 177.3981 \]
\[ \lambda_{10} = 204.6027 \]

The first eigenvector for the 3 elements case and the second and third for the 27 elements case are shown in figure 20. For the 16 elements case the second and third eigenvectors are less accurate than the ones shown. We note that the eigenvalue (0.1), although smaller than the eigenvalue (0.2) can not be obtained using only one quarter of the plate as employed here. This problem was solved using the shell program and a plate with radius 1.0.

e) Circular cantilever shell

This shell is shown in figure 21, where the material properties are also included. This structure has previously been studied by many investigators (A-3, B-2,3, H-2,3, N-2, O-1,2) and experimental results are also available (O-1). According to reference (B-2) it forms a stringent test on programs involving very thin elements. Table 5 includes values from the above references as well as the values obtained with the Semiloof shell element. Figure 22 shows a convergence study for the first eigenvalue employing different types of elements. As can be seen,
the performance of the Semiloof shell element compares favourably with that of the other elements. We note that the degrees of freedom of the Semiloof element refereed to in table 5 and figure 22 are the total number of degrees of freedom in the problem although in some of the references these are reduced for solution purposes by using an eigenvalue economiser.

Some eigenvectors for this problem are given in figures 23 to 26 and are in good agreement with the ones given in reference (0-2). Some higher order eigenvalues for the same problem are included in table 6.

f) Skew cantilever plate

This cantilever plate with a skew angle of $45^\circ$ and a side length of 1.0 was solved with the plate program using both quadrilateral and triangular elements. Results are given in table 7 together with other solutions from reference (B-4). The values for the triangular elements were obtained using a 3 point midside rule (Z-1) for the integration of the stiffness and mass matrices.

For the quadrilateral elements the values given on table 7 refer to a 4 point Gauss integrating rule; using a 5 point and a 9 point integrating rule the first eigenvalue was calculated, respectively, as

$$\lambda_1^5 = 3.5155 \quad \lambda_1^9 = 3.6479$$

which do not differ much from the value 3.4981 given in the table. The first 3 eigenvectors obtained are given in figure 27.

g) Square Cantilever Plate

A plate with a side length of 2.0 was considered for this problem. Meshes with 4, 9 and 16 elements were used. The results for the first ten eigenvalues, for both plate and shell elements, are given
in table 8. As should be expected only very marginal differences exist between the values yielded by the plate and shell elements. Figure 28 shows a convergence study for the first five eigenvalues. It shows that even relatively coarse meshes give quite accurate eigenvalues. Table 9 compares the results obtained using for the same mesh (16 elements) different types of integration. It can be seen that, in general, the accuracy increases with the order of integration, the results for the 3x3 Gauss point case being approximately twice as accurate as for the 2x2 Gauss point case.

The first five eigenvectors are shown in figures 29 to 31. Figure 32 gives the approximate theoretical nodal lines. As it can be seen the eigenvectors obtained compare well with the ones theoretically expected.

h) Cylindrical chimney

The last problem considered in this section refers to a cylindrical chimney with the base built in and having the following characteristics:

Length: 72.0
Radius: 6.0
Thickness: 0.01
Young's Modulus: $10^7$
Poisson's ratio: 0.3
Mass density: $2.59 \times 10^{-4}$

Only one quarter of the shell was considered, and, as such, only certain eigenvalues can be detected. Representing by $m$ and $n$ respectively the longitudinal and circumferential wave number, the model
employed is able to detect eigenvalues corresponding to n even and m odd or even. In the finite element mesh used, 20 elements were considered arranged in 10 layers along the shell length. Under these conditions the following eigenvalues were obtained:

\[
\lambda_1 = 19.8 \text{ cycles/second} \\
\lambda_2 = 39.4 \\
\lambda_3 = 41.6
\]

which compare reasonably well with the experimental values given in reference (5-2):

\[
\lambda_1 = 26.3 \text{ cycles/second} \ (m=1, n=2) \\
\lambda_2 = 37.3 \ (m=1, n=4) \\
\lambda_3 = 56.9 \ (m=2, n=4)
\]

It should be noted that the 20 element mesh used is a relatively coarse one, since only 2 elements are considered in the circumferential direction.

In chapter VII the free vibration of a circular reinforced shell will be considered.

6. VIBRATIONS OF INITIALLY STRESSED SHELLS

6.1 Problem Formulation

As seen in section (4) the consideration of the nonlinear part of the Green's strains (27) generates an additional term in the strain energy equation and adds another matrix, geometric matrix, to the equilibrium equations. If this matrix is taken into consideration for the establishment of the equations of motion (section 5.1) the following eigenvalue problem is obtained:
\[
\left[ \begin{bmatrix} K_E \end{bmatrix} + \begin{bmatrix} K_G \end{bmatrix} - \lambda^2 \begin{bmatrix} K_M \end{bmatrix} \right] \{q\} = \{0\} \tag{56}
\]

This is the same eigenvalue problem as the one represented by (52) and allows the analysis of the vibrations of initially stressed bodies.

6.2 Numerical Results

For the testing of the finite element program for this type of solution the problem of a cantilever plate subject to membrane stresses was considered. The following dimensions and material properties were assumed:

\[
E = 3000.0 \\
\nu = 0.3 \\
\text{side length} = 2.0 \\
\text{thickness} = 0.05
\]

The plate was discretized into 4 finite elements as shown in figure 33. A constant state of stress was induced along the Y direction and using option D of the program (section 2.2) the buckling stress was calculated to be \(-0.4033\). In the sequence of analyses performed, compressive and tensile stresses ranging from \(-0.396\) up to 720.0 were applied to the structure and the first 3 frequencies of vibration were calculated. These values are given in table 10 where it can be seen that the frequencies of vibration increase with the tensile stresses and decrease with the increase of the compressive stresses. Figure 32 shows a plot of the first frequency of vibration versus the applied stress. It can be seen that as the stress approaches the buckling value the frequency tends to zero. This is physically understandable because in the neighbourhood of the buckling point any small physical disturbance
must be capable of inducing the plate to vibrate.

7. CONCLUSIONS

The Semiloof shell and plate elements have been used for the solution of a wide range of eigenvalue problems. Comparisons were made with theoretical or other existing solutions and it was shown that the elements yield a good representation of the lower part of the eigenvalue spectrum. When compared with other elements the Semiloof element proved to be very competitive; reasonably accurate results being obtained with very coarse meshes.

The eigenvalue solution system developed proved to be very versatile and efficient, keeping the computer core storage requirement at practically the same level as that required for a simple static solution.
CHAPTER IV - EIGENVALUE ANALYSIS

BIBLIOGRAPHY

A-1 Anderson, R. G.
'A finite element eigenvalue solution system'

A-2 Anderson, R. G., Irons, B. M., Zienkiewicz, O. C.
'Vibration and stability of plates using finite elements'

A-3 Ahmad, S., Anderson, R. G. and Zienkiewicz, O. C.
Vibration of Thick Curved Shells with particular reference to
Turbine Blades

B-1 Baker, E. H. Kovalevsky, L., Risk, F. L.
'Structural Analysis of shells'

B-2 Bossak, M., Zienkiewicz, O. C.
'Free Vibration of Initially stressed Solids, with particular
reference to Centrifugal force effects in rotating machinery'

B-3 Bridle, M. D. J.
'Vibrations of thick plates and shells'

B-4 Barton, M. V.
'Vibration of rectangular and skew cantilever plates'

C-1 Clough, R. W.
'Analysis of Structural Vibrations and Dynamic Response'
In Recent Advances in Matrix Methods of Structural Analysis and

C-2 Clough, R. W., Penzien, J.
'Dynamics of structures'

D-1 Desai, C. S. and Abel, J. F.
'Introduction to the Finite Element Method'

G-1 Gourlay, A. R. and Watson, G. A.
Computational methods for matrix eigenproblems

G-2 Gupta, K. K.
Solution of Eigenvalue Problems by Sturm Sequence Method
International Journal for Numerical Methods in Engineering,
G-3 Guimarães, A.
Lícões de Algebra Linear
Universidade do Porto

H-1 Henshell, R. D. and Ong, J. H.
Automatic Masters for Eigenvalue Economization.
Int. J. of Earthquake Engineering and Structural Dynamics

H-2 Hofmeister, L. D. and Evensen, D. A.
Vibration Problems using Isoparametric Shell Elements
International Journal for Numerical Methods in Engineering
1972, 5, 142-145.

H-3 Henshell, R. D., Neale, B. K., Warburton, G. B.
A new hybrid cylindrical shell finite element

I-1 Irons, B. M.
A frontal solution program for finite element analysis
International Journal for Numerical Methods in Engineering, 2,
5-32, 1970.

M-1 Martins, R. A. F.
Finite Element Eigenvalue Solution Employing the Semiloof Shell

M-2 Martins, R. A. F., Owen, D. R. J.
Structural instability and natural vibration analysis of thin arbitrary
shells by use of the Semiloof element
Int. J. of Numerical Methods in Engineering (to be published)

N-1 Noble, B.
Numerical Methods

N-2 Neale, B. K.
Vibration of Shell Structures

O-1 Olson, M. D. and Lindberg, C. M.
Dynamic Analysis of Shallow Shells with a doubly curved triangular
finite element.

O-2 Olson, M. D. and Lindberg, C. M.
Vibration analysis of cantilevered curved plates using a new
cylindrical shell finite element
Second Conference on Matrix Methods in Structural Mechanics,
Wright-Patterson A.F. Base, Ohio, 1968.

P-1 Peters, G. and Wilkinson, J. H.
Eigenvalues of $Ax = \lambda Bx$ with band symmetric A and B
P-2 Przemieniecki, J. S.
   Theory of Matrix structural analysis

S-1 Stokey, W. F.
   Vibration of systems having distributed mass and elasticity

S-2 Sharma, C. B.
   Natural frequency of clamped-free circular cylindrical shells

T-1 Timoshenko, S., Young, D. H., Weaver, W.
   Vibration problems in engineering

W-1 Wilkinson, J. H.
   The algebraic eigenvalue problem

W-2 Wittrick, W.H. and Williams, F. H.
   New Procedures for structural eigenvalue calculations
   Fourth Australian Conference on the Mechanics of Structures

Z-1 Zienkiewicz, O. C.
   The finite element method in engineering science
<table>
<thead>
<tr>
<th>Boundary Conditions</th>
<th>Order of the Eigenvalue</th>
<th>Theoretical Value</th>
<th>1 Element</th>
<th>2x2 Elements</th>
<th>4x4 Elements</th>
<th>5x5 Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero Rot.</td>
<td>(1.1)</td>
<td>1.0224176</td>
<td>1.03606</td>
<td>1.02345</td>
<td>1.022486</td>
<td>1.022446</td>
</tr>
<tr>
<td></td>
<td>(3.1)</td>
<td>5.112088</td>
<td>3.13340</td>
<td>5.0115</td>
<td>5.0129</td>
<td>5.0383</td>
</tr>
<tr>
<td></td>
<td>(3.3)</td>
<td>9.2017</td>
<td>9.93287</td>
<td>9.4864</td>
<td>9.2459</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(5.1)</td>
<td>13.291</td>
<td>16.85639</td>
<td>11.9225</td>
<td>12.899</td>
<td></td>
</tr>
<tr>
<td>Zero Disp.</td>
<td>(1.2)</td>
<td>2.556044</td>
<td>2.1858</td>
<td>2.52056</td>
<td>2.5394</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3.2)</td>
<td>6.64571</td>
<td>9.1777</td>
<td>6.6513</td>
<td>6.6176</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1.4)</td>
<td>8.69055</td>
<td>94.9965</td>
<td>7.7468</td>
<td>8.4378</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(3.4)</td>
<td>12.78022</td>
<td></td>
<td>11.5858</td>
<td>12.8339</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(5.2)</td>
<td>14.82506</td>
<td></td>
<td>13.0600</td>
<td>14.5424</td>
<td></td>
</tr>
<tr>
<td>Zero Rot.</td>
<td>(2.2)</td>
<td>4.089671</td>
<td>4.975</td>
<td>4.1451</td>
<td>4.09273</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2.4)</td>
<td>10.224176</td>
<td>95.303</td>
<td>8.7433</td>
<td>10.0813</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(4.4)</td>
<td>16.35868</td>
<td></td>
<td>12.5336</td>
<td>16.5886</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2.6)</td>
<td>20.44836</td>
<td></td>
<td>19.8865</td>
<td>20.046</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(4.6)</td>
<td>26.5828</td>
<td></td>
<td>35.7160</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE 1 - Eigenvalues corresponding to the vibrations of a simply supported square plate
<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>395</th>
<th>4</th>
<th>3</th>
<th>2</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>365.4</td>
<td>340.2</td>
<td>275.7</td>
<td>264.8</td>
<td>247.9</td>
<td></td>
</tr>
<tr>
<td>375.0</td>
<td>353.0</td>
<td>284.9</td>
<td>271.8</td>
<td>254.0</td>
<td></td>
</tr>
<tr>
<td>365.8</td>
<td>340.8</td>
<td>275.9</td>
<td>264.9</td>
<td>247.9</td>
<td></td>
</tr>
<tr>
<td>375.0</td>
<td>353.0</td>
<td>284.9</td>
<td>271.8</td>
<td>254.0</td>
<td></td>
</tr>
</tbody>
</table>

**Table 2:** First three Eigenvalues for a Simply Supported Square Plate

<table>
<thead>
<tr>
<th>Element</th>
<th>4,098961</th>
<th>2,556044</th>
<th>1,022918</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type</td>
<td>THIRD</td>
<td>SECOND</td>
<td>FIRST</td>
</tr>
<tr>
<td>Number</td>
<td>16</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>Theoretical</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
## TABLE 3 Clamped square plate

<table>
<thead>
<tr>
<th>Eigenvalue Number</th>
<th>Computed Values</th>
<th>Theoretical Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.4465</td>
<td>7.4566</td>
</tr>
<tr>
<td>2</td>
<td>14.6682 (D)</td>
<td>15.2095</td>
</tr>
<tr>
<td>3</td>
<td>22.5941</td>
<td>22.4320</td>
</tr>
<tr>
<td>4</td>
<td>24.7488</td>
<td>27.2739</td>
</tr>
<tr>
<td>5</td>
<td>26.7440</td>
<td>27.4003</td>
</tr>
<tr>
<td>6</td>
<td>33.0285 (D)</td>
<td>34.2167</td>
</tr>
<tr>
<td>7</td>
<td>36.0231 (D)</td>
<td>-</td>
</tr>
</tbody>
</table>

## TABLE 4 Square plate with 2 adjacent edges clamped

<table>
<thead>
<tr>
<th>Eigenvalue Number</th>
<th>Experimental (Ref. 0-2)</th>
<th>4x4 f. elem. (175 DOF) (0-2)</th>
<th>4x4 semiloof el. (275 DOF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>531.1</td>
<td>534.3</td>
<td>543.2</td>
</tr>
<tr>
<td>7</td>
<td>743.2</td>
<td>781.5</td>
<td>727.6</td>
</tr>
<tr>
<td>8</td>
<td>751.2</td>
<td>792.2</td>
<td>785.9</td>
</tr>
</tbody>
</table>

## TABLE 6 Cantilever Shell (frequencies in cycles/sec.)
<table>
<thead>
<tr>
<th>Eigenvalue Number</th>
<th>Reference (B-4)</th>
<th>2 Triangles</th>
<th>8 Triangles</th>
<th>32 Triangles</th>
<th>16 Quadrilaterals</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.31 Th</td>
<td>2.8376</td>
<td>3.4862</td>
<td>3.6574</td>
<td>3.4981</td>
</tr>
<tr>
<td>3</td>
<td>18.21 Tn</td>
<td>11.9186</td>
<td>19.9164</td>
<td>21.7308</td>
<td>21.2642</td>
</tr>
<tr>
<td>4</td>
<td>20.69 Tn</td>
<td>16.6642</td>
<td>23.5996</td>
<td>25.4591</td>
<td>24.5133</td>
</tr>
<tr>
<td>5</td>
<td>34.47 Tn</td>
<td>37.8871</td>
<td>40.9357</td>
<td>44.8539</td>
<td>40.7388</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>45.8403</td>
<td>48.0723</td>
<td>64.6215</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>46.3659</td>
<td>60.7893</td>
<td>68.8513</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>50.6804</td>
<td>68.1523</td>
<td>84.0602</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>59.3181</td>
<td>84.1030</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>68.2658</td>
<td>90.9571</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Th - Theoretical values
Tn - Test values (non corrected)

TABLE 7 Frequencies of vibration (rad/sec) for a skew cantilever plate (45°)
TABLE 8  Cantilever square plate, ten first vibrating modes  
2x2 Gauss Rule

(Mesh of 4x4 Shell Elements with 3 Types of Integration)

<table>
<thead>
<tr>
<th>Number of the Eigenval.</th>
<th>4 Semiloof Shell Elements</th>
<th>9 Semiloof Shell Elements</th>
<th>16 Semiloof Shell Elements</th>
<th>16 Semiloof Plate Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.6896</td>
<td>0.7052</td>
<td>0.7110</td>
<td>0.7110</td>
</tr>
<tr>
<td>2</td>
<td>1.6850</td>
<td>1.7270</td>
<td>1.7410</td>
<td>1.7415</td>
</tr>
<tr>
<td>3</td>
<td>4.1026</td>
<td>4.1781</td>
<td>4.2602</td>
<td>4.2568</td>
</tr>
<tr>
<td>4</td>
<td>5.6428</td>
<td>5.4751</td>
<td>5.5354</td>
<td>5.5249</td>
</tr>
<tr>
<td>5</td>
<td>6.0647</td>
<td>6.2565</td>
<td>6.3469</td>
<td>6.3292</td>
</tr>
<tr>
<td>6</td>
<td>8.6583</td>
<td>11.0804</td>
<td>10.9247</td>
<td>11.0242</td>
</tr>
<tr>
<td>7</td>
<td>10.5733</td>
<td>11.9593</td>
<td>12.0014</td>
<td>12.3539</td>
</tr>
<tr>
<td>8</td>
<td>11.2263</td>
<td>13.4511</td>
<td>13.3539</td>
<td>13.2386</td>
</tr>
<tr>
<td>10</td>
<td>18.2857</td>
<td>18.2857</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE 9  Cantilever square plate, frequencies of vibration
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>720.0</td>
<td>2.6858</td>
<td>8.4889</td>
<td>9.9757</td>
</tr>
<tr>
<td>360.0</td>
<td>2.6636</td>
<td>8.3921</td>
<td>8.7779</td>
</tr>
<tr>
<td>72.0</td>
<td>2.5184</td>
<td>5.5358</td>
<td>8.0415</td>
</tr>
<tr>
<td>7.2</td>
<td>1.8534</td>
<td>2.7183</td>
<td>6.1116</td>
</tr>
<tr>
<td>0.72</td>
<td>1.0469</td>
<td>1.8389</td>
<td>4.6049</td>
</tr>
<tr>
<td>0.072</td>
<td>0.7405</td>
<td>1.7018</td>
<td>4.1457</td>
</tr>
<tr>
<td>0.0</td>
<td>0.6896</td>
<td>1.6850</td>
<td>4.1026</td>
</tr>
<tr>
<td>-0.00072</td>
<td>0.6890</td>
<td>1.6850</td>
<td>4.1026</td>
</tr>
<tr>
<td>-0.072</td>
<td>0.6319</td>
<td>1.6696</td>
<td>4.0265</td>
</tr>
<tr>
<td>-0.12</td>
<td>0.5889</td>
<td>1.6577</td>
<td>4.0050</td>
</tr>
<tr>
<td>-0.24</td>
<td>0.4561</td>
<td>1.6281</td>
<td>3.8715</td>
</tr>
<tr>
<td>-0.36</td>
<td>0.2397</td>
<td>1.6026</td>
<td>3.7382</td>
</tr>
<tr>
<td>-0.39</td>
<td>0.1333</td>
<td>1.6051</td>
<td>3.8399</td>
</tr>
<tr>
<td>-0.396</td>
<td>0.0986</td>
<td>1.6103</td>
<td>3.4687</td>
</tr>
</tbody>
</table>

**TABLE 10**
Fig. 2 - Flow chart for the eigenvalue solution
CIRCUNFERENCIAL PROFILE, Z = 11.5 m

Below: Longitudinal Profiles

\[ \theta = 45^\circ \quad \theta = 63^\circ \quad \theta = 81^\circ \quad \theta = 99^\circ \quad \theta = 117^\circ \quad \theta = 135^\circ \]

Fig. 4 - Cylindrical Shell
Fig. 5 - Cylindrical Shell
FIG. 6 PREDICTION OF THE LOWEST FREQUENCY (1,1) FOR A SQUARE, SIMPLY SUPPORTED PLATE BY VARIOUS ELEMENTS.

<table>
<thead>
<tr>
<th>No. Elem</th>
<th>D.O.F.</th>
<th>Error(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9</td>
<td>1.34</td>
</tr>
<tr>
<td>4</td>
<td>44</td>
<td>0.098</td>
</tr>
<tr>
<td>16</td>
<td>192</td>
<td>0.0098</td>
</tr>
</tbody>
</table>

(ANALYSIS OF 1/4)

FIG. 7 PREDICTIONS OF THE FREQUENCIES (1,1), (1,2) AND (2,2) WITH THE SEMILOOF ELEMENT (S.S. SQUARE PLATE).

NUMERICAL VALUES
<table>
<thead>
<tr>
<th>No. Elem</th>
<th>D.O.F.</th>
<th>Error(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>21.6</td>
</tr>
<tr>
<td>4</td>
<td>45</td>
<td>1.35</td>
</tr>
<tr>
<td>16</td>
<td>179</td>
<td>0.08</td>
</tr>
</tbody>
</table>

(NUMBER OF ELEM.)

NUMERICAL VALUES
<table>
<thead>
<tr>
<th>No. Elem</th>
<th>D.O.F.</th>
<th>Error(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9</td>
<td>-14.5</td>
</tr>
<tr>
<td>4</td>
<td>44</td>
<td>-1.37</td>
</tr>
<tr>
<td>16</td>
<td>192</td>
<td>-0.66</td>
</tr>
</tbody>
</table>

(NUMERICAL VALUES)

(ANALYSIS OF 1/4)
A - One element in the quadrant:

B - Four elements in the quadrant:

C - Sixteen elements in the quadrant:

Fig. 8 Eigenvector corresponding to the third eigenvalue (2,2) for a simply supported square plate.
A - One element in the quadrant:

B - Four elements in the quadrant:

C - Sixteen elements in the quadrant:

Fig. 9 Eigenvector corresponding to the first eigenvalue for a simply supported square plate.
A - One element in the quadrant:

B - Four elements in the quadrant:

C - Sixteen elements in the quadrant:

Fig. 10 Eigenvector corresponding to the second eigenvalue (1, 2) for a simply supported square plate.
FIG. 11 - SIMPLY SUPPORTED SQUARE PLATE
**EIGENVECTOR (3.1)**

**EIGENVECTOR (3.3)**

*Fig. 12  Simply Supported Square Plate*
Fig. 13  Simply supported square plate
A - EIGENVECTOR (5-2)

B - EIGENVECTOR (6.1)

C - EIGENVECTOR (5.4)

FIG. 14  SIMPLY SUPPORTED SQUARE PLATE
A - SECOND MODE

- NODAL POINTS
- NODAL LINES
- DEFORMED SHAPE

B - FOURTH MODE

Figure 15
SIXTH VIBRATING MODE

- NODAL POINTS
--- NODAL LINES
--- DEFORMED SHAPE

THEORETICAL, REF. 5-1

FIGURE 16
A - FIRST VIBRATING MODE

□ - NODAL POINTS

---- NODAL LINES

--- DEFORMED SHAPE

B - SECOND VIBRATING MODE

THEORETICAL SECOND MODE, REF. S-1

FIGURE 17
A - FIFTH VIBRATING MODE

- NODAL POINTS
- NODAL LINES
- DEFORMED SHAPE

THEORETICAL FIFTH MODE
REF. S-1

B - TENTH VIBRATING MODE

FIGURE 18
A - CONVERG. STUDY

-8
-6
-4
-2
2
4
6
8

(1,0)

(0,0)

ERROR (%)

(0,2)

3

16

27

NUMBER OF ELEMENTS

S - NUMBER OF NODAL CIRCLES
N - NUMBER OF NODAL DIAMETERS

B - NUMERICAL VALUES

<table>
<thead>
<tr>
<th>EIGENVALUES</th>
<th>THEOR.</th>
<th>3 ELEM.</th>
<th>16 ELEM.</th>
<th>27 ELEM.</th>
</tr>
</thead>
<tbody>
<tr>
<td>FIRST</td>
<td>VALUE</td>
<td>ERROR</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0,0)</td>
<td>8.461</td>
<td>-0.99%</td>
<td>8.433</td>
<td>8.455</td>
</tr>
<tr>
<td>SECOND</td>
<td>VALUE</td>
<td>ERROR</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(0.2)</td>
<td>28.873</td>
<td>1.18</td>
<td>28.771</td>
<td>28.814</td>
</tr>
<tr>
<td>THIRD</td>
<td>VALUE</td>
<td>ERROR</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1.0)</td>
<td>32.967</td>
<td>7.35</td>
<td>32.300</td>
<td>32.435</td>
</tr>
</tbody>
</table>

FIGURE 19
A - SECOND EIGENVECTOR, (0, 2)

□ - NODAL POINTS
--- NODAL LINES

B - FIRST EIGENVECTOR, (0, 0)

C - THIRD EIGENVECTOR, (1, 0)

FIGURE 20
CHAPTER IV-EIGENVALUE ANALYSIS

MATERIAL PROPERTIES:

\[ E = 30 \times 10^6 \text{ lb/} \text{in}^2 \]

\[ \nu = 0.3 \]

\[ \rho = 7.35 \times 10^{-4} \text{lb.sec}^2/\text{in}^4 \]

(DIMENSIONS IN INCHES)

**FIG. 21 CYLINDRICAL CANTILEVER SHELL.**

**FIG. 22 PREDICTIONS OF THE FIRST EIGENVALUE FOR THE SHELL OF FIGURE 21 USING DIFFERENT FINITE ELEMENTS.**
A - THIRD VIBRATING MODE

B - EIGHTH VIBRATING MODE

FIGURE 23
FIG. 24 FIRST EIGENVECTOR FOR THE SHELL OF FIG. 21

FIG. 25 NODAL LINES FOR THE FIFTH EIGENVECTOR FOR THE SHELL OF FIG. 21
A - FOURTH MODE

B - SIXTH MODE

Q - NODAL POINTS
--- NODAL LINES
--- DEFORMED SHAPE

FIGURE 26
A - FIRST MODE

B - SECOND MODE

C - THIRD MODE

THEOR., REF. B-4

THEOR., REF. B-4

45°

(THEOR. NOT KNOWN)

FIG. 27 - SKEW CANTILEVER PLATE
A - CONVERGENCE STUDY

B - NUMERICAL VALUES

<table>
<thead>
<tr>
<th>EIGENVALUES</th>
<th>THEOR.</th>
<th>2x2 ELEM.</th>
<th>3x3 ELEM.</th>
<th>4x4 ELEM.</th>
</tr>
</thead>
<tbody>
<tr>
<td>FIRST { \begin{align*} \text{VALUE} &amp; : 0.7239 \ \text{ERROR} &amp; : -0.7896 - 4.7% \end{align*} }</td>
<td>0.6896 - 4.7%</td>
<td>0.7052 - 2.6</td>
<td>0.7110 - 1.8</td>
<td></td>
</tr>
<tr>
<td>SECOND</td>
<td>1.7708</td>
<td>1.6850 - 4.8</td>
<td>1.7270 - 2.5</td>
<td>1.7410 - 1.7</td>
</tr>
<tr>
<td>THIRD</td>
<td>4.4421</td>
<td>4.1026 - 7.6</td>
<td>4.1781 - 5.9</td>
<td>4.2602 - 4.1</td>
</tr>
<tr>
<td>FOURTH</td>
<td>5.6893</td>
<td>5.6428 - 0.8</td>
<td>5.4751 - 3.8</td>
<td>5.5354 - 2.7</td>
</tr>
</tbody>
</table>

FIGURE 28
A - FIRST VIBRATING MODE

B - SECOND MODE

FIGURE 29
A - Third Mode

B - Fourth Mode

FIGURE 30
Fifth Vibrating Mode

0 - Nodal Points

--- Deformed Shape

Figure 31

Figure 32

Theor. Nodal Lines
**FIGURE 33** - STRESSED CANTILEVER PLATE
CHAPTER V - GEOMETRICALLY NONLINEAR ANALYSIS

CHAPTER V

GEOMETRICALLY NONLINEAR ANALYSIS

1. INTRODUCTION

The linear character of the analysis presented in the second and third chapters of this work resulted from the fact that:

a) The relation defining the strains in terms of the displacements was assumed to be linear

b) The relation defining the stresses in terms of strains was assumed to be linear.

This chapter is concerned with the nonlinearity resulting from the non-occurrence of condition a). This is usually referred to as "geometric nonlinearity".

The need for consideration of this type of nonlinearity is particularly relevant for shell structures and slender structures in general. In fact, while, for simple and massive structures, the linear methods of analysis have generally yielded accurate enough results, slender structures require more sophisticated methods of analysis if their actual behaviour is to be properly represented. Structures of this type are used with increasing frequency in all fields of engineering, and it is not surprising then that during the last 15 or 20 years great developments have taken place in the field of analysis of geometrically nonlinear structures. These developments have been closely related to the parallel development of the finite element method, without which it would be practically impossible to solve nonlinear problems of practical significance. However, in spite of the large number of papers published on nonlinear mechanics, this subject is still in its infancy (0-1).
The work reported in this chapter refers to the application of the Semiloof shell element to the solution of geometrically nonlinear plate and shell problems, a finite element computer program having been developed for this purpose. A brief presentation of the methods of analysis is first given, the method of analysis used in this work being then presented in more detail. Some results for typical plate and shell problems are shown and compared with other known solutions.

2. METHODS OF ANALYSIS

The geometrically nonlinear analysis of plates and shells can be treated in two essentially different ways. In the first one a nonlinear problem is mathematically formulated and then solved numerically; in the second one the nonlinear problem is solved by successive corrections to a classical linearized problem.

The solution of a shell problem by the first method requires, naturally, that a nonlinear shell theory be used. Reference (S-2) presents the derivation of an exact theory for the large deflection of a thin arbitrary shell and then, making some simplifications, several approximate theories are derived. Strain-displacement relations are given, among others, for a small strains approximation, small strains and moderately small rotations, as well as for the well known Marguerre's shallow shell theory.

Any of these theories can be formulated in terms of finite elements and various results are shown in the literature using this type of approach. Geometrically nonlinear analysis of plates based on the Von Karman equations are presented in references (W-1), (T-3) and (B-1). This last reference presents also results for shallow shells using the Marguerre equations. Further results for shallow shells are shown in references (C-1), (D-1) and (S-1) amongst others.
The type of approach presented is usually based on a Lagrangian formulation. Stresses and strains are therefore referred to the original configuration of the structure. For the case of the Von Karman equations for plates the finite element formulation yields the following incremental equations (W-1)

\[ d(P) = [K_E + K_G + K_D] \, d(\delta) \]  

(1)

where \( P \) and \( \delta \) represent vectors of nodal forces and nodal displacements respectively. The stiffness matrix results from the superposition of 3 stiffness matrices:

- The usual infinitesimal stiffness matrix, \( K_E \)
- The geometric or initial stress matrix, \( K_G \)
- The initial displacement matrix, \( K_D \)

A salient feature of this formulation is that the non-linearity depends, through \( K_D \), on the initial displacements of the structure.

In the second method of analysis, the external loads are changed in each step of the solution and the coordinate system "continuously" updated. Depending on the basic simplifications and assumptions, different types of formulations (some actually including nonlinear terms) can be found in the literature connected with this type of approach. Because a local coordinate system is used, no initial displacement matrix is used in this formulation.

The incremental method of analysis developed in this work falls into this last category. This method possesses the advantage of being easily implemented for the particular case of the Semiloof shell element. This results from the fact that the Semiloof shell element is a general shell element, with the basic shape functions and stresses being referred to local systems of axes.
The next section deals with incremental methods in general and describes in detail the method of solution used.

3. THE INCREMENTAL METHOD

3.1 General

The incremental method of analysis in continuum mechanics is related to the development of the theory of elasticity for a medium subject to a system of initial stresses. This theory is more involved than the classical theory of elasticity, and, in view of the various avenues open for its formulation, no consensus of opinion seems to have been reached by the various researchers in this field.

The first investigations in this theory date back to Cauchy (T-1) who derived a correct form of constitutive equations by superposing infinitesimal displacement gradients on large initial deformations (Y-1). Reference (C-1) presents a theory in which the superposed displacements themselves are infinitesimal.

One of the difficulties involved in the derivation of the theory of elasticity for a medium under initial stress, lies in the difficulty of separating the geometrical rotations from the actual physical rotations. In order to separate these two components reference (B-2) presents a theory in which the stresses are referred to axes whose orientation varies from point to point and depend on the local rotation of the material. Pure deformations and rotations are therefore separated, and the constitutive equations involve only pure deformations.

A brief account of the developments in the theory of elasticity for a medium subject to a system of initial stresses can be found in reference (T-1).
In the finite element literature connected with the application of the incremental method to geometrically nonlinear problems, different methods of approach can also be found. The original work in this field was presented by Turner et al in the classical paper (T-2). This paper described the concept of using linear incremental steps to approximate large deflection problems and showed that a new class of stiffness matrix had to be considered, this resulting from the superposition of the classical stiffness matrix, $K_E$, with an "initial stress" or "geometric" stiffness matrix, $K_C$, which is a function of the state of stress in the element. This reference presents also the derivatives of this additional stiffness matrix for axial force member, and a triangle in plane stress. Many other papers followed this one, whose description will not be attempted here. A survey of the work up to 1969 can be found in reference (M-2).

Most of the earlier papers on geometric nonlinearity were mainly based on a physical and intuitive approach to the problem. More recently however, some authors have presented more general formulations of the incremental equations. Ref. (Y-2) presents a formulation leading to equations with a linear and a nonlinear part and employs the linear part with a purely incremental method to solve some axisymmetric shell problems. This method seems particularly attractive because of the fact that no interactions are used to reach equilibrium. It is possible however, that the smaller increments that have to be used, make the method more uneconomical.

There are several ways of obtaining incremental formulations for geometrically nonlinear problems. The formulation presented here (see appendix) follows that presented in reference (M-3). More general formulations have been presented by Oden and by Hibbit et al in references (O-1) and (H-1). Much of the earlier work on the application of finite elements to geometrically nonlinear problems used the geometric matrix to
correct the linear stiffness matrix at the end of each successive load increment. As pointed out in (0-1) this procedure is incorrect unless a new material frame of reference is established in the deformed element at the end of each increment. This was done by Murray and Wilson (M-1) who considered one coordinate system connected with each element, this coordinate system being updated for the successive positions of the element. Rigorously however, the use of one coordinate system per element is only adequate for constant strain elements (H-1). For higher order elements because of the possibility of rotation within the element a local coordinate system must be considered at each material point. This means, in practice, that a local coordinate system must be considered at each integrating point.

The program developed here uses an incremental and iterative approach similar to the one of references (M-1,3) but considers one set of axes at each integrating point. The determination of the local axes at the integrating points was discussed in chapter III and they do not need to be considered explicitly in the formulation.

Reference (D-1) presents, by both Eulerian and Lagrangian approaches, an analysis of shell structures subject to large displacements and small strains. It concludes that the updating procedure is a combination of both approaches. In the examples given in this reference solutions obtained with a fully Lagrangian approach are compared with the ones yielded by an updating procedure. It can be seen that, for the case of an arch under a concentrated load, a better rate of convergence is obtained with the updating procedure, while the contrary is the case for a simply supported square plate subject to uniform pressure. This is attributed to the "bowing" effect of the elements when a small number of elements are used.
We note that no such difficulty was experienced here. In fact figure 6 shows that for this problem very similar solutions are obtained using meshes of 4 and 16 elements.

3.2 Stiffness matrices

The method of analysis employed in this work employs an incremental stiffness matrix resulting from the superposition of the following two matrices:

- The infinitesimal stiffness matrix \([K_E(x)]\) that depends on the element position as implied by considering it a function of \(x\)
- The geometric matrix \([K_G(\sigma, x)]\) that depends on the element position, as well as on the state of stress within the element.

The derivation of these two matrices for a current position of the element was discussed in chapters III and IV and will not be considered further.

The solution is incremental and iterative and is based on achieving an equilibrium between the applied forces and the equilibrating forces acting on each element. For a particular load increment \(\{\Delta P\}\) to which corresponds an increment of displacement \(\{\Delta \delta\}\) the following system of equations is solved for the whole structure:

\[
[K_E(x) + K_G(\sigma, x)]\{\Delta \delta\} = \{\Delta P\}
\] (2)

The incremental formulations presented in references (0-1) and (H-1) derive linear incremental stiffness matrices that depend on the state of stress in the element for the particular increment considered. This results from the fact that in the establishment of the constitutive equations the initial stresses are taken into account. In the plastic range, the slope of the stress strain curve can be of the same order of magnitude as the stress itself and so the additional terms appearing in
the constitutive equations, due to the stresses, must be taken into consideration. In the elastic range, however, the magnitude of these terms is very small when compared with the elastic modulus matrix \([D]\) and can be neglected (H-1), as is the case in most of the incremental large deformations solutions reported. The incremental linear stiffness matrix becomes therefore identical to the infinitesimal stiffness matrix \([K_E(x)]\). The geometric matrices of references (O-1) and (H-1) are also identical to the geometrical matrix \([K_G(\sigma, x)]\).

3.3 Method of solution

The method of solution used here is essentially a modified Newton-Raphson method. The load is applied in increments and within each increment iterations are performed until equilibrium is reached. Compared with a linear, purely incremental method, this method has the advantage that equilibrium is reached for each increment and the disadvantage that the iterations needed to achieve equilibrium make it more uneconomical. A detailed description of the algorithm as it was programmed is now presented.

(a) Consider each load increment until the total load required is reached.

For each load increment proceed as follows:

(b) Read for each particular increment:
   - The load factor
   - The tolerance allowed for the unbalanced forces
   - The maximum number of iterations allowed

(c) Iterate until the required accuracy is achieved or until the maximum number of iterations allowed is reached.

For each iteration perform the following calculations:
(d) Consider each element in the structure and define
   - The applied load vector
   - The stiffness matrix

(e) Assemble element stiffness matrices and load vectors for
    the total number of elements and reduce the system of equations

(f) Backsubstitute through the equations and determine
    - displacements
    - reactions

(g) Accumulate the displacements with the previous ones (if any)
    and calculate
    - The total displacements for the increment
    - The total displacements from the initial position

(h) Add calculated reactions to the load array

(i) Use the new element position and the incremental displacements
    to calculate successively
    - The "stress" matrix, \([B^T][D]\)
    - The corresponding "stresses"
    - The nodal forces corresponding to these "stresses"
      (equivalent nodal forces)

(j) Compare the applied load with the equivalent nodal forces
    just determined and check whether the required tolerance
    was reached

(k) Calculate unbalanced forces. These are equal to the
    difference between the applied forces and the equivalent
    nodal forces
(1) Load the structure with the unbalanced forces and solve
(as from d) until the difference between the applied forces
and the equivalent nodal forces is arbitrarily small
(as defined by the given tolerance)

(m) Consider a new load increment and repeat process (as
from b).

As a final point we note that the computer program written
for this chapter can consider a stiffness matrix resulting from the
superposition of \( K_E(x) \) with \( K_E(\sigma, x) \) as defined in (2) or, if desired,
only \( K_E(x) \) is considered. The reason for using only \( K_E(x) \), results
from the fact that, since the procedure is iterative and aimed at
achieving a balance of forces, it is not necessary to calculate "exactly"
the stiffness matrix \((M-1)\). The results obtained using each one of these
procedures are very similar. It is possible however that for problems
with higher membrane stresses significant differences can be observed.

3.4 Convergence criteria

As mentioned before the convergence criterion is based on
achieving equilibrium between the applied forces and the "equivalent"
nodal forces. For the definition of the criterion it will be assumed
that the applied nodal forces and the equivalent nodal forces are
defined respectively by the vectors \( \{P\} \) and \( \{F\} \), the dimension of these
vectors being equal to the number of degrees of freedom, \( n \), in the structure.

Defining the Euclidean norms of the unbalanced forces and
of the applied forces by respectively

\[
A = \left[ \sum_{I=1}^{n} (P(I) - F(I))^2 \right]^{\frac{1}{2}}
\]
and
\[ B = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (P(I))^2} \]

the norm of the residual sum ratio is given in percentage by

\[ C = 100.0 \frac{A}{B} \]

The maximum residual for each iteration as defined by

\[ R = \max_{I=1,n} (|P(I) - F(I)|) \]

was also calculated, but, in all the problems solved, the value of C was the one always used to decide whether to terminate or to pursue the iterative process. Typically the process was stopped as soon as C reached a value less than 1.0.

4. RESULTS

Some plate and shell problems were solved using the finite element program developed in this chapter.

Results for a clamped uniformly loaded square plate are shown in figures 2 to 4. The following material properties were used.

- Young's modulus, \( E = 3000.0 \)
- Poisson's ratio, \( v = 0.316 \)
- Plate thickness, \( t = 0.05 \)
- Side length, \( a = 4.0 \)

In order to study the convergence of the solution, three types of meshes (with 1, 4 and 16 elements in the quadrant) were used in this problem. Figure 2 shows plots of the load versus central deflection and load versus stresses at the centre for the 1 and 4 element meshes. The classical
solution by Levy (Ref. L-1) is also shown in this figure as well as a finite element solution from reference (W-1). This last solution is based on a Lagrangian approach to solve the nonlinear Von Karman equations. As it transpires from this figure reasonably accurate results are obtained even for these very coarse meshes. The convergence towards the Levy solution is also evident from this figure. A better comparison with this solution is obtained using a 16 elements mesh (figure 3). Figure 4 shows deflection profiles along the meridian for different partial loads. The linear deflection profile for full load is also incorporated in this figure, showing the considerable difference that exists between the linear and nonlinear solutions. Finally, table 1 gives the norm of the residual sum ratio (as defined in section 3.4) for the various iterations needed to achieve equilibrium. It can be seen that the convergence is more difficult for the first two increments for which six iterations were needed to obtain a residual less than 1%. For the last three increments, although they are 50% bigger than the initial ones, only 3 iterations were necessary.

In fig. 5 results are given for a simply supported square plate. This includes graphs for the load versus central deflection and load versus stress at the centre are plotted. The results shown were obtained using a 16 finite element mesh and the same material properties as for the clamped plate. This figure includes as well the solution from reference (L-2). A fairly good agreement is observed between the two solutions, the differences being more pronounced for the stresses.

Figure 6 shows some deflection profiles obtained with the 16 element mesh, and compares them with the ones obtained with a mesh of 4 elements. From this figure a good agreement between these two sets of profiles is observed.

Regarding the convergence of solution, table 2, this is now slower than for the clamped square plate. Taking the first increment for instance, its size is approximately half the value used for the clamped plate, and, even so, for the same type of accuracy, almost twice as many iterations were
necessary. Table 2 shows the residuals for both the cases of 4 and 16 element meshes; it can be seen that these are practically equal. This fact can possibly be used to estimate the necessary size of the increments for a particular problem by solving first the same problem with a very coarse mesh.

Results for a clamped uniformly loaded circular plate are shown in figure 7. The following material properties were considered:

\[
\begin{align*}
E &= 3000.0 \\
v &= 0.3 \\
t &= 0.05 \\
\text{radius, } a/2 &= 1.0
\end{align*}
\]

A mesh with 3 elements, as shown in figure 5, was used. This figure includes plots of the load versus central deflection, load versus stress at the centre and load versus stress at the edge. Results are also presented from references (W-1) and (W-2). These are shown in full line in the figure and were obtained using the Ritz method. In spite of the coarse mesh used relatively accurate results were obtained for both displacements and stresses in the present analysis.

Finally figure 8 gives the results obtained for a clamped cylindrical shell with the following material properties:

\[
\begin{align*}
E &= 450000.0 \\
v &= 0.3 \\
t &= 0.125 \\
\text{radius, } r &= 100.0 \\
\text{angle, } \theta &= 0.2 \text{ rad} \\
\text{Length, } \ell &= 20.0
\end{align*}
\]
Results from other references (M-4), (C-1), (G-2), (S-1) are included in this figure, and, as it can be seen, the results obtained with the semiloof shell element compare well with these solutions.

Solutions for two more shell problems obtained with the program for geometrically nonlinear analysis of shells will be reported in Chapter VII.

Having reported the results obtained for various structures, the computer times needed to solve a typical problem on a CDC 7600 computer are now given. These are for the clamped square plate as follows:

<table>
<thead>
<tr>
<th>Elements</th>
<th>1 element</th>
<th>4 elements</th>
<th>16 elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>8 seconds</td>
<td>28 seconds</td>
<td>110 seconds</td>
</tr>
</tbody>
</table>

It is seen from Table 1 that the results for this problem were obtained to within an accuracy of less than 0.1%. If results were sought to within an accuracy of 1% the computer times would be reduced (see Table 1) to approximately 70% of the above values, becoming then

<table>
<thead>
<tr>
<th>Elements</th>
<th>1 element</th>
<th>4 elements</th>
<th>16 elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>6 seconds</td>
<td>20 seconds</td>
<td>77 seconds</td>
</tr>
</tbody>
</table>

As a final point we note that in view of the different approaches, different formulations and different methods of solution that can be found in the literature connected with the geometrically nonlinear analysis of shells, an "absolute" identity of solutions would not be possible, as it is often impossible to say which is the right solution for a particular problem. We note for instance that reference (C-1) considers the solution for the clamped uniformly loaded square plate presented in reference (N-3) as more accurate than the classical solution by Levy. That solution gives slightly higher values for the centre deflection as are in fact also given by the solution presented in this work.
Perhaps we can conclude by stating, as in reference (C-2),
that to say that a problem is geometrically nonlinear, is just to say what
it is not, and not what it is.

5. CONCLUSIONS

The Semiloof shell element has been applied to the solution
of various plate and shell problems. For these problems a good agreement
was observed between the solutions yielded by the Semiloof element and
other known solutions.

The computer program developed uses an incremental and
iterative approach that proved to be particularly convenient for the case
of the semiloof shell element. This results from the particularities of
the element formulation, mainly from the fact that shape functions and
stresses are referred to local systems of axes.

Although more expensive than a purely incremental method the
method developed has the advantage of being easily implemented.
APPENDIX

INCREMENTAL EQUILIBRIUM EQUATIONS

Consider a deformable body in its path of deformation from an initial configuration A to a final configuration B (figure 1). This deformation is referred to a global coordinate system $x_i$ and A represents an equilibrium intermediate configuration in which the body has volume $V_0$ and area $A_0$ and is subject to a system of forces $P$. Configuration B is supposed to be as close to A as desired and in equilibrium under a system of loads $P + \Delta P$.

For configuration A the displacement of the current point of the body, referred to the global system $x_i$ will be represented by $\bar{u}_i$ and for configuration B by $\bar{u}_i + u_i$. If a local coordinate system $X_i$, attached to the body, is used, these displacements will be represented respectively by $\bar{U}_i$ and $\bar{U}_i + U_i$. Although no restrictions are imposed on the displacement gradients it is assumed that the engineering strains remain small during the deformation. In these circumstances the product of an increment in Green's strain tensor $(\Delta E_{ij})$ and Kirchhoff's stress tensor $(S_{ij})$ represents work (M=3).

For configurations A and B, respectively, the strains are defined in terms of displacements by

$$2E_{ij} = \bar{u}_{ij} + \bar{u}_{ji} + \bar{u}_{ik} \bar{u}_{kj}$$  \hspace{1cm} (1)

and

$$2(E_{ij} + \Delta E_{ij}) = (\bar{u}_{ij} + u_{ij})_j + (\bar{u}_{ji} + u_{ji})_i +$$

$$ + (\bar{u}_{ik} + u_{ik}) \bar{u}_{kj} + (\bar{u}_{kj} + u_{kj}) \bar{u}_{ik}$$  \hspace{1cm} (2)

The stresses for configurations A and B will be defined respectively by

...
\[ S_{ij} = C_{ijk \ell} E_{k \ell} \] (3)

and

\[ S_{ij} + \Delta S_{ij} = C_{ijk \ell} (E_{k \ell} + \Delta E_{k \ell}) \] (4)

where \( C_{ijk \ell} \) is assumed to be constant.

The increment of strain can be calculated by subtracting equation (1) from equation (2). This results in:

\[ \Delta E_{ij} = \tfrac{1}{2} (u_{i,j} + u_{j,i} + u_k,i + u_k,j + u_k,i + u_k,j + u_{k,i} + u_{k,j}) \] (5)

Regarding the displacements \( u_i \) as variables and applying the principle of virtual displacements to configuration B the following expression is obtained:

\[ \int_{V_o} (S_{ij} + \Delta S_{ij}) \delta(\Delta E_{ij}) \, dV_o = \int_{S_o} (T_i + \Delta T_i) \delta u_i \, dS_o \] (6)

where \( T_i \) is the Kirchoff stress vector acting on the body surface.

In configuration A a similar expression holds:

\[ \int_{V_o} S_{ij} \delta(\Delta E_{ij}) \, dV_o = \int_{S_o} T_i \delta u_i \, dS_o \] (7)

The incremental equilibrium equation can now be obtained (B-2) by taking the difference between the two equilibrium equations (6) and (7):

\[ \frac{1}{2} \int_{V_o} S_{ij} \delta(u_{k,i} + u_{k,j}) \, dV_o + \int_{V_o} \Delta S_{ij} \delta(\Delta E_{ij}) \, dV_o = \int_{S_o} \Delta T_i \delta u_i \, dS_o \] (8)
These integrals can be evaluated in the usual manner of the finite element method: by subdividing the structure into finite elements and "summing" the integrals corresponding to each element. At the same time, for small engineering strains, equations (1) to (5) can be rewritten in terms of displacements referred to the local coordinate system. If \( \bar{u}_i \) and \( u_i \) are replaced respectively by \( \bar{U}_i \) and \( U_i \) expressions (8) and (5) become

\[
\sum \frac{1}{2} \int_{V_o} S_{ij} \delta(u_{k,i} u_{k,j}) \, dv_o + \sum \int_{V_o} \Delta S_{ij} \delta(\Delta E_{ij}) \, dv_o = \\
= \sum \int_{S_o} \Delta T_i \delta u_i \, ds_o \tag{9}
\]

and

\[
\Delta E_{ij} = \frac{1}{2} \{ U_{i,j} + U_{j,i} + U_{k,i} U_{k,j} + \bar{U}_{k,i} U_{k,j} \} + U_{k,i} + U_{k,j} + \bar{U}_{k,i} + \bar{U}_{k,j} \}
\tag{10}
\]

In this expression the product terms may be reduced to negligible quantities by using a sufficiently fine finite element mesh. Therefore the first term of equation (9) yields only the geometric matrix and the second term the classical infinitesimal stiffness matrix. The term on the right hand side represents the work done by the increments of generalized external loading, \( \Delta P \), on virtual variations of displacement increments.

The actual formulation of the element matrices for the Semiloof shell element can be found in chapters III and IV.
BIBLIOGRAPHY

B-1 Bergan, P. G., Clough, R. W.
Large deflection analysis of plates and shallow shells using the
finite element method.

B-2 Biot, M. A.
Mechanics of incremental deformations

C-1 Brebbia, C., Connor, J.
Geometrically nonlinear finite element analysis.

C-2 Cook, R. D.
Concepts and applications of finite element analysis

D-1 Dupuis, G. A., Hibbitt, H. D., McNamara, S. F., Marçal, P. V.
Nonlinear material and geometric behaviour of shell structures

G-1 Green, A. E., Rivlin, R. S., Shield, R. T.
General theory of small elastic deformations superimposed on
finite elastic deformations

G-2 Gallagher, R. H.
The finite element method in shell stability analysis

H-1 Hibbitt, H. D., Marçal, P. V., Rice, J. R.
A finite element formulation for problems of large strain and
large displacement

L-1 Levy, S.
Square plate with clamped edges under normal pressure producing
large deflections
National advisory committee for aeronautics, Technical Note No. 847.

L-2 Levy, S.
Bending of rectangular plates with large deflections
National advisory committee for aeronautics, Technical Note No. 846.

M-1 Murray, D. W., Wilson, E. L.
Finite element large deflection analysis of plates
Proceedings of the ASCE, Journal of the engineering mechanics
M-2 Martin, H. C.
Finite elements on the analysis of geometrically nonlinear problems
In: Recent advances in matrix methods of structural mechanics
and design, Tokyo, 1969.

M-3 Murray, D. W., Wilson, E. L.
Finite element post-buckling analysis of thin elastic plates

M-4 Morim, N.
Nonlinear analysis of thin shells

O-1 Oden, J. T.
Finite elements of nonlinear continua

S-1 Sabir, A. B., Lock, A. C.
The application of finite elements to the large deflection
geometrically nonlinear behaviour of cylindrical shells
Conference on Variational Methods, University of Southampton, 1973.

S-2 Sanders, J. L.
Nonlinear theories for thin shells

T-1 Truesdell, C.
The mechanical foundations of elasticity and fluid dynamics

T-2 Turner, M. J., Dill, E. H., Martin, H. C., Melosh, R. J.
Large deflections of structures subjected to heating and external
load
Journal of the aero/space sciences, 1960, p. 97, 106.

T-3 Tadahiko Kawai, Yoshimura, N.
Analysis of large deflection of plates by the finite element method
International Journal for Numerical Methods in Engineering,

W-1 Wood, R. D.
The application of finite element methods to geometrically
nonlinear structural analysis

W-2 Weil, N. A., Newmark, N. H.
Large deflections of elliptical plates

*3 Wei-Zang, C., Yek Kai-Yuan
On the large deflection of rectangular plates
Publication of the international association of bridge and
structural engineering, 1957.

Y-1 Yaghmai, S.
Incremental analysis of large deformations in mechanics of solids
with applications to axisymmetric shells of revolution
Ph.D. dissertation, Department of Civil Engineering, University
of California, 1968.
Y-2 Yaghmai, S., Popov, E. P.
Incremental analysis of large deflections of shells of revolution
Figure 1
Fig. 2 Stresses and displacements at the centre of a clamped square plate (uniform load)
Fig. 3 Stresses and displacements for an uniformly loaded square plate with clamped edges.
Fig. 4 Deflection profiles along AB for a clamped square plate (uniform load)

Table 1: Norm of residual sum ratio (clamped square plate uniformly loaded, 16 elements)
Figure 5 Stresses and displacements at the centre of a simply supported, uniformly loaded square plate.
<table>
<thead>
<tr>
<th>LOAD</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>12.288</td>
<td>46.19</td>
<td>74.6</td>
<td>72.2</td>
<td>29.2</td>
<td>20.2</td>
<td>10.5</td>
<td>6.4</td>
<td>3.6</td>
<td>2.1</td>
<td>1.2</td>
<td>0.7</td>
</tr>
<tr>
<td>24.596</td>
<td>52.6</td>
<td>74.0</td>
<td>71.3</td>
<td>28.8</td>
<td>20.0</td>
<td>10.3</td>
<td>6.3</td>
<td>3.5</td>
<td>2.1</td>
<td>1.2</td>
<td>0.7</td>
</tr>
<tr>
<td>45.056</td>
<td>172.7</td>
<td>39.8</td>
<td>19.4</td>
<td>7.6</td>
<td>3.3</td>
<td>1.4</td>
<td>0.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>81.920</td>
<td>172.3</td>
<td>39.5</td>
<td>19.2</td>
<td>7.5</td>
<td>3.2</td>
<td>1.3</td>
<td>0.6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>143.36</td>
<td>121.4</td>
<td>34.8</td>
<td>15.3</td>
<td>6.1</td>
<td>2.6</td>
<td>1.1</td>
<td>0.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>204.80</td>
<td>121.0</td>
<td>34.6</td>
<td>15.1</td>
<td>6.0</td>
<td>2.5</td>
<td>1.0</td>
<td>0.4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>286.72</td>
<td>105.9</td>
<td>33.0</td>
<td>13.5</td>
<td>5.3</td>
<td>2.2</td>
<td>0.9</td>
<td>0.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>46.6</td>
<td>105.0</td>
<td>32.7</td>
<td>13.4</td>
<td>5.2</td>
<td>2.1</td>
<td>0.8</td>
<td>0.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>43.9</td>
<td>90.8</td>
<td>28.3</td>
<td>11.8</td>
<td>2.8</td>
<td>0.7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>49.1</td>
<td>49.1</td>
<td>28.3</td>
<td>11.8</td>
<td>2.8</td>
<td>0.7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9.8</td>
<td>10.2</td>
<td>2.3</td>
<td>2.3</td>
<td>0.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>23.3</td>
<td>46.6</td>
<td>4.3</td>
<td>2.3</td>
<td>0.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 6  Simply supported uniformly loaded square plate.  

Deflection profiles along AB.
Figure 7 - Stresses and displacements for an uniformly loaded circular plate \((v = 0.3)\), with clamped edges.
SHELL THICKNESS: 0.125 in.
\[ \theta = 0.1 \text{ rad.} \]
\[ E = 450000.0 \text{ PSI} \]
\[ \nu = 0.3 \]

**Fig. 8 - Cylindrical Shell**
1. INTRODUCTION

It is a well known fact that the design of structures based on linear theories can lead to structures with a considerable reserve of strength. Although no general conclusions can be made when comparing a linear analysis with a geometrically non linear one (see Chapters V and VII), an elasto-plastic analysis, when compared with a linear analysis leads usually to more economical structures. It is therefore natural that considerable effort had been directed towards the establishment of a theoretical basis to elasto-plastic analysis, together with the development of numerical techniques capable of dealing with problems of practical significance.

Small deformations elasto-plastic theory was essentially complete by the end of the fifth decade of this century (B-1). By this time, however, only solutions for a limited number of problems were available. The advent of the electronic computer and the parallel development of the finite element method completely changed this picture, allowing solutions to be obtained for a much wider range of structures.

The initial effort concerned with the application of finite element techniques to elasto-plastic problems was mainly devoted to two dimensional (M-2,3, P-1, Z-1, Y-1) and three dimensional problems (A-1). The analysis of plates with both membrane and bending stresses is reported in references (A-2,3, B-1), shells of revolution in reference (K-1) and more recent papers refer to the application of finite element elasto-plastic techniques to arbitrary shells (D-1), (B-2). Some of these solutions include both material and geometrical non linearities.
In this chapter the semiloof plate and shell elements are applied to the elasto-plastic analysis of plates and arbitrary shells. Two computer programs were developed for this purpose. The first one uses the semiloof plate element and, as such, only plates in pure bending can be treated. The second one includes the semiloof shell element, thereby allowing the treatment of general shells, and, in particular, plates subjected to both membrane and bending stresses. This second program may employ the usual stiffness matrix for the semiloof shell element or also take into account the effect of the geometric matrix on the stiffness of the element.

It is intended in this chapter to give an overall view of the problem of plasticity and therefore we commence with the consideration of yielding conditions in general, and for shell structures in particular. Having defined the limit of the field of work, attention is then given to the definition of the stress-strain relations for the plastic domain. The methods of solution are then considered and the results obtained for both plates and shells presented and discussed.

2. YIELDING CONDITIONS

2.1 General

In the elasto-plastic analysis of a structure two fundamental problems arise. The first one concerns defining when a certain point of the structure has reached the plastic state. For some simple cases of straining a quick answer to this problem can be obtained experimentally. For more complicated cases of two and three dimensional straining and, in particular, for the case of shell structures, this represents a problem of considerable complexity. The second problem which has to
be considered, is the definition of the behaviour of the structure once that the plastic state is reached in whole or part of the structure. In this section only the first problem will be considered.

The definition of the elasto-plastic boundary must certainly be based on the state of stress at the particular point under consideration. If an isotropic body is considered, this boundary can be defined as a function of the principal stresses

$$F(\sigma_1, \sigma_2, \sigma_3) = 0$$

If it is assumed that no plastic deformations are induced by states of isotropic straining, the yield condition (1) can be rewritten as

$$F(T_2, T_3) = 0$$

where $T_2$ and $T_3$ are the second and third invariants of the deviatoric stresses (A-4).

In the space $(\sigma_1, \sigma_2, \sigma_3)$, Westergaard space, (1) represents a surface, termed the yield surface, which refer to points which are on the point of yielding or have previously become plastic. In this space, line Ob, making equal angles with axes $O\sigma_1$, $O\sigma_2$ and $O\sigma_3$, Figure 1, represents states of isotropic deformation. States of stress with the same deviatoric stresses are represented by straight lines parallel to Ob. The yielding surface is therefore a cylindrical surface with generators parallel to Ob, and, as such, is completely defined by its directrix. A plane (plane $\pi$) normal to Ob and passing through the origin of the coordinates defines by intersection with the yielding surface a yielding curve (Meldahl representation). This yielding curve must obey
the following conditions:

- if the material is isotropic the yielding curve must
  intersect the three axes \( \sigma_1, \sigma_2, \sigma_3 \) at the same distance
  from the origin.

- if \( E_{\text{tension}} = E_{\text{compression}} \) is constant, the yielding curve
  must be symmetric in relation to the coordinate planes.

This implies that 0 must be the centre of the yielding curve and that
this must lie between two hexagons similar to \( S_1 S_3 S_2 S_1' S_3' S_2' \) and
\( P_1 P_3 P_2 P_1' P_3' P_2' \) of Figure 2 (A-4).

The definition of the actual yield conditions must be established,
experimentally by studying states of stress included in one of the 12
segments shown on this figure.

A considerable number of yield conditions, some of them only of
historical interest, are referred to in the literature (A-4, H-1).
For ductile materials, however, two criteria are usually accepted as
representing the actual behaviour of the material. These are the
Tresca and the Von Mises criteria.

Note: Reference (H-3) describes some tests made on soft brass that
seem to indicate that the yielding surface has a local bulge disrupting
the smooth cylindrical pattern referred to above. However, more recent
work (F-1) using more accurate devices discards this local bulging
hypothesis and concludes that hydrostatic pressure has little or no
effect on the yield behaviour of the material. The yield surface can
therefore, as described above, be represented by a cylindrical surface
in three dimensional stress space.
2.2 Tresca and Von Mises Conditions

According to Tresca's yielding condition, plastic deformation occurs whenever the maximum tangential stress reaches a certain value. If \( \sigma_\xi \) and \( \sigma_\eta \) represent principal stresses this condition can therefore be defined by six relationships of the type:

\[
\sigma_\xi - \sigma_\eta = \text{constant}
\]  

(3)

Each one representing a plane in the Westergaard space. The corresponding yield curve can be represented by the hexagon \( S_1 S_3' S_2 S_1' S_3 S_2' \) of Figure 2. For two dimensional problems this yielding surface is represented by two lines parallel to the \( \sigma \)-axes.

There is experimental evidence that this yielding condition is not applicable to brittle materials. In fact, the Lüders lines (where its determination is possible) show a different orientation for these materials than the one predicted by the Tresca law. From the point of view of applications, it very often leads to complex mathematical formulations (A-4).

The Von Mises yielding condition is expressed by

\[
-T_2 - q^2 = 0
\]

(4)

It represents a cylinder in the Westergaard space, the yielding curve being a circumference with radius \( \sqrt{2}q \).

If the constant of expression (3) and \( q \) in expression (4) are suitably chosen the criteria of Tresca and Von Mises can be made to agree with each other and with experience for a particular state of stress that can be chosen arbitrarily. Usually the circle is made to pass through the corners of the hexagon, by taking the constant in (3)
as the yield stress in simple tension. The maximum difference between
the two criteria is therefore of approximately 15%.

Although the Von Mises yield condition has a purely mathematical
character, some physical interpretations are possible. According to
Henky this criteria reflects the fact that the yielding begins when the
elastic energy of distortion reaches a critical value. On the other hand,
Nadai proved that yielding begins with the Von Mises criterion when the
shear stresses acting over the octahedral planes reaches a certain value.

Whatever physical interpretation is adopted, this yielding
condition has the advantage of simplifying the calculations (A-4).
Also, for most metals, the Von Mises law fits the data more closely than
Tresca's (H-1). In the next section some yield surfaces for shells
composed of material obeying the Von Mises yield criterion will be
considered.

2.3 Yield Surfaces for Shells

In the case of a shell structure both membrane and bending stresses
are generally present and it is not a simple matter, even after choosing
a yield criteria for the material, to decide what combination of stresses
will originate yielding. At the same time the yield surface no longer
exhibits the simple character that it does in the \((\sigma_1, \sigma_2)\) space, once
that four dimensions must now be used. The little experimental evidence
available also makes a reasonable choice of a yield surface for this
type of structure difficult.

A derivation of a yield surface for a shell can be made either
by attempting an exact solution according to a particular yield criteria
or, on the other hand, some simpler models and simplifications can be introduced. The complete formulation leads usually to great mathematical difficulties making the formulation impracticable for the solution of problems of practical significance. The fact that the material yield condition used (Von Mises condition e.g.) already represents an approximation possibly does not justify the difficulties experienced in obtaining exact formulations. For approximate yield conditions, recourse is made to some bounding theorems providing upper and lower bounds on the yield point load. An alternative approach consists of considering an ideal shell with modified material, so that the approximate yield condition for the real shell becomes the exact yield condition for the ideal shell (H-2).

The yield conditions for shells are usually presented in terms of the following stress resultants:

\[ \begin{align*}
N_x &= \sigma^m x t; \\
N_y &= \sigma^m y t; \\
N_{xy} &= \sigma^m xy t \\
M_x &= \sigma^b x \frac{t^2}{4}; \\
M_y &= \sigma^b y \frac{t^2}{4}; \\
M_{xy} &= \sigma^b xy \frac{t^2}{4}
\end{align*} \tag{5} \]

where \( t \) is the shell thickness and \( m \) and \( b \) refer to membrane and bending stresses respectively.

The derivation of yielding conditions for shells is the subject of various papers (N-2, C-1,2) but the first investigation in this field appears to have been carried out by Ilyushim in 1948. Reference (R-1) gives a short description of this work and compares a number of yielding conditions from various sources. Some results from this reference will now be presented.
The following dimensionless stress resultant and bending moments are first defined:

\[ n_x = \frac{N_x}{N_0} \quad n_y = \frac{N_y}{N_0} \quad m_{xy} = \frac{N_{xy}}{N_0} \]
\[ m_x = \frac{M_x}{M_0} \quad m_y = \frac{M_y}{M_0} \quad m_{xy} = \frac{M_{xy}}{M_0} \]

(7) \hspace{1cm} (8)

where \( N_0 = \sigma_0 t \) is the uniaxial yield force and \( M_0 = \frac{1}{4} \sigma_0 t^2 \) is the uniaxial yield moment.

Considering the following quadratic forms:

\[ Q_t = n_x^2 + n_y^2 - n_x n_y + 3 n_{xy}^2 \]
\[ Q_m = m_x^2 + m_y^2 - m_x m_y + 3 m_{xy}^2 \]
\[ Q_{tm} = n_x m_x - \frac{1}{2} n_x m_y - \frac{1}{2} n_y m_x + n_y m_y + 3 n_{xy} m_{xy} \]

(9) \hspace{1cm} (10) \hspace{1cm} (11)

The yield conditions are given as functions of this form:

\[ F(Q_t, Q_m, Q_{tm}) = 0 \]

(12)

In Ilyushim's work two yield surfaces are considered:

\[ Q_t + Q_m = 1 \]
\[ Q_t + Q_m + |Q_{tm}|/\sqrt{3} = 1 \]

(Y1) \hspace{1cm} (Y2)

The yielding surface for sandwich shells presented in reference (N-2) and the yield surface of reference (M-4) are also considered. In the above notations these read, respectively,

\[ Q_t + Q_m + 2 |Q_{tm}| = 1 \]
\[ Q_m = (1 - Q_t)^2 \]

(Y3) \hspace{1cm} (Y4)

The yielding condition from reference (R-2) is

\[ Q_t + \frac{3}{4} Q_m = 1 \]

(Y5)
and is discarded in this study as it is 15% unconservative for a purely bending state and is, in one case, 22.5% unconservative.

Representing by $P_1$ the limit load for yielding condition $Y_1$, and using Schwartz inequality the yield surfaces can be related to one another:

$$P_3 \leq P_2 \leq 1.246 P_3$$  \hspace{1cm} (13)

$$P_2 \leq P_1 \leq 1.135 P_2$$  \hspace{1cm} (14)

$$P_3 \leq P_1 \leq 1.414 P_3$$  \hspace{1cm} (15)

They can also be related to the exact yield condition (with limit load $P_o$) the following bounds being indicated in reference (R-1):

$$0.955 P_o \leq P_1 \leq 1.155 P_o$$  \hspace{1cm} (16)

$$0.833 P_o \leq P_4 \leq P_o$$  \hspace{1cm} (17)

$$0.939 P_o \leq P_2 \leq 1.034 P_o$$  \hspace{1cm} (18)

$$0.8 P_o \leq P_3 \leq P_o$$  \hspace{1cm} (19)

As it transpires from these inequalities, the best bounds are given by yield surface $Y_2$, and, in view of the physical approximations made in the first place, searching for accuracy greater than that given by (2) may not be justified (R-1). When compared with $Y_1$ and $Y_4$ this yield surface has however the disadvantage (shared with $Y_3$) of containing two surfaces in the $(Q_t, Q_m, Q_{tn})$ space.

Yield surface ($Y_1$) has only one surface and its maximum lack of safety occurs only in the neighbourhood of a particular point of the original yield surface. According to reference (R-1) it will lead to reliable results in the vast majority of cases. A check could however be made for yield violation.
This can be made using a more accurate expression, similar to the one derived in reference (I-1):

\[
Q_t + \frac{1}{2} Q_m - \frac{0.25 (Q_t Q_m - Q_{tm}^2)}{Q_t + 0.48 Q_m} + \sqrt{0.25 Q_m^2 + Q_{tm}^2} = 1 \quad (Y6)
\]

for which

\[
0.999 P_o \leq P_6 \leq 1.0005 P_o \quad (20)
\]

2.4 Strain Hardening

Many materials exhibit an increase in yield stress when subject to continuing plastic deformation. This hardening phenomenon implies that the yielding surface changes continually with the development of the hardening. Various theories exist to interpret mathematically this phenomenon. The isotropic hardening theory assumes that during the plastic deformation the yielding surface expands uniformly from the original yielding surface, keeping however the same origin, shape and orientation. The loading surface can therefore be expressed by

\[
F(\sigma, \kappa) = 0 \quad (21)
\]

where \(\sigma\) lists the relevant stresses, and \(\kappa\) is a hardening parameter on which the position of the yield surface depends. Figure 3 illustrates, for a two dimensional problem, the yielding surfaces when the state of stress moves from point 1 to point 2. If the body is unloaded and then reloaded in the reverse direction, yielding will occur at point 3. This theory fails therefore to explain the Bauschinger effect. It is however commonly used as it leads to reliable results for conditions of monotonic loading (A-5).

The Bauschinger effect is taken into account in the kinematic hardening theory. This theory, due to Prager (P-2, P-3), assumes that
during plastic deformation the yield surface translates as a rigid body, maintaining however the original size, shape and orientation (Figure 4). An ideal Bauschinger effect for completely reversed loading conditions is now predicted. According to Prager, for a particular state of stress the increments of translation of the yield surface occur in the direction of the exterior normal to the surface. Other interpretations are however possible for the definition of the increment of translation (A-5).

The isotropic and kinematic hardening theories can be combined in a theory, retaining the essential features of both of them. The yield surface can be expressed by

\[ F(\sigma, \varepsilon_p, \kappa) = 0 \]  \hspace{1cm} (22)

where \( \varepsilon_p \) is the accumulated plastic strain (N-1).

3. PLASTICITY RELATIONS

3.1 General

When a body undergoes a deformation within the elastic range, there is a linear relationship between stresses and strains (Hooke's law). For plastic deformations however, the final state of the body depends on the path of deformation and no such unique relationship exists. Consequently the behaviour of a body in the plastic range has to be described by differential relations contrary to the elastic field where finite relations can be obtained.

Various theories exist to define the plastic behaviour of a body and in view of the usually accepted fact that no variation of volume occurs during a plastic deformation, all these theories postulate a
certain relationship between the plastic strains and the deviatoric stress components. According to the way in which the relation is established two kinds of theories have been presented by various researchers. These are the Flow Theories, due mainly to Reuss, Prandtl, Von Mises, Levy and Saint-Venant and the Deformation Theories due mainly to Hencky, Nadai and Ilyushin.

In Hencky's theory the components of the total plastic strain are considered to be proportional to the corresponding deviatoric stress components. This theory is more mathematically convenient for some problems but it can lead to unexpected conclusions: namely (H-1) that the plastic strain can be changed during unloading and reloading. This type of relations have however been used, particularly by Russian writers, for applications where the strains are small.

In Reuss's equations, the components of the incremental plastic strains are considered to be proportional to the deviatoric stress components. They are considered by Hill to be mathematically more consistent and physically more appropriate and are in fact, the only ones used throughout his monograph (H-1). The same view is taken in the text of Prager and Hodge (P-4). These incremental forms will also be used in this work and for this reason they will now be considered in detail.

3.2 Fundamental Assumptions

The plastic flow theory is based on the following four hypotheses:

a. The body is isotropic
b. The relative variation of volume is small and represents an elastic deformation proportional to the mean normal stress, that is

\[ \varepsilon = 3 \kappa \sigma \]  

or \[ d\varepsilon = 3 \kappa d\sigma \]  

where

\[ \varepsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z \]  \hspace{1cm} (25)  
\[ \sigma = \frac{1}{3} (\sigma_x + \sigma_y + \sigma_z) \]  \hspace{1cm} (26)  
\[ \kappa = (1-2\nu)/E \]  \hspace{1cm} (27)  

This means that the plastic component of the mean normal strain must vanish:

\[ \varepsilon^p = \frac{1}{3} (\varepsilon_x^p + \varepsilon_y^p + \varepsilon_z^p) = 0 \]  \hspace{1cm} (28)  

and that the plastic strain deviator is identical to the plastic strain.

c. The total increments, \( d\varepsilon_{ij} \), of the strain result from the sum of the increments of the elastic deformation, \( d\varepsilon_{ij}^e \), and those due to plastic deformation, \( d\varepsilon_{ij}^p \), that is

\[ d\varepsilon_{ij} = d\varepsilon_{ij}^e + d\varepsilon_{ij}^p \]  \hspace{1cm} (29)  

the elastic increments of the deformation are defined by Hooke's law:

\[ d\varepsilon_{ij}^e = \frac{1+\nu}{E} \left[ d\sigma_{ij} - 3\nu \frac{\delta_{ij}}{1+\nu} d\sigma \right] \]  \hspace{1cm} (30)  

d. During plastic flow, the increments of plastic strain (or plastic strain deviation) are at any instant proportional to the instantaneous stress deviations, that is:

\[ d\varepsilon_{ij}^p = d\lambda \cdot S_{ij} \]  \hspace{1cm} (31)  

where

\[ S_{ij} = \sigma_{ij} - \sigma\delta_{ij} \]  \hspace{1cm} (32)  

and \( d\lambda \) is a scalar factor.
This hypothesis can be presented (M-5), (K-2) in a different way: it is assumed that the increments of plastic deformations can be derived from a plastic potential \( Q \). This is usually referred to as the 'normality principle' because it implies that in the space of stresses and strains the vector of incremental strain is normal to the plastic potential surface. Equation (31) can therefore be expressed by

\[
\begin{align*}
  d\varepsilon_{ij}^p &= d\lambda \frac{\partial Q}{\partial \sigma_{ij}} \\
  \text{(33)}
\end{align*}
\]

If the plastic potential coincides with the yield function (associated plasticity law) this expression can be rewritten as:

\[
\begin{align*}
  d\varepsilon_{ij}^p &= d\lambda \frac{\partial F}{\partial \sigma_{ij}} \\
  \text{(34)}
\end{align*}
\]

3.3 Stress-Strain Relations

Considering the plastic potential to which the normality principle is applicable as defined in (N-1):

\[
Q(\sigma, \varepsilon_p, \kappa_0) = 0 \\
\text{(35)}
\]

and using the normality principle:

\[
\begin{align*}
  d\varepsilon_p &= d\lambda \frac{\partial Q}{\partial \sigma} = d\lambda \bar{a} \\
  \text{(36)}
\end{align*}
\]

where \( \bar{a} \) is a vector containing the derivatives of \( Q \) with respect to the relevant stresses:

\[
\bar{a} = \left\{ \begin{array}{c}
\frac{\partial Q}{\partial \sigma_x} \\
\frac{\partial Q}{\partial \sigma_y} \\
\vdots
\end{array} \right\} \\
\text{(37)}
\]
Considering yield function (22), and since during plastic flow
\[ F = \text{constant} = 0, \quad dF = 0, \quad \text{i.e.} \]

\[ dF = \left( \frac{\partial F}{\partial \sigma} \right)^T d\sigma + \frac{\partial F}{\partial \kappa} d\kappa + \left( \frac{\partial F}{\partial \varepsilon_p} \right)^T d\varepsilon_p = 0 \]  
(38)

considering

\[ a = \left\{ \begin{array}{c} \frac{\partial F}{\partial \sigma_x} \\ \frac{\partial F}{\partial \sigma_y} \\ \vdots \end{array} \right\} \]

(39)

and

\[ A = - \frac{1}{d\lambda} \left( \frac{\partial F}{\partial \kappa} d\kappa + \left( \frac{\partial F}{\partial \varepsilon_p} \right)^T d\varepsilon_p \right) \]

(40)

equation (38) can be rewritten as

\[ a^T d\sigma - A d\lambda = 0 \]

(41)

Using hypothesis (c) of the previous section and expression (36)

\[ d\varepsilon = D^{-1} d\sigma + d\lambda \bar{a} \]

(42)

or

\[ d^T d\varepsilon = A d\lambda + d\lambda \bar{\beta} \]

(43)

where \( D \) represents the elasticity matrix and

\[ d = Da, \quad \bar{\beta} = a^T \bar{\sigma}, \quad \bar{d} = D \bar{a} \]

(44)

Equation (43) gives the plastic multiplier as

\[ d\lambda = \frac{1}{A\beta} d^T d\varepsilon \]

(45)

A relationship between the infinitesimal increments of stress
and the corresponding increments of strain can now be obtained by
substituting this value of $d\lambda$ into (42) and rearranging:

\[ d = (D - D_p) \frac{d\varepsilon}{\varepsilon} = D_{ep} \frac{d\varepsilon}{\varepsilon} \]  

(46)

where

\[ D_p = \frac{1}{\lambda + \beta} d d^T \]  

(47)

The essential non-linearity is evident from equation (46) with the elasto-plastic matrix $D_{ep}$ being dependent on the state of total stress.

4. FINITE ELEMENT FORMULATIONS

4.1 General

Various methods of solution can be found in the current literature (M-6, 7, Z-1, N-1, P-1) to deal with elasto-plastic problems. They solve the basically non-linear problem of plasticity through a series of linear problems.

The tangent modulus method is based on the linearity of the incremental laws of plasticity. The load is applied incrementally and, at each stage, new coefficients are obtained for the equilibrium equations. The stiffness matrix is therefore continuously updated.

The initial strains method is based on the idea of modifying the elastic equations of equilibrium to take into account the fact that plastic strains do not cause any change in stress. This method fails if ideal plasticity is postulated or if the degree of hardening is small (Z-1), (M-6).

The initial stress method uses the fact that the increments of strain prescribe uniquely the stress system. An iterative process is
used to distribute elastically the "initial stresses" throughout the structure. In this method a constant stiffness matrix is used throughout the solution and according to reference (N-1) it is often more economical than the tangent modulus method. This aspect is possibly very important when using the semiloof shell element in view of the relatively complex formulation of the stiffness matrices. Because the initial stress method uses a constant positive definite matrix at each step of the iteration, it allows the formulation to include strain softening materials and non associative laws.

In the work reported here the initial stress method was used, and as such this method will now be considered in more detail.

4.2 The Initial Stress Method

In the first place, for a particular increment of load an elastic problem is solved to determine an increment of strain $\Delta \varepsilon'$ and of stress $\Delta \sigma'$ at each gauss point. However, for the increment of strain found the increment of stress will not, in principle, be the exact one. If the correct increment of stress corresponding to a given strain $\Delta \varepsilon'$ is $\Delta \sigma$ a set of body forces must then be provided to equilibrate the initial stress system $\Delta \sigma' - \Delta \sigma$. If the body force system is then removed allowing the structure to deform further an additional set of strain and stress increments are originated, these, probably, exceeding those allowed by the non linear relationship. A redistribution of body forces is then necessary and the process will continue until convergence is achieved.
For a particular load increment the process can be summarised as follows (2-1):

1. Solve linear problem and compute elastic increments of stress $\Delta \sigma'_1$ and strain $\Delta \varepsilon'_1$.

2. Calculate
   
   \[ \sigma' = \sigma_0 + \Delta \sigma'_1 \tag{48} \]
   
   where $\sigma_0$ are the stresses at the start of increment. (At this stage if yielding did not occur another load increment is considered or the process stopped).

3. If the point was plastic at the start of the increment calculate

   \[ \Delta \sigma'_1 = D_{ep} \Delta \varepsilon'_1 \tag{49} \]

   ($D_{ep}$ being computed using stresses $\sigma'$) and calculate the stresses to be equilibrated by body forces:

   \[ \Delta \sigma''_1 = \Delta \sigma'_1 - \Delta \sigma'_1 \tag{50} \]

   Also store the current stress

   \[ \sigma = \sigma' - \Delta \sigma''_1 \tag{51} \]

   and the current strain

   \[ \varepsilon = \varepsilon' + \Delta \varepsilon'_1 \]

4. If the point was not plastic at the start of the increment find the intermediate stress value at which yield begins and compute increment $\Delta \sigma_1$ by using (49). Then proceed as in (3).

5. Compute the equivalent nodal forces for each element

   \[ p^e_1 = \int B^T \Delta \sigma''_1 \, d\text{Vol} \]
6. Resolve using original elastic properties and load system $P$ to find $\Delta \sigma'_2$ and $\Delta \varepsilon'_2$.

7. Repeat steps 2 to 6.

This iterative process is stopped when the equivalent nodal forces reach sufficiently small values. Another increment of load is then applied. When the structure approaches collapse the convergence becomes slower, and collapse is deemed to have occurred if convergence cannot be achieved.

5. RESULTS

Two computer programs for the elasto-plastic analysis of plates and shells structures were developed in this chapter. The first program incorporates the semiloof plate element and a yield surface described by $Q_t = 1$. This yield surface can be derived from any of the first 3 reported in section 3.3 by simply removing the terms containing membrane stresses. The second program incorporates the semiloof shell element and although any of the yield surfaces described in section 3.3 can be incorporated in the program, the results presented were obtained by the use of yield surface Y1. In this program two kinds of stiffness matrices can be considered by a suitable choice of a parameter. These are the usual, linear stiffness matrix ($\kappa_E$) and a stiffness matrix for which the geometric matrix is taken into account ($\kappa_E + \kappa_G$). The geometric matrix is introduced in order to take into account the effects of the membrane loading on the bending stiffness of the element (A-3).
Elements of this matrix are functions of the membrane stresses and the out-of-plane derivatives and its determination was discussed in Chapter IV. The geometric matrix can be calculated at the beginning of each increment and used during the increment as a constant or it can be recalculated for each iteration of the increment. The shell program has therefore three possibilities:

Method 1 - Uses the classical stiffness matrix $\kappa_E$ during all the computations.

Method 2 - Uses stiffness matrix $\kappa_E + \kappa_G$, $\kappa_G$ being constant for each increment (equal to its value at the beginning of the increment)

Method 3 - Uses stiffness matrix $\kappa_E + \kappa_G$, $\kappa_G$ being updated for each iteration.

The first problem solved with the plate program was a simply supported circular plate uniformly loaded. Elastic ideally plastic material behaviour was assumed and the following material properties were employed:

- Young's modulus, $E = 10000000.0$
- Poisson's ratio $\nu = 0.24$
- Yield stress $\sigma = 16000.0$
- Radius $r = 1.0$
- Thickness $t = 0.05$

In order to test the performance of the element two finite element idealisations, one with 3 and the other with 16 elements, were employed. These are shown in Figure 5. This figure also shows the yielding sequence for the gauss points. The numbers on these meshes
refer to the number of the increment at which plastification occurred. It can be seen that the yielded zone progresses from the centre of the plate to the periphery until collapse is achieved. Figure 5 includes as well the plot of load versus centre deflection. The full line represents the 16 elements case, the 3 elements case being represented by small circles. A good agreement between the two cases is observed. The biggest disagreements occur on the almost horizontal branch of the curve. This can be seen by comparing Tables I and II where the numerical values corresponding to this graph are given. These tables also show the type of convergence observed with the program, this being more difficult as the graph becomes horizontal as the collapse load is approached. This collapse load occurs at a load of approximately 6.6 which compares well with the value 6.5 of reference (A-3). The results of this reference (and the corresponding meshes) are shown in Figure 6 together with the present solution for the 3 elements case. The results obtained for meshes a and b are virtually the same (A-3) but, as the figure shows, solution varies considerably with the different load increment sizes considered. The deflection profiles obtained with the semiloof plate elements are shown in Figure 7. A good agreement is generally observed between the two meshes. In this figure the difference of load between curves b and c, and curves c and d is the same ($\Delta \rho = 0.95$); still the difference of deflections is quite different because of the influence of the plastic zones on the deformation. Figure 8 shows the variation of circumferential and radial moments for some of the increments. As it can be seen the shape of the circumferential moments changes considerably from the linear solution corresponding to increment 2 to the non linear solution of increment 20.
Also using the plate program the problem of an uniformly loaded simply supported square plate was solved. The following material properties were considered:

\[
E = 10000000.0 \\
\nu = 0.3 \\
\sigma = 144000.0 \\
2a = 4.0 \quad \text{side length} \\
t = 0.05
\]

As for the circular plate, 2 meshes (with 4 and 16 elements) were used and comparisons made between them and other solutions. Figure 9 shows the load deflection curve and the yield sequence at the gauss points for the 4 element case. Unlike the other two solutions included in this graph, the almost horizontal branch of the load deflection curve is contained within the upper and lower bounds indicated in reference (H-4). The yield sequence and load deflection curve for the 16 elements case are given in Figure 10. Table 3 gives the numerical values corresponding to this curve. The deflection profiles obtained with both meshes are shown in Figure 11. These profiles practically coincide for the linear case (e), the difference becoming greater when the yielding develops. Figure 12 gives the distribution of moments along the diagonal AB. A good agreement can be observed between the two meshes.

The shell program was first tested by solving the problem of a square plate with 3 edges simply supported, the fourth edge free and subject to an uniform load. In view of the symmetry only one half of the plate was considered, the following material properties having been assumed:
E = 10000000.0
ν = 0.3
σ = 144000

side length 2a = 2.0
t = 0.05

The half plate was discretized into 8 finite elements as shown in Figure 13. This figure also shows the load versus central deflection curve. The nearly horizontal branch of this curve (collapse) can be seen to lie between the bounds given by (H-4).

As a further test of the shell program a problem was solved involving only membrane stresses. This is the deep cantilever beam shown on Figure 14. This figure includes also the solution for the same problem given in reference (Z-1). The disagreement between these two solutions (although not large) should be expected in view of the very coarse mesh used in the present solution.

The cylindrical shell roof whose linear solution was presented in Chapter III was also analysed with the elasto-plastic program for shells. It was solved using meshes with 6 and 12 elements in the quadrant, the results obtained being very similar. Figure 15 includes results from references (B-2, C-3, D-1) as well as the solution obtained with 6 semiloof shell elements. Although no bounds for this problem are known to the author, the solution obtained here compares well with the other solutions plotted in this figure. It should also be noted that the two convergence studies included (B-2,C-3) show a reduction of the collapse load with increasing number of elements. This seems
to be consistent with the fact that a lower curve is obtained with the semiloof element. Table 4 gives details of the solutions included in Figure 15. The load-deflection curve given in Figure 15 was obtained using Method 3, the curve for Method 1 being very similar to this one.

The last problem solved with the shell program consists of a plate subject to both membrane and bending loads. For this problem the difference between Methods 1 and 3 is larger than in the previous problem. Load deflection curves for both methods are given in Figure 16 where the yield sequence for the gauss points and material properties are also included. Table 5 includes the numerical values for both methods as well as the type of convergence observed. It can be seen that after the fourth increment a better convergence is obtained using Method 3.

6. CONCLUSIONS

The semiloof plate and shell elements have been successfully applied for the elasto-plastic analysis of structures. The elements show a good convergence rate, the solutions obtained with a small number of elements being very similar to the ones where a larger number of elements was used.

For all the problems solved the solutions obtained lay within all known bounds. For problems where bounds were not available a good comparison with other finite element solutions was also observed.

It can therefore be said that, in spite of the unorthodox formulations of these elements, they yield for elasto-plastic problems reliable and rapid convergent solutions.

The initial stress method as developed in (2-1) also proved to be a very efficient method for the elasto-plastic analysis of plates and shells.
CHAPTER VI - ELASTO-PLASTIC ANALYSIS

BIBLIOGRAPHY

A-1 Argyris, J.H.  
'Elasto-plastic matrix displacement analysis of three-dimensional continua'  

A-2 Armen, H., Pifko, A. and Levine, H.S.  
'A finite element method for the plastic bending analysis of structures'  

A-3 Armen, H., Pifko, A. and Levine, H.S.  
'Finite element analysis of structures in the plastic range'  

A-4 Aranjo, F.C.  
'Elasticidade e plasticidade'  

A-5 Armen, H., Levine, H.S. and Pifko, A.B.  
'Plasticity - theory and finite element applications'  

B-1 Bergan, P.G. and Clough, R.W.  
'Elasto-plastic analysis of plates using the finite element method'  

B-2 Bäcklund, J. and Wennerström, H.  
'Finite element analysis of elasto-plastic shells'  

B-3 Bhaumik, A.K. and Hanley, J.T.  
'Elasto-plastic plate analysis by finite differences'  

C-1 Calladine, C.R.  
'On the derivation of yield conditions for shells'  

C-2 Crisfield, M.A.  
'On an approximate yield criterion for thin shells'  
Transport and Road Research Laboratory, Laboratory Report 658, 1974.

C-3 Cormeau, I.  


H-3 Hu, L.W., Markowitz, J. and Bartush, T.A. 'A triaxial experiment on yield condition in plasticity' Exp. Mech., 23 (1966) p. 58, 64.


I-1 Ivanov, G.V. 'Inzhenernyi Zhurnal Mekhanika Tverdogo Tela, No. 6, 74, 1967.


M-5 Massonet, C.  

M-6 Marcal, P.V.  
'A comparative study of numerical methods of elastic-plastic analysis'  

M-7 Marcal, P.V.  
'Finite element analysis with material non-linearities, theory and practice'  
In: Recent Advances in Matrix Methods of Structural Analysis (U.S. - Japan Seminar), Tokyo, 1969.

N-1 Nayak, G.C. and Zienkiewicz, O.C.  
'Elastic-plastic stress analysis. A generalization for various constitutive relations including strain softening'  

N-2 Nakamura, T.  
'Limit analysis of non-symmetric sandwich shells'  

P-1 Pope, G.G.  
'The application of the matrix displacement method in plane elasto-plastic problems'  

P-2 Prager, W.  
'The theory of plasticity: a survey of recent achievements'  

P-3 Prager, W.  
'A new method of analysing stress and strains in work hardening plastic solids'  

P-4 Prager, W. Hodge, P.  
'The theory of perfectly plastic solids'  

R-1 Robinson, M.  
'A comparison of yield surfaces for thin shells'  

R-2 Rozenblyum, V.I.  

<table>
<thead>
<tr>
<th>Load Factor</th>
<th>Total Load</th>
<th>Centre Deflection</th>
<th>Initial Residual</th>
<th>Number of Iterations</th>
<th>Final Residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.375</td>
<td>3.375</td>
<td>0.0559</td>
<td>Linear</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>4.550</td>
<td>4.875</td>
<td>0.0753</td>
<td>&quot;</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>0.325</td>
<td>0.0807</td>
<td>0.0872</td>
<td>3.18</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>0.150</td>
<td>0.0832</td>
<td>3.05</td>
<td>1</td>
<td>0.05</td>
</tr>
<tr>
<td>5</td>
<td>0.150</td>
<td>0.0914</td>
<td>3.41</td>
<td>5</td>
<td>0.59</td>
</tr>
<tr>
<td>6</td>
<td>0.175</td>
<td>0.0971</td>
<td>3.70</td>
<td>6</td>
<td>0.64</td>
</tr>
<tr>
<td>7</td>
<td>0.125</td>
<td>0.1027</td>
<td>2.93</td>
<td>6</td>
<td>0.66</td>
</tr>
<tr>
<td>8</td>
<td>0.150</td>
<td>0.1097</td>
<td>3.49</td>
<td>7</td>
<td>0.59</td>
</tr>
<tr>
<td>9</td>
<td>0.150</td>
<td>0.1167</td>
<td>3.40</td>
<td>7</td>
<td>0.58</td>
</tr>
<tr>
<td>10</td>
<td>0.150</td>
<td>0.1245</td>
<td>2.16</td>
<td>7</td>
<td>0.67</td>
</tr>
<tr>
<td>11</td>
<td>0.150</td>
<td>0.1422</td>
<td>2.48</td>
<td>13</td>
<td>0.64</td>
</tr>
<tr>
<td>12</td>
<td>0.150</td>
<td>0.1612</td>
<td>2.51</td>
<td>13</td>
<td>0.69</td>
</tr>
<tr>
<td>13</td>
<td>0.075</td>
<td>0.1709</td>
<td>1.52</td>
<td>9</td>
<td>0.69</td>
</tr>
<tr>
<td>14</td>
<td>0.075</td>
<td>0.1806</td>
<td>1.50</td>
<td>9</td>
<td>0.68</td>
</tr>
<tr>
<td>15</td>
<td>0.0075</td>
<td>0.1813</td>
<td>0.68</td>
<td>1</td>
<td>0.68</td>
</tr>
<tr>
<td>16</td>
<td>0.0075</td>
<td>0.1820</td>
<td>0.71</td>
<td>1</td>
<td>0.71</td>
</tr>
<tr>
<td>17</td>
<td>0.0075</td>
<td>0.1828</td>
<td>0.73</td>
<td>1</td>
<td>0.73</td>
</tr>
<tr>
<td>18</td>
<td>0.0075</td>
<td>0.1835</td>
<td>0.81</td>
<td>1</td>
<td>0.81</td>
</tr>
<tr>
<td>19</td>
<td>0.0075</td>
<td>0.1844</td>
<td>0.90</td>
<td>1</td>
<td>0.90</td>
</tr>
<tr>
<td>20</td>
<td>0.0075</td>
<td>0.1853</td>
<td>0.98</td>
<td>1</td>
<td>0.98</td>
</tr>
<tr>
<td>21</td>
<td>0.0075</td>
<td>0.1946</td>
<td>1.07</td>
<td>10</td>
<td>1.00</td>
</tr>
<tr>
<td>22</td>
<td>0.0075</td>
<td>0.2206</td>
<td>1.10</td>
<td>25</td>
<td>1.12</td>
</tr>
<tr>
<td>23</td>
<td>0.0075</td>
<td>0.2497</td>
<td>1.23</td>
<td>25</td>
<td>1.25</td>
</tr>
<tr>
<td>24</td>
<td>0.0075</td>
<td>0.2818</td>
<td>1.35</td>
<td>25</td>
<td>1.37</td>
</tr>
<tr>
<td>25</td>
<td>0.0075</td>
<td>0.3169</td>
<td>1.48</td>
<td>25</td>
<td>1.50</td>
</tr>
<tr>
<td>26</td>
<td>0.0075</td>
<td>0.3551</td>
<td>1.60</td>
<td>25</td>
<td>1.62</td>
</tr>
<tr>
<td>27</td>
<td>0.0075</td>
<td>0.3963</td>
<td>1.73</td>
<td>25</td>
<td>1.75</td>
</tr>
<tr>
<td>28</td>
<td>0.0075</td>
<td>0.4405</td>
<td>1.85</td>
<td>25</td>
<td>1.87</td>
</tr>
</tbody>
</table>

**TABLE 1** - Centre deflection and convergence for an uniformly loaded circular plate, 3 semilooF plate elements.
<table>
<thead>
<tr>
<th>Load Factor</th>
<th>Total Load</th>
<th>Centre Deflection</th>
<th>Initial Residual</th>
<th>Number of Iterations</th>
<th>Final Residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.375</td>
<td>0.0558</td>
<td>Linear</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>4.550</td>
<td>0.0752</td>
<td>&quot;</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>4.875</td>
<td>0.0806</td>
<td>&quot;</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>5.025</td>
<td>0.0832</td>
<td>1.74</td>
<td>3</td>
<td>0.68</td>
</tr>
<tr>
<td>5</td>
<td>5.175</td>
<td>0.0865</td>
<td>1.70</td>
<td>4</td>
<td>0.50</td>
</tr>
<tr>
<td>6</td>
<td>5.325</td>
<td>0.0910</td>
<td>4.00</td>
<td>6</td>
<td>0.50</td>
</tr>
<tr>
<td>7</td>
<td>5.500</td>
<td>0.0969</td>
<td>4.21</td>
<td>7</td>
<td>0.61</td>
</tr>
<tr>
<td>8</td>
<td>5.625</td>
<td>0.1016</td>
<td>2.92</td>
<td>5</td>
<td>0.70</td>
</tr>
<tr>
<td>9</td>
<td>5.775</td>
<td>0.1095</td>
<td>4.38</td>
<td>8</td>
<td>0.66</td>
</tr>
<tr>
<td>10</td>
<td>5.925</td>
<td>0.1174</td>
<td>4.12</td>
<td>8</td>
<td>0.62</td>
</tr>
<tr>
<td>11</td>
<td>6.075</td>
<td>0.1270</td>
<td>2.28</td>
<td>8</td>
<td>0.64</td>
</tr>
<tr>
<td>12</td>
<td>6.225</td>
<td>0.1397</td>
<td>1.96</td>
<td>10</td>
<td>0.65</td>
</tr>
<tr>
<td>13</td>
<td>6.375</td>
<td>0.1543</td>
<td>2.43</td>
<td>10</td>
<td>0.67</td>
</tr>
<tr>
<td>14</td>
<td>6.450</td>
<td>0.1650</td>
<td>1.31</td>
<td>9</td>
<td>0.70</td>
</tr>
<tr>
<td>15</td>
<td>6.525</td>
<td>0.1806</td>
<td>1.38</td>
<td>13</td>
<td>0.68</td>
</tr>
<tr>
<td>16</td>
<td>6.5325</td>
<td>0.1815</td>
<td>0.72</td>
<td>1</td>
<td>0.72</td>
</tr>
<tr>
<td>17</td>
<td>6.5400</td>
<td>0.1825</td>
<td>0.76</td>
<td>1</td>
<td>0.76</td>
</tr>
<tr>
<td>18</td>
<td>6.5475</td>
<td>0.1836</td>
<td>0.80</td>
<td>1</td>
<td>0.80</td>
</tr>
<tr>
<td>19</td>
<td>6.5550</td>
<td>0.1848</td>
<td>0.84</td>
<td>1</td>
<td>0.84</td>
</tr>
<tr>
<td>20</td>
<td>6.5625</td>
<td>0.1860</td>
<td>0.87</td>
<td>1</td>
<td>0.87</td>
</tr>
<tr>
<td>21</td>
<td>6.5700</td>
<td>0.1873</td>
<td>0.93</td>
<td>1</td>
<td>0.93</td>
</tr>
<tr>
<td>22</td>
<td>6.5775</td>
<td>0.1886</td>
<td>0.97</td>
<td>1</td>
<td>0.97</td>
</tr>
<tr>
<td>23</td>
<td>6.5850</td>
<td>0.1913</td>
<td>1.00</td>
<td>2</td>
<td>0.97</td>
</tr>
<tr>
<td>24</td>
<td>6.5925</td>
<td>0.1941</td>
<td>1.01</td>
<td>2</td>
<td>0.97</td>
</tr>
<tr>
<td>25</td>
<td>6.6000</td>
<td>0.1955</td>
<td>0.99</td>
<td>1</td>
<td>0.99</td>
</tr>
<tr>
<td>26</td>
<td>6.6075</td>
<td>0.2330</td>
<td>1.11</td>
<td>25</td>
<td>1.13</td>
</tr>
<tr>
<td>27</td>
<td>6.6150</td>
<td>0.2735</td>
<td>1.20</td>
<td>25</td>
<td>1.21</td>
</tr>
<tr>
<td>28</td>
<td>6.6225</td>
<td>0.3171</td>
<td>1.29</td>
<td>25</td>
<td>1.30</td>
</tr>
<tr>
<td>29</td>
<td>6.6300</td>
<td>0.3636</td>
<td>1.37</td>
<td>25</td>
<td>1.38</td>
</tr>
</tbody>
</table>

**TABLE 2** - Centre deflection and convergence for an uniformly loaded circular plate, 16 semiloof plate elements
<table>
<thead>
<tr>
<th>Load Factor</th>
<th>Total Load</th>
<th>Centre Deflection</th>
<th>Initial Residual</th>
<th>Number of Iterations</th>
<th>Final Residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.66(6)</td>
<td>0.66(6)</td>
<td>0.0649</td>
<td>Linear</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.16(6)</td>
<td>0.83(3)</td>
<td>0.0815</td>
<td>9.92</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>0.08(3)</td>
<td>0.91(6)</td>
<td>0.0919</td>
<td>11.77</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>0.01(6)</td>
<td>0.93(3)</td>
<td>0.0952</td>
<td>3.98</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>0.01(6)</td>
<td>0.950</td>
<td>0.0993</td>
<td>3.22</td>
<td>6</td>
</tr>
<tr>
<td>6</td>
<td>0.01(6)</td>
<td>0.96(6)</td>
<td>0.1036</td>
<td>3.26</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>0.01(6)</td>
<td>0.98(3)</td>
<td>0.1095</td>
<td>3.48</td>
<td>11</td>
</tr>
<tr>
<td>8</td>
<td>0.00(3)</td>
<td>0.98(6)</td>
<td>0.1113</td>
<td>1.55</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>0.00(3)</td>
<td>0.990</td>
<td>0.1147</td>
<td>1.57</td>
<td>18</td>
</tr>
<tr>
<td>10</td>
<td>0.00(3)</td>
<td>0.99(3)</td>
<td>0.1185</td>
<td>1.65</td>
<td>20</td>
</tr>
<tr>
<td>11</td>
<td>0.00(3)</td>
<td>0.99(6)</td>
<td>0.1225</td>
<td>1.63</td>
<td>21</td>
</tr>
<tr>
<td>12</td>
<td>0.00(3)</td>
<td>1.000</td>
<td>0.1269</td>
<td>1.66</td>
<td>23</td>
</tr>
<tr>
<td>13</td>
<td>0.00(3)</td>
<td>1.00(3)</td>
<td>0.1319</td>
<td>1.66</td>
<td>25</td>
</tr>
<tr>
<td>14</td>
<td>0.00(3)</td>
<td>1.00(6)</td>
<td>0.1376</td>
<td>1.61</td>
<td>25</td>
</tr>
<tr>
<td>15</td>
<td>0.00(3)</td>
<td>1.010</td>
<td>0.1439</td>
<td>1.62</td>
<td>25</td>
</tr>
</tbody>
</table>

**TABLE 3** - Simply supported square plate, 16 elements mesh
<table>
<thead>
<tr>
<th>Mesh</th>
<th>Element Type</th>
<th>Reference</th>
<th>Degrees of Freedom</th>
</tr>
</thead>
<tbody>
<tr>
<td>4x5x2</td>
<td>Hellan-Morley Triangles 10 layers</td>
<td>B-2</td>
<td>156</td>
</tr>
<tr>
<td>12x16x2</td>
<td>&quot;</td>
<td>&quot;</td>
<td>1264</td>
</tr>
<tr>
<td>16 noded isoparametric 2x2x6 Gauss integral</td>
<td>C-3</td>
<td>48</td>
<td></td>
</tr>
<tr>
<td>2x2</td>
<td>&quot;</td>
<td>&quot;</td>
<td>126</td>
</tr>
<tr>
<td>3x3</td>
<td>&quot;</td>
<td>&quot;</td>
<td>240</td>
</tr>
<tr>
<td>4x4</td>
<td>&quot;</td>
<td>&quot;</td>
<td>390</td>
</tr>
<tr>
<td>4x5x2</td>
<td>Dupuis curved triangles</td>
<td>D-1</td>
<td>270</td>
</tr>
<tr>
<td>2x3</td>
<td>Semiloof</td>
<td>This analysis</td>
<td>106</td>
</tr>
</tbody>
</table>

**TABLE 4**
<table>
<thead>
<tr>
<th>Incr. Load Factor</th>
<th>Method 3</th>
<th>Method 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number Iter.</td>
<td>Final Residual</td>
</tr>
<tr>
<td>1</td>
<td>0.520</td>
<td>Linear</td>
</tr>
<tr>
<td>2</td>
<td>0.570</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>0.600</td>
<td>11</td>
</tr>
<tr>
<td>4</td>
<td>0.620</td>
<td>16</td>
</tr>
<tr>
<td>5</td>
<td>0.625</td>
<td>8</td>
</tr>
<tr>
<td>6</td>
<td>0.630</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>0.633</td>
<td>12</td>
</tr>
<tr>
<td>8</td>
<td>0.636</td>
<td>15</td>
</tr>
<tr>
<td>9</td>
<td>0.639</td>
<td>15</td>
</tr>
<tr>
<td>10</td>
<td>0.642</td>
<td>15</td>
</tr>
<tr>
<td>11</td>
<td>0.645</td>
<td>15</td>
</tr>
</tbody>
</table>

**TABLE 5**
Figure 5
Uniformly loaded, simply supported circular plate. Load versus centre deflection and yielding sequence.
--- ○ --- 3 SEMILOOF PLATE ELEMENTS

UNIFORMLY LOADED SIMPLY SUPPORTED CIRCULAR PLATE:
LOAD vs CENTER DEFLECTION FOR VARIOUS LOAD INCREMENTS,
REFER.(A-3)

FIG. 6 CIRCULAR PLATE IDEALIZATIONS, REF.(A-3)
Figure 7  Deflection profiles for a simply supported circular plate uniformly loaded

\( \rho = \frac{p a^2}{M_0} \)

(a) \( \rho = 6.525 \)
(b) \( \rho = 6.450 \)
(c) \( \rho = 5.500 \)
(d) \( \rho = 4.550 \)
Figure 8  Moment distribution for a simply supported uniformly loaded circular plate

Circumferential moment

(Numbers refer to increment number)

Radial moment

16 element mesh

3 element mesh, increment 14.

(20 practically coincident with 14)
CHAPTER VI - ELASTO-PLASTIC ANALYSIS

Figure 9. Simply supported square plate

Center deflection $\frac{wD}{4M_0a^2}$

Lower bound $: 1.106$
Upper bound $: 1.036$
(Ref. H-4.)

Last increment: load $= 1.037$, defl. $= 0.2846$

Finite differences solution
Reference B-3

Solution, with 4 semiload plate elements

Finite elements solution
36 element mesh, ref. A-3
Material properties:

\[ E = 10^7, \quad \nu = 0.3 \]
\[ M_0 = 60, \quad \text{thickness} = 0.05 \]
side length = 4.0

(a) Yielding sequence for Gauss points

(b) Load versus central deflection

Figure 10 - Uniformly loaded simply supported square plate, 16 element mesh
Figure 11 - Deflection profiles along AC (simply supported square plate)
Figure 12 - Simply supported, uniformly loaded square plate. Moment distribution along diagonal AB.
Figure 13 - Uniformly loaded square plate with three simple supports and one free edge.

Load \((p_a^2/6M_b)\)

Deflection \((\omega D/4 M_o a^2)\)

- **Upper bound**: 0.654 (H-4)
- **Lower bound**: 0.595 (H-4)

Collapse load for ref. (A-3)
$P_c$ - collapse load as from beam theory

Figure 14 - Deep cantilever beam
DETAILS IN TABLE 4

FIG. 15 CYLINDRICAL SHELL ROOF
Material properties:
E = 10^7; \quad v = 0.3
Yield stress = 144000
Side length = 4.0
Thickness = 0.05
Load for load factor = 1.0
Out of plane pressure = 150
In plane force per unit length = 3600

Yielding sequence for Gauss points:

<table>
<thead>
<tr>
<th>6</th>
<th>5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

Figure 16
CHAPTER VII - TYPICAL INDUSTRIAL APPLICATION

CHAPTER VII
APPLICATION OF THE PREVIOUS PROGRAMS TO A TYPICAL INDUSTRIAL PROBLEM

1. INTRODUCTION

The Semiloof shell element has been used with the programs developed in the previous chapters, to solve a wide range of problems. For most of the problems solved, some kind of solution was already known and good agreement was observed between the solutions obtained in this work and other known solutions. The test problems presented in the previous chapters correspond to geometries and loading situations which are relatively simple.

As a further test on the Semiloof element itself and on the other programs developed in this work, a "large" shell problem will be solved in this chapter. This problem is tackled using various approaches, namely linear analysis, instability and vibration analysis and geometrically nonlinear analysis, showing the different results that different approaches to the same problem can yield. Furthermore the influence on the results of the changing of some boundary conditions is also studied.
2. PROBLEM DEFINITION

2.1 Structure

The structure to be analysed is a reinforced cylindrical tank with a floating roof used for the storage of crude oil and erected at a place subject to high winds. The tank, figure 1, which is symmetric in relation to a vertical plane, is 23 metres high and has a diameter of 76 metres. The walls are composed of 10 layers of steel plate each 2.3 m in height and with the following thicknesses from bottom to top:

<table>
<thead>
<tr>
<th>Layer</th>
<th>Thickness</th>
<th>Layer</th>
<th>Thickness</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0352 m</td>
<td>6</td>
<td>0.0166</td>
</tr>
<tr>
<td>2</td>
<td>0.0315</td>
<td>7</td>
<td>0.0125</td>
</tr>
<tr>
<td>3</td>
<td>0.0278</td>
<td>8</td>
<td>0.0122</td>
</tr>
<tr>
<td>4</td>
<td>0.0240</td>
<td>9</td>
<td>0.0100</td>
</tr>
<tr>
<td>5</td>
<td>0.0203</td>
<td>10</td>
<td>0.0100</td>
</tr>
</tbody>
</table>

There are three circumferential stiffeners with the following characteristics:

<table>
<thead>
<tr>
<th>Stiffener</th>
<th>Distance from top</th>
<th>$I_z$</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.07 m</td>
<td>497000 cm$^4$</td>
<td>170.0 cm$^2$</td>
</tr>
<tr>
<td>B</td>
<td>3.79</td>
<td>1480</td>
<td>34.6</td>
</tr>
<tr>
<td>C</td>
<td>7.76</td>
<td>1480</td>
<td>34.6</td>
</tr>
</tbody>
</table>

where $I_z$ represents the second moment of area in relation to an axis parallel to z (figure 1) and passing through the centre of mass of the cross section of the stiffner, A being the cross-sectional area.
The base of the tank is reinforced by an annular plate, 0.011 m thick and 1 m wide, the outside circumference of which is welded to the bottom of the vertical walls. The remaining portion of the base is made from 0.0064 m steel plate.

Throughout the structure the following material properties are assumed:

- Young's modulus: \( 200 \times 10^6 \text{ KN/m}^2 \)
- Maximum working stress: \( 240 \times 10^3 \text{ KN/m}^2 \)
- Yield stress: \( 360 \times 10^3 \text{ KN/m}^2 \)
- Poisson's ratio: 0.30
- Density: \( 8300 \text{ kg/m}^3 \)

### 2.2 Loads

The analyses carried out for the structure described above refer only to wind load and the load resulting from its own weight. The gravity load was applied as described in Chapter 3. The wind load (API650) results in a normal pressure which is considered to be constant in a vertical direction and whose circumferential variation is given by (Ref. A-2)

\[
p(\theta) = K \sum_{n=0}^{9} (a_n \cos(n\theta))
\]

where \( \theta \) is defined in Fig. 1.

The coefficients of this series are defined to be:

- \( a_0 = -0.5944 \)
- \( a_1 = 0.2482 \)
- \( a_2 = 0.9041 \)
- \( a_3 = 0.3757 \)
- \( a_4 = -0.0248 \)
- \( a_5 = 0.0000 \)
- \( a_6 = 0.0269 \)
- \( a_7 = 0.0287 \)
- \( a_8 = 0.0101 \)
- \( a_9 = 0.0047 \)
The factor \( K = \frac{1}{2} \rho v^2 \) where \( \rho \) is the density of air and \( v \) is the wind velocity. For a wind speed of 64 m/s, \( K \) equals 2.51 KN/m\(^2\).

The circumferential distribution of pressure is shown in figure 2 where it can be seen that in a frontal semi-angle of approximately 30° the tank is subjected to a positive pressure whereas on the remaining part the pressure is negative and almost constant on the last 40 degrees.

2.3 Boundary conditions

In view of the symmetry of the load in relation to a diametral plane only half of the structure need be analysed. The boundary conditions along the vertical sections corresponding to \( \theta = 0^\circ \) and \( \theta = 180^\circ \) are therefore zero normal rotations and zero displacement along the y axis.

Regarding the boundary conditions along the intersection between the vertical walls and the base, two models are employed. In the first (Model I) the base of the tank is considered pinned (all 3 displacements restrained) along the periphery. A linear analysis with this model showed that in a frontal semiangle of approximately 33° the vertical reactions are directed downwards. As no support for these reactions is provided by the method of construction employed a second model of analysis (Model II) is considered in which the front part of the base is allowed to lift within a semiangle of 36°. The base is represented by only the annular plate whose inner boundary is completely pinned in both models. Results from both models give the vertical reactions directed upwards along this inner boundary which justifies this restraint condition. It will be seen later that Model II induces more severe stressing conditions than Model I.

24. Finite element discretization

Two different types of mesh were considered. In the first
(Mesh A) the structure was discretized by 110 elements with a total number of 1601 degrees of freedom. This mesh is shown in figure 3, where it can be seen that the tank is divided vertically into 11 rows of elements (3 corresponding to the stiffeners and 1 to the base) and circumferentially into ten equal rows of elements each one corresponding to a central angle of 18°. In each circumferential row the elements are considered to be of constant thickness equal to the weighted thicknesses of the layers of steel plate which they simulate. These thicknesses are given in figure 1.

In the second mesh (Mesh B) only 36 elements are considered (567 degrees of freedom). Each circumferential row has 6 elements, the first 3 with a central angle of 24° and the remaining elements have a central angle of 36°. In this mesh the elements are vertically larger than for mesh A. The element thicknesses are considered variable and the thickness of the Gauss points are interpolated using the actual thickness prescribed at the element nodes. As the lower stiffeners are relatively small they are not considered in mesh B, resulting in a saving of 12 elements.

The need for mesh B was mainly dictated by the necessity of carrying out a geometrically nonlinear analysis, for which mesh A, having 1601 degrees of freedom, would prove too costly. The linear solution for mesh B is also presented, in order to give an assessment of the errors introduced by the coarser mesh.

3. LINEAR ANALYSIS

3.1 Model I

The linear analysis was carried out for a load factor of 3.5 corresponding to a wind speed of 120 m/s, which is far in excess of the maximum wind speed expected.
Figures 4 and 5 show deflection profiles for the following sections:

- Vertical section for $\theta = 0^\circ$ degrees
- Horizontal section for $z = 23.00$ metres
- Horizontal section for $z = 21.93$

It is seen from these figures that the largest displacements occur for $\theta = 0^\circ$ (inwards) and $\theta = 75^\circ$ (outwards), the displacements of the rear half of the tank being much smaller than the ones in the front portion.

The displacements obtained with mesh B are also shown in these figures and generally good agreement with the ones obtained with mesh A is evident. Some small differences should be expected since the two lower stiffeners were not considered in the coarser mesh and also owing to the difficulty of representing the actual variation of the thickness (by steps) along the tank walls.

Table 1 gives the maximum stress at each element of mesh A. These are the combined stresses resulting from both membrane and bending effects and are referred to local axes whose direction cosines are also included in the table. For elements 1 to 10 there are two stresses with the same absolute value and opposite signs. This means that these elements are subjected to pure bending. In fact since the base is fully pinned to the foundation at all points, no membrane stresses can be transmitted to the base. The entry in the last column of table 1 refers to the number of the Gauss point at which the maximum stress occurs. The numeration of the Gauss points is made anti-clockwise; the first one being (figure 3) the one on the left hand side of the figure closer to the bottom of the figure. The letters B, T, O, I refer to the face of the element where the maximum stress occurs and refer respectively to "bottom", "top", "outside" and "inside". The values of these stresses, are seen from this table to be comfortably
below the working stress for steel indicated in the previous section.

In figures 6 and 7, the maximum stresses obtained along some circumferential sections are plotted. The values obtained using mesh A are drawn in full line while the circles give the values obtained with mesh B. A good agreement is generally evident.

The plots of figures 6, 7 and the ones shown for the displacements (figures 4, 5) seem to reinforce the opinion expressed in reference (A-1) that reliable results can be obtained in a minicomputer using a relatively small number of Semiloof elements. In fact, the results reported above show that mesh B would have given sufficient accuracy from the engineering point of view.

As a further test for the consistency of the problem solved a balance of vertical forces was made. The total weight of the tank being 4873 KN, it is expected to obtain this value as the total upward reaction. The reaction total is in fact 4868 KN giving a difference of only 0.1%.

As mentioned in the previous section, the model being discussed gives reactions directed downwards in a semiangle of 33° in the front part of the base. The next section examines the behaviour of the tank when this part of the base is allowed to lift off from the foundation.

3.2 Model II

As for Model I, the linear analysis was carried out for a wind speed of 120 m/s. Figures 8 and 8a show deflection profiles for the following sections:
- Vertical section for \( \theta = 0 \) degrees
- Horizontal section for \( z = 23.0 \) metres
- Horizontal section for \( z = 21.93 \) metres

It is observed from these figures that the rear part of the tank undergoes displacements of the same order of magnitude as in model I. In the front part, however, the displacements are much larger and for \( \theta = 0^0 \) there is lift-off of the outer edge of the base of approximately 0.06 m. At the top of the tank, for \( \theta = 0^0 \), the vertical deflection is approximately equal to the value observed at the base but becomes practically zero for \( \theta = 36^0 \).

The maximum stress in each element and its location are given in table 2, where the same conventions as for Model I apply. These are also plotted in figures 9a to 9f for various circumferential sections. The comparison of these figures with the corresponding ones for Model I, show that in the front part of the tank, a "disruption" of the smooth pattern observed for Model I occurs. Not only is this pattern disrupted but also the stresses become much larger when compared with the previous ones. For the circumferential section \( z = 22.1561 \) m, for example, the maximum stress given by Model II is almost 6 times greater than the value calculated using Model I. The position of maximum stress is also generally altered.

In figure 9b it can be seen that a considerable difference exists, within a semiangle of 45 degrees, between the stresses calculated using meshes A or B. It should however be noticed that the values shown in this figure refer to a section located near one of the stiffeners which was not included in mesh B, which explains the discrepancy observed. The
reason why this discrepancy for Model I is much smaller is also understandable: because in Model II the bottom is allowed to lift the stiffeners have to play a bigger role in preventing the deformation inwards of the front part of the tank.

Although the stresses for Model II are considerably higher than for Model I the maximum stress (element 101) is only marginally higher than the maximum working stress. From the point of view of a linear analysis the structure can therefore be considered to be stable, so much so that the values reported refer to a considerably high load factor.

3.3 Additional Models of Analysis

Two other models of analysis were used but will not be referred to in detail. In the first one, the base is allowed to lift in a frontal semi-angle of 54°. For this condition the point of the shell corresponding to $\theta = 0^\circ$, $z = 23.0$ m lifts 32 cm and moves 96 cm in the wind direction. The maximum stress becomes $640 \times 10^3$ KN/m$^2$, that is well above the maximum working stress.

In the second supplementary model, the bottom is left free in a front semi-angle of 72°. The displacement of the point referred above becomes even bigger than in the previous case, being 145 cm upwards and 273 cm in the wind direction, stresses as big as $1200 \times 10^3$ KN/m$^2$ can now be observed in the upper part of the tank.

4. ELASTIC STABILITY ANALYSIS

It is well known that the load carrying capacity of a structure is not always conveniently defined by means of a linear analysis. Phenomena of elastic stability (and others) can drastically reduce the load carrying capacity of certain structures to values much smaller than the ones yielded by a linear analysis. This being particularly true for many types of thin
shells, it is natural that the problem of the elastic stability of the shell in question be considered. In this section the stability of the shell will be approached as an eigenvalue problem. This type of approach was formulated in chapter IV, and the computer program used in this study will be the one developed in that chapter. Both models I and II will be used, and, in what follows, the eigenvalues will be referred to the load factor corresponding to the design wind speed. So an eigenvalue $\lambda = 1.0$ will imply that buckling occurs for a value of the factor $K = 2.51$ or a wind speed of 64 m/s. In this section, unless otherwise stated, the values given refer to mesh A without the intermediary stiffeners. These stiffeners were primarily eliminated to reduce the time of computation.

4.1 Model II

The first computer job to be run for this problem, referred to Model II and gave an eigenvalue surprisingly low. According with it the structure would buckle for a load factor approximately equal to 0.00875. Scrutiny of the eigenvector showed this buckling mode to correspond to a local instability in the front part of the upper stiffener. This buckling mode is represented in figure 10, where the out of plane deformation of the stiffener is represented. In view of the high compressive stresses to which this part of the stiffener is subjected, and once that no vertical stiffening brackets are provided for the stiffener itself, it can be said that this low eigenvalue should be expected. In the real structure, however, these stiffening brackets are provided and only for economical considerations they were not included in the analysis.
In order to eliminate this source of local instability, an attempt was made to modify the actual configuration of the stiffener, making it thicker but smaller. However, if the stiffener is simulated to a beam, this poses the impossible problem of keeping both values $EA$ and $EI_z$ as in the original stiffener. Perhaps not very convincingly, it was decided to consider a stiffener with a cross section of $0.1 \text{ m} \times 0.5 \text{ m}$, $0.5 \text{ m}$ being the length along the z axes; this resulted in an $I_z$ approximately 120 smaller than the original one, the area $A$, however, being approximately 3 times bigger. For this model a first eigenvalue $\lambda_1 = 0.42$ was found. The corresponding eigenvector is given in figure 11 and corresponds to the buckling of the front part of the base. As the nodes in the base-wall intersection are free up to $\theta = 36^0$ this eigenvalue is also understandable. The linear analysis with the modified stiffener gives, however, in the front part of the tank, much bigger displacements than the ones calculated using the original stiffener (for $\theta = 0$ degrees, $z = 23 \text{ m}$, e.g., x-displacement becomes 10 times bigger). This is natural since the value of $I_z$ is considerably reduced. On the other hand the maximum stress for element 101 becomes approximately 3 times smaller. This shows the difficulty of making this approximation representative of the behaviour of the original structure and as such another avenue will be considered.

In this third avenue the stiffener was kept in its original form, but its out-of-plane buckling possibility was eliminated. This was achieved by constraining the out-of-plane displacements of the free part of the stiffener to the values obtained from the linear analysis. The in-plane movement was let free, so allowing the stiffening effect to be considered. With this type of boundary conditions the following eigenvalues were determined:
\[ \lambda_1 = 0.053 \quad \lambda_2 = 0.056 \quad \lambda_3 = 0.074 \]

The corresponding eigenvectors are shown in figures 12 to 14 where it can be seen that the stiffener has in fact a marked restraining effect. The program has also shown that 15 eigenvalues exist from the first one up to a load factor of 3.5. This, together with the relatively low value obtained for the first eigenvalue shows that from the point of view of an eigenvalue analysis the structure as represented by Model II is relatively unstable.

In view of the low eigenvalues obtained it was decided to consider in the analysis the small stiffeners that had previously been eliminated. With these stiffeners on, a first eigenvalue \( \omega_1 = 0.0805 \) was calculated, that, of course, is not drastically higher than \( \lambda_1 \). However, a review of some literature connected with the buckling of stiffened shells (B-1), shows that the values obtained for the buckling load can be very deceiving and that a structure can be perfectly usable at loads above the one corresponding to the first eigenvalue.

4.2 Model I

In the buckling analysis carried out using this model, the out-of-plane movement of the stiffener was restrained to the values given by the linear analysis. The first 2 eigenvalues were calculated, these being

\[ \beta_1 = 0.501 \quad \beta_2 = 0.581 \]

The first buckling mode is shown in figure 15. It was also found that 7 eigenvalues exist after the first one and until a load factor of 3.5. A comparison of models I and II shows that for Model I the first eigenvalue is almost 10 times greater than for Model II and that the first part of the spectra of eigenvalues is more dislocated to the region of higher
loads. This certainly makes the structure as represented by Model I more stable than if Model II is used.

5. FREE VIBRATION ANALYSIS

A free vibration analysis was carried out for Model II only, and all 3 stiffeners were considered. The possible out-of-plane vibration of the wind girder was eliminated by restricting to zero all the $z$-displacements and rotations on the outward side of this stiffener. Using this assumption the natural frequency of the first vibrating mode was found to be

$$\gamma_1 = 8.57 \text{ cycles/sec}$$

The corresponding mode of vibration is given in Figure 16. This figure shows the vibration profile of the upper circumferential section and of the longitudinal sections corresponding to values of $\theta$ of $0^0$, $90^0$ and $180^0$. Other circumferential and longitudinal sections are basically identical to the ones shown. As the figure shows, the circumferential stiffeners have a marked effect on the vibration mode. The fact that the front part of the tank is unrestrained vertically does not seem to influence the frequency of vibration significantly. Apart from a negligible $z$ component for the vertical section corresponding to $\theta = 0^0$, the vibrating mode obtained shows a remarkable symmetry in relation to the shell axes.

6. GEOMETRICALLY NONLINEAR ANALYSIS

This analysis was carried out using mesh B and both models I and II. The method of analysis is presented in Chapter V, and the finite element computer program used is also the one developed in this chapter. In order not to increase the computer times excessively the tolerance for convergence (chapter V, section 3.4) was set at 10\% for
all the load increments. Since the load used for the linear analysis is
well above the design load, only a part of this load was considered for
the nonlinear analysis. The applied load will usually be referred to as a
"Load Factor", a load factor of 1.0 e.g. corresponding to $K = 2.51$ and a
wind speed of 64 m/s.

6.1 Model I

The analysis employing this model was carried out up to a
load factor of 1.575. Table 3 shows the increments of load considered,
these increasing from 0.07 up to 0.175, and the type of convergence observed.
The convergence can be seen to be relatively fast with only 3 or 4 iterations
being needed for equilibrium.

Figures 17 to 20 show curves of displacement versus load for
various points of the structure, these being:-

- Figure 17 - nodal points along longitudinal section $\theta = 0^0$
- Figure 18 - nodal points along longitudinal section $\theta = 72^0$
- Figure 19 - nodal points along longitudinal section $\theta = 180^0$
- Figure 20 - nodal points along the wind girder

An analysis of these figures shows that considerable differences
can exist between the displacements obtained from the linear computations and
nonlinear large deflection analysis. This is particularly true for points
situated in the intermediate position between the base and the wind
girder, showing generally a "loss of stiffness" in this area. For the wind
girder, figure 20 there is no great difference between the linear and non-
linear cases. This is possibly due to the fact that in-plane deformation
is predominant in this region.
6.2 Model II

The analysis of this model was primarily carried out up to a load factor of 2.345. Displacement versus load curves for some representative points of the structure are given in figures 21 and 23 to 25, as follows:

- nodal points along longitudinal section \( \theta = 0^0 \) on figure 21
- nodal points along longitudinal section \( \theta = 72^0 \) on figure 23
- nodal points along longitudinal section \( \theta = 180^0 \) on figure 24
- nodal points on the stiffener for \( \theta = 72^0, 180^0 \) on figure 25

Considering the front section of the tank (\( \theta = 0^0 \)), figure 21 a, b, shows an initial loss of stiffness, a subsequent recuperation and another more pronounced fall in the gradient of the load-deflection curve. This low gradient can also be observed in figure 21 c, d, e, f. It is interesting to note the very high loss of stiffness in the \( z \) direction observed in figure 21 f which occurs from the commencement of loading. This is consistent with the extremely low buckling mode corresponding to a local buckling in the wind girder calculated in section 4. It should also be noted that the very high displacements observed in figure 21 f correspond in physical terms to a reduction in the effect of the stiffener, as shown in figure 22.

Points on or above the wind girder (figures 23 a, b and 24 a, b) also show very low load/displacement gradients, especially for load factors above 1.4. Points below the wind girder (figures 23 c, d, 24 c, d) show low load/displacement gradients from the commencement of loading, with a stiffening action developing at higher load values.

Finally an analysis of figure 25 (points on the wind girder) show low initial load/displacement gradients followed by a recuperation
of stiffness. It should be noticed that in the real structure the wind
girder is itself reinforced. If this reinforcement were to be included
in analysis the character of the curves of figures 21f and 25 would
most certainly be changed.

At this point perhaps it is interesting to note that low
performance elements can for certain cases present some advantages over
the more powerful high performance elements. This happens when, like
in this case, the structure has fine geometrical detail that, if represented
by high performance elements would make the analysis too uneconomical.

Figure 26 and 27 show some deflection profiles for load factors
of 0.35 and 1.0. A study of the vertical profiles corresponding to
\( \theta = 72^0 \) and \( \theta = 180^0 \) gives a better understanding of figures 23 a, 24 a. In
fact these figures show an initial reversal in the direction of the
displacements when the linear and the nonlinear cases are compared.
As figure 27 suggests this appears to be due to the fact that the
displacements and rotations of the shell below the wind girder have a
strong influence on the behaviour of the shell above the stiffener. This
seems to be natural, in view of the smallness of the part of the shell
situated above the stiffener.

The stresses corresponding to a load factor 1.0 (design load)
are all below the maximum allowable stress. The maximum stress occurs in
the wind girder where a combined stress of \( 150 \times 10^3 \) KN/m\(^2\) is obtained.
Table 4 gives the membrane, bending and combined hoop stresses for the
Gauss integrating points along the stiffener. The stresses in the shell
itself are smaller than those in the wind girder, the maximum combined
stress being equal to \( 90 \times 10^3 \) KN/m\(^2\) in compression. The other stresses
in the shell are generally well below this value. Table 5 gives the longi-
tudinal stresses on the windward side of the shell for a load factor of 1.0.
Convergence of the solution for Model II was found to be slower than for Model I. In fact, after the program failed to converge for an increment in the load factor of 0.07, it was decided to consider all the load increments equal to 0.035. With these load increments convergence was achieved usually after 3 iterations.

This relative difficulty in achieving convergence and the much lower load/displacement gradients observed for Model II show, once again, that the lifting of the base of the tank has a profound detrimental effect on the behaviour of the structure.

The loading of Model II was increased incrementally in excess of the value corresponding to a load factor of 2.345. Increase in deflections in the critical regions of the structure occurred at an accelerating rate. For example the deflection in the x direction at the top of the tank for $\theta = 0^0$ increased from approximately 0.4 m to 1.2 m as the load factor increased from 2.345 to 3.43. It can be concluded that the structure is no longer serviceable under such gross distortions. At this load level, the stresses at various points in the structure are well in excess of the plastic limit.

7. CONCLUSIONS

The Semiloof shell element has been applied to solve a "large" reinforced thin shell problem using different types of approaches. No numerical difficulties were experienced in obtaining any of the solutions, and for the analysis where two types of meshes were used a good correlation of results was generally observed.

It was shown that an eigenvalue analysis of a reinforced shell can lead to a very low buckling load this not representing the actual load
carrying capacity of the structure. On the other hand a simple linear analysis can also lead to results considerably different from the ones yielded by a full nonlinear analysis.

The influence on the behaviour of the structure of the changing of some boundary conditions was also studied and it was shown the structure to be more stable for a particular case of boundary conditions.
BIBLIOGRAPHY

A-1 Albuquerque, F.
A beam element for use with the Semiloof shell element

A-2 Adams, J. H.
The study of wind girder requirements for large A.P.I.650
floating roof tanks. A.P.I. Refining, 40th Midyear Meeting,

B-1 Brush, D. O. and Almroth, B. O.
<table>
<thead>
<tr>
<th>Element</th>
<th>Max. combined stress (KN/m²)</th>
<th>Direction cosines</th>
<th>Gauss point</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>± 31504</td>
<td>-0.978 -0.207 0.000</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>± 29461</td>
<td>-0.956 0.293 0.000</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>± 29193</td>
<td>-0.801 0.598 0.000</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>± 28660</td>
<td>-0.571 0.821 0.000</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>± 28485</td>
<td>-0.022 1.000 0.000</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>± 28223</td>
<td>0.289 0.957 0.000</td>
<td>2</td>
</tr>
<tr>
<td>7</td>
<td>± 28949</td>
<td>0.573 0.819 0.000</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>± 25261</td>
<td>0.802 0.598 0.000</td>
<td>2</td>
</tr>
<tr>
<td>9</td>
<td>± 29199</td>
<td>0.956 0.293 0.000</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>± 30899</td>
<td>0.977 -0.212 0.000</td>
<td>2</td>
</tr>
<tr>
<td>11</td>
<td>18560</td>
<td>0.014 0.216 0.946</td>
<td>11</td>
</tr>
<tr>
<td>12</td>
<td>-18032</td>
<td>-0.373 -0.593 0.713</td>
<td>20</td>
</tr>
<tr>
<td>13</td>
<td>18480</td>
<td>0.488 0.589 0.644</td>
<td>40</td>
</tr>
<tr>
<td>14</td>
<td>21904</td>
<td>0.763 0.480 0.432</td>
<td>40</td>
</tr>
<tr>
<td>15</td>
<td>21487</td>
<td>0.967 0.244 0.075</td>
<td>40</td>
</tr>
<tr>
<td>16</td>
<td>20111</td>
<td>0.866 -0.222 -0.428</td>
<td>30</td>
</tr>
<tr>
<td>17</td>
<td>16053</td>
<td>0.803 -0.321 -0.50</td>
<td>40</td>
</tr>
<tr>
<td>18</td>
<td>10638</td>
<td>0.609 -0.505 -0.612</td>
<td>40</td>
</tr>
<tr>
<td>19</td>
<td>5532</td>
<td>0.450 -0.714 -0.536</td>
<td>40</td>
</tr>
<tr>
<td>20</td>
<td>4370</td>
<td>0.066 -0.996 0.058</td>
<td>30</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Element</th>
<th>Max. combined stress (KN/m²)</th>
<th>Direction cosines</th>
<th>Gauss point</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>14153</td>
<td>0.113 0.446 0.888</td>
<td>20</td>
</tr>
<tr>
<td>22</td>
<td>16659</td>
<td>0.388 0.616 0.685</td>
<td>20</td>
</tr>
<tr>
<td>23</td>
<td>20162</td>
<td>0.645 0.535 0.546</td>
<td>21</td>
</tr>
<tr>
<td>24</td>
<td>21045</td>
<td>0.904 0.362 0.226</td>
<td>30</td>
</tr>
<tr>
<td>25</td>
<td>21482</td>
<td>0.966 0.244 0.081</td>
<td>40</td>
</tr>
<tr>
<td>26</td>
<td>20580</td>
<td>0.968 -0.064 -0.243</td>
<td>40</td>
</tr>
<tr>
<td>27</td>
<td>16791</td>
<td>0.796 -0.318 -0.515</td>
<td>11</td>
</tr>
<tr>
<td>28</td>
<td>11173</td>
<td>0.611 -0.507 -0.608</td>
<td>11</td>
</tr>
<tr>
<td>29</td>
<td>5454</td>
<td>0.491 -0.781 -0.396</td>
<td>40</td>
</tr>
<tr>
<td>30</td>
<td>4413</td>
<td>0.066 -0.997 0.033</td>
<td>30</td>
</tr>
<tr>
<td>31</td>
<td>-16249</td>
<td>0.066 0.984 -0.164</td>
<td>41</td>
</tr>
<tr>
<td>32</td>
<td>16808</td>
<td>0.374 0.595 0.711</td>
<td>20</td>
</tr>
<tr>
<td>33</td>
<td>22836</td>
<td>0.672 0.557 0.458</td>
<td>31</td>
</tr>
<tr>
<td>34</td>
<td>27975</td>
<td>0.904 0.361 0.227</td>
<td>31</td>
</tr>
<tr>
<td>35</td>
<td>28065</td>
<td>0.968 0.245 0.053</td>
<td>41</td>
</tr>
<tr>
<td>36</td>
<td>27408</td>
<td>0.969 -0.065 -0.237</td>
<td>41</td>
</tr>
<tr>
<td>37</td>
<td>21264</td>
<td>0.850 -0.340 -0.403</td>
<td>41</td>
</tr>
<tr>
<td>38</td>
<td>13290</td>
<td>0.684 -0.567 -0.459</td>
<td>41</td>
</tr>
<tr>
<td>39</td>
<td>7292</td>
<td>0.504 -0.801 -0.322</td>
<td>41</td>
</tr>
<tr>
<td>40</td>
<td>6152</td>
<td>0.066 -0.997 0.039</td>
<td>31</td>
</tr>
</tbody>
</table>

TABLE 1  STRESSES FOR MODEL I
<table>
<thead>
<tr>
<th>Gauss point</th>
<th>Direction cosines</th>
<th>Element Max. stress (MN/m²)</th>
<th>Element Max. combined stress (MN/m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>41</td>
<td>0.2225</td>
<td>0.890</td>
<td>0.396</td>
</tr>
<tr>
<td>42</td>
<td>0.309</td>
<td>0.773</td>
<td>0.554</td>
</tr>
<tr>
<td>43</td>
<td>0.2876</td>
<td>0.580</td>
<td>0.164</td>
</tr>
<tr>
<td>44</td>
<td>0.36759</td>
<td>0.366</td>
<td>0.164</td>
</tr>
<tr>
<td>45</td>
<td>0.36865</td>
<td>0.094</td>
<td>0.094</td>
</tr>
<tr>
<td>46</td>
<td>0.3596</td>
<td>0.982</td>
<td>0.175</td>
</tr>
<tr>
<td>47</td>
<td>0.2761</td>
<td>0.827</td>
<td>0.351</td>
</tr>
<tr>
<td>48</td>
<td>0.17640</td>
<td>0.827</td>
<td>0.351</td>
</tr>
<tr>
<td>49</td>
<td>0.10655</td>
<td>0.516</td>
<td>0.248</td>
</tr>
<tr>
<td>50</td>
<td>0.591</td>
<td>0.235</td>
<td>0.021</td>
</tr>
<tr>
<td>51</td>
<td>2.3641</td>
<td>0.235</td>
<td>0.021</td>
</tr>
<tr>
<td>52</td>
<td>2.8215</td>
<td>0.461</td>
<td>0.887</td>
</tr>
<tr>
<td>53</td>
<td>2.3779</td>
<td>0.717</td>
<td>0.867</td>
</tr>
<tr>
<td>54</td>
<td>3.9220</td>
<td>0.352</td>
<td>0.062</td>
</tr>
<tr>
<td>55</td>
<td>4.0974</td>
<td>0.997</td>
<td>0.089</td>
</tr>
<tr>
<td>56</td>
<td>3.9852</td>
<td>0.352</td>
<td>0.062</td>
</tr>
<tr>
<td>57</td>
<td>2.7990</td>
<td>0.997</td>
<td>0.089</td>
</tr>
<tr>
<td>58</td>
<td>1.7195</td>
<td>0.705</td>
<td>0.649</td>
</tr>
<tr>
<td>59</td>
<td>1.1746</td>
<td>0.528</td>
<td>0.494</td>
</tr>
<tr>
<td>60</td>
<td>1.2200</td>
<td>0.0113</td>
<td>0.094</td>
</tr>
<tr>
<td>Element</td>
<td>Max. combined stress (KN/m²)</td>
<td>Direction cosines</td>
<td>Gauss point</td>
</tr>
<tr>
<td>---------</td>
<td>-----------------------------</td>
<td>-------------------</td>
<td>-------------</td>
</tr>
<tr>
<td>81</td>
<td>- 35894</td>
<td>0.238 0.942</td>
<td>- 0.235</td>
</tr>
<tr>
<td>82</td>
<td>- 23372</td>
<td>0.345 0.862</td>
<td>- 0.372</td>
</tr>
<tr>
<td>83</td>
<td>32930</td>
<td>0.757 0.628</td>
<td>0.182</td>
</tr>
<tr>
<td>84</td>
<td>50183</td>
<td>0.844 0.531</td>
<td>0.068</td>
</tr>
<tr>
<td>85</td>
<td>56990</td>
<td>0.968 0.245</td>
<td>- 0.058</td>
</tr>
<tr>
<td>86</td>
<td>46333</td>
<td>0.965 -0.244</td>
<td>- 0.097</td>
</tr>
<tr>
<td>87</td>
<td>32602</td>
<td>0.921 -0.369</td>
<td>- 0.127</td>
</tr>
<tr>
<td>88</td>
<td>18763</td>
<td>0.762 0.632</td>
<td>- 0.143</td>
</tr>
<tr>
<td>89</td>
<td>11632</td>
<td>0.363 0.908</td>
<td>- 0.211</td>
</tr>
<tr>
<td>90</td>
<td>11472</td>
<td>0.245 0.969</td>
<td>-0.033</td>
</tr>
<tr>
<td>91</td>
<td>61568</td>
<td>0.971 -0.238</td>
<td>0.000</td>
</tr>
<tr>
<td>92</td>
<td>- 60101</td>
<td>0.966 -0.258</td>
<td>0.000</td>
</tr>
<tr>
<td>93</td>
<td>59741</td>
<td>0.818 0.575</td>
<td>0.000</td>
</tr>
<tr>
<td>94</td>
<td>- 61380</td>
<td>0.603 0.798</td>
<td>0.000</td>
</tr>
<tr>
<td>95</td>
<td>- 60084</td>
<td>0.003 1.000</td>
<td>0.000</td>
</tr>
<tr>
<td>96</td>
<td>- 60506</td>
<td>0.319 0.947</td>
<td>0.000</td>
</tr>
<tr>
<td>97</td>
<td>- 60004</td>
<td>0.593 0.805</td>
<td>0.000</td>
</tr>
<tr>
<td>98</td>
<td>- 58949</td>
<td>0.813 0.583</td>
<td>0.000</td>
</tr>
<tr>
<td>99</td>
<td>- 59853</td>
<td>0.829 0.559</td>
<td>0.000</td>
</tr>
<tr>
<td>100</td>
<td>- 60825</td>
<td>0.979 0.202</td>
<td>0.000</td>
</tr>
</tbody>
</table>

**TABLE 1 CONTINUED**
<table>
<thead>
<tr>
<th>Element</th>
<th>Max. combined stress (KN/m²)</th>
<th>Direction cosines</th>
<th>Gauss Point</th>
<th>Element</th>
<th>Max. combined stress (KN/m²)</th>
<th>Direction cosines</th>
<th>Gauss Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>40988</td>
<td>-0.540 0.841 0.000</td>
<td>3T</td>
<td>21</td>
<td>-15897</td>
<td>0.066 0.987 0.145</td>
<td>40</td>
</tr>
<tr>
<td>2</td>
<td>40003</td>
<td>-0.611 0.911 0.000</td>
<td>4T</td>
<td>22</td>
<td>37889</td>
<td>0.058 0.091 0.994</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>±29287</td>
<td>-0.613 0.790 0.000</td>
<td>2</td>
<td>23</td>
<td>64696</td>
<td>0.288 0.348 0.893</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>±28755</td>
<td>-0.574 0.819 0.000</td>
<td>1</td>
<td>24</td>
<td>-28047</td>
<td>-0.175 -0.110 0.978</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>±28551</td>
<td>-0.022 1.000 0.000</td>
<td>2</td>
<td>25</td>
<td>21320</td>
<td>0.985 0.066 -0.160</td>
<td>30</td>
</tr>
<tr>
<td>6</td>
<td>±28716</td>
<td>0.289 0.957 0.000</td>
<td>2</td>
<td>26</td>
<td>20180</td>
<td>0.966 -0.064 -0.251</td>
<td>40</td>
</tr>
<tr>
<td>7</td>
<td>±28944</td>
<td>0.573 0.819 0.000</td>
<td>2</td>
<td>27</td>
<td>16363</td>
<td>0.812 -0.325 0.485</td>
<td>11</td>
</tr>
<tr>
<td>8</td>
<td>±29262</td>
<td>0.812 0.598 0.000</td>
<td>2</td>
<td>28</td>
<td>11135</td>
<td>0.620 -0.514 -0.592</td>
<td>11</td>
</tr>
<tr>
<td>9</td>
<td>±29199</td>
<td>0.956 0.293 0.000</td>
<td>2</td>
<td>29</td>
<td>5464</td>
<td>0.490 -0.977 -0.390</td>
<td>40</td>
</tr>
<tr>
<td>10</td>
<td>±30898</td>
<td>0.977 -0.213 0.000</td>
<td>2</td>
<td>30</td>
<td>4429</td>
<td>0.066 -0.997 0.038</td>
<td>30</td>
</tr>
<tr>
<td>11</td>
<td>18229</td>
<td>0.086 0.340 0.937</td>
<td>2I</td>
<td>31</td>
<td>-20232</td>
<td>0.064 0.958 0.280</td>
<td>40</td>
</tr>
<tr>
<td>12</td>
<td>30090</td>
<td>-0.171 -0.271 0.947</td>
<td>30</td>
<td>32</td>
<td>46265</td>
<td>0.066 0.106 0.992</td>
<td>20</td>
</tr>
<tr>
<td>13</td>
<td>81424</td>
<td>0.278 0.335 0.900</td>
<td>40</td>
<td>33</td>
<td>61134</td>
<td>0.344 0.415 0.842</td>
<td>10</td>
</tr>
<tr>
<td>14</td>
<td>-22393</td>
<td>-0.243 -0.097 0.965</td>
<td>2I</td>
<td>34</td>
<td>-34939</td>
<td>-0.178 -0.112 0.978</td>
<td>10</td>
</tr>
<tr>
<td>15</td>
<td>21484</td>
<td>0.967 0.245 -0.065</td>
<td>40</td>
<td>35</td>
<td>28629</td>
<td>0.968 0.245 -0.050</td>
<td>41</td>
</tr>
<tr>
<td>16</td>
<td>20332</td>
<td>0.945 -0.063 -0.321</td>
<td>40</td>
<td>36</td>
<td>26478</td>
<td>0.974 -0.065 -0.218</td>
<td>41</td>
</tr>
<tr>
<td>17</td>
<td>16178</td>
<td>0.805 -0.322 -0.499</td>
<td>40</td>
<td>37</td>
<td>20805</td>
<td>0.864 -0.346 -0.366</td>
<td>41</td>
</tr>
<tr>
<td>18</td>
<td>10873</td>
<td>0.604 -0.501 -0.620</td>
<td>40</td>
<td>38</td>
<td>13329</td>
<td>0.689 -0.571 -0.445</td>
<td>41</td>
</tr>
<tr>
<td>19</td>
<td>5685</td>
<td>0.440 -0.700 -0.563</td>
<td>40</td>
<td>39</td>
<td>7316</td>
<td>0.504 -0.801 -0.321</td>
<td>41</td>
</tr>
<tr>
<td>20</td>
<td>4364</td>
<td>0.056 -0.995 0.070</td>
<td>30</td>
<td>40</td>
<td>5953</td>
<td>0.245 -0.969 -0.015</td>
<td>41</td>
</tr>
</tbody>
</table>

TABLE 2 STRESSES FOR MODEL II
<table>
<thead>
<tr>
<th>Element</th>
<th>Max. combined stress (KN/m²)</th>
<th>Direction Cosines</th>
<th>Gauss Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>41</td>
<td>- 27320</td>
<td>0.245 0.968 0.050</td>
<td>20</td>
</tr>
<tr>
<td>42</td>
<td>43342</td>
<td>0.037 0.058 0.998</td>
<td>20</td>
</tr>
<tr>
<td>43</td>
<td>66069</td>
<td>0.386 0.468 0.796</td>
<td>10</td>
</tr>
<tr>
<td>44</td>
<td>37066</td>
<td>0.839 0.528 0.130</td>
<td>41</td>
</tr>
<tr>
<td>45</td>
<td>30176</td>
<td>0.968 0.245 -0.054</td>
<td>41</td>
</tr>
<tr>
<td>46</td>
<td>33891</td>
<td>0.985 -0.066 -0.160</td>
<td>41</td>
</tr>
<tr>
<td>47</td>
<td>26919</td>
<td>0.886 -0.354 -0.300</td>
<td>41</td>
</tr>
<tr>
<td>48</td>
<td>17677</td>
<td>0.717 -0.595 -0.362</td>
<td>41</td>
</tr>
<tr>
<td>49</td>
<td>10471</td>
<td>0.515 -0.819 -0.253</td>
<td>41</td>
</tr>
<tr>
<td>50</td>
<td>2674</td>
<td>0.245 -0.969 -0.009</td>
<td>41</td>
</tr>
<tr>
<td>51</td>
<td>- 71801</td>
<td>0.304 0.953 0.000</td>
<td>1T</td>
</tr>
<tr>
<td>52</td>
<td>76230</td>
<td>0.320 0.947 0.000</td>
<td>2T</td>
</tr>
<tr>
<td>53</td>
<td>83737</td>
<td>0.778 0.628 0.000</td>
<td>1T</td>
</tr>
<tr>
<td>54</td>
<td>40653</td>
<td>0.931 0.365 0.000</td>
<td>2B</td>
</tr>
<tr>
<td>55</td>
<td>39856</td>
<td>0.989 0.149 0.000</td>
<td>3B</td>
</tr>
<tr>
<td>56</td>
<td>40342</td>
<td>0.986 -0.151 0.000</td>
<td>1B</td>
</tr>
<tr>
<td>57</td>
<td>27010</td>
<td>0.920 -0.391 0.000</td>
<td>1T</td>
</tr>
<tr>
<td>58</td>
<td>17251</td>
<td>0.758 -0.650 0.000</td>
<td>4B</td>
</tr>
<tr>
<td>59</td>
<td>11780</td>
<td>0.518 -0.856 0.000</td>
<td>4B</td>
</tr>
<tr>
<td>60</td>
<td>12269</td>
<td>0.122 -0.993 0.000</td>
<td>2B</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Element</th>
<th>Max. combined stress (KN/m²)</th>
<th>Direction Cosines</th>
<th>Gauss Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>61</td>
<td>- 72062</td>
<td>0.207 0.820 0.533</td>
<td>20</td>
</tr>
<tr>
<td>62</td>
<td>- 58523</td>
<td>0.361 0.902 0.237</td>
<td>41</td>
</tr>
<tr>
<td>63</td>
<td>70664</td>
<td>0.630 0.523 0.574</td>
<td>21</td>
</tr>
<tr>
<td>64</td>
<td>58149</td>
<td>0.831 0.523 0.188</td>
<td>41</td>
</tr>
<tr>
<td>65</td>
<td>46917</td>
<td>0.996 0.066 -0.064</td>
<td>31</td>
</tr>
<tr>
<td>66</td>
<td>42172</td>
<td>0.989 -0.066 -0.131</td>
<td>41</td>
</tr>
<tr>
<td>67</td>
<td>33670</td>
<td>0.915 -0.366 -0.171</td>
<td>41</td>
</tr>
<tr>
<td>68</td>
<td>21117</td>
<td>0.755 -0.626 -0.196</td>
<td>41</td>
</tr>
<tr>
<td>69</td>
<td>12825</td>
<td>0.528 -0.839 -0.133</td>
<td>41</td>
</tr>
<tr>
<td>70</td>
<td>16554</td>
<td>0.245 -0.969 -0.028</td>
<td>42</td>
</tr>
<tr>
<td>71</td>
<td>- 95962</td>
<td>0.330 0.944 0.000</td>
<td>1T</td>
</tr>
<tr>
<td>72</td>
<td>93047</td>
<td>0.258 0.966 0.000</td>
<td>2T</td>
</tr>
<tr>
<td>73</td>
<td>91191</td>
<td>0.811 0.585 0.000</td>
<td>1T</td>
</tr>
<tr>
<td>74</td>
<td>56571</td>
<td>0.763 0.546 0.000</td>
<td>4T</td>
</tr>
<tr>
<td>75</td>
<td>51314</td>
<td>0.992 0.125 0.000</td>
<td>2B</td>
</tr>
<tr>
<td>76</td>
<td>46421</td>
<td>0.994 -0.106 0.000</td>
<td>13</td>
</tr>
<tr>
<td>77</td>
<td>34160</td>
<td>0.930 -0.368 0.000</td>
<td>1T</td>
</tr>
<tr>
<td>78</td>
<td>23011</td>
<td>0.787 -0.621 0.000</td>
<td>4T</td>
</tr>
<tr>
<td>79</td>
<td>16067</td>
<td>0.523 -0.852 0.000</td>
<td>4B</td>
</tr>
<tr>
<td>80</td>
<td>16409</td>
<td>0.104 -0.995 0.000</td>
<td>2B</td>
</tr>
</tbody>
</table>

*TABLE 2 continued*
<table>
<thead>
<tr>
<th>Element No.</th>
<th>Max. Combined Stress (ksi)</th>
<th>101</th>
<th>201</th>
<th>301</th>
<th>401</th>
<th>501</th>
<th>601</th>
<th>701</th>
<th>801</th>
<th>901</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>81</td>
<td>179140</td>
<td>0.215</td>
<td>0.680</td>
<td>0.851</td>
<td>20</td>
<td>101</td>
<td>1.222</td>
<td>0.099</td>
<td>0.066</td>
<td>0.046</td>
<td>0.024</td>
</tr>
<tr>
<td>82</td>
<td>195050</td>
<td>0.523</td>
<td>0.331</td>
<td>0.190</td>
<td>30</td>
<td>102</td>
<td>0.055</td>
<td>0.045</td>
<td>0.055</td>
<td>0.055</td>
<td>0.055</td>
</tr>
<tr>
<td>83</td>
<td>128810</td>
<td>0.640</td>
<td>0.550</td>
<td>0.290</td>
<td>40</td>
<td>103</td>
<td>0.769</td>
<td>0.769</td>
<td>0.769</td>
<td>0.769</td>
<td>0.769</td>
</tr>
<tr>
<td>84</td>
<td>8017</td>
<td>0.610</td>
<td>0.245</td>
<td>0.069</td>
<td>41</td>
<td>104</td>
<td>0.927</td>
<td>0.927</td>
<td>0.927</td>
<td>0.927</td>
<td>0.927</td>
</tr>
<tr>
<td>85</td>
<td>16440</td>
<td>0.967</td>
<td>0.967</td>
<td>0.967</td>
<td>42</td>
<td>105</td>
<td>0.999</td>
<td>0.999</td>
<td>0.999</td>
<td>0.999</td>
<td>0.999</td>
</tr>
<tr>
<td>86</td>
<td>50313</td>
<td>0.851</td>
<td>0.851</td>
<td>0.851</td>
<td>43</td>
<td>106</td>
<td>0.371</td>
<td>0.371</td>
<td>0.371</td>
<td>0.371</td>
<td>0.371</td>
</tr>
<tr>
<td>87</td>
<td>18590</td>
<td>0.362</td>
<td>0.362</td>
<td>0.362</td>
<td>44</td>
<td>107</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>88</td>
<td>15133</td>
<td>0.359</td>
<td>0.359</td>
<td>0.359</td>
<td>45</td>
<td>108</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>89</td>
<td>11519</td>
<td>0.362</td>
<td>0.362</td>
<td>0.362</td>
<td>46</td>
<td>109</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>90</td>
<td>11575</td>
<td>0.362</td>
<td>0.362</td>
<td>0.362</td>
<td>47</td>
<td>110</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>91</td>
<td>269650</td>
<td>0.362</td>
<td>0.362</td>
<td>0.362</td>
<td>48</td>
<td>111</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>92</td>
<td>190950</td>
<td>0.362</td>
<td>0.362</td>
<td>0.362</td>
<td>49</td>
<td>112</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>93</td>
<td>195580</td>
<td>0.362</td>
<td>0.362</td>
<td>0.362</td>
<td>50</td>
<td>113</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>94</td>
<td>108700</td>
<td>0.362</td>
<td>0.362</td>
<td>0.362</td>
<td>51</td>
<td>114</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>95</td>
<td>67150</td>
<td>0.362</td>
<td>0.362</td>
<td>0.362</td>
<td>52</td>
<td>115</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>96</td>
<td>61239</td>
<td>0.362</td>
<td>0.362</td>
<td>0.362</td>
<td>53</td>
<td>116</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>97</td>
<td>60251</td>
<td>0.362</td>
<td>0.362</td>
<td>0.362</td>
<td>54</td>
<td>117</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>98</td>
<td>59044</td>
<td>0.362</td>
<td>0.362</td>
<td>0.362</td>
<td>55</td>
<td>118</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>99</td>
<td>59845</td>
<td>0.362</td>
<td>0.362</td>
<td>0.362</td>
<td>56</td>
<td>119</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Table 2 continued
<table>
<thead>
<tr>
<th>Increm. number</th>
<th>Increm. Load</th>
<th>Total Load</th>
<th>Initial Residual</th>
<th>Number of iterations</th>
<th>Final Residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.07</td>
<td>0.07</td>
<td>188.7</td>
<td>4</td>
<td>7.2</td>
</tr>
<tr>
<td>2</td>
<td>0.07</td>
<td>0.14</td>
<td>162.8</td>
<td>4</td>
<td>6.2</td>
</tr>
<tr>
<td>3</td>
<td>0.07</td>
<td>0.21</td>
<td>200.0</td>
<td>4</td>
<td>3.8</td>
</tr>
<tr>
<td>4</td>
<td>0.07</td>
<td>0.28</td>
<td>285.1</td>
<td>3</td>
<td>8.3</td>
</tr>
<tr>
<td>5</td>
<td>0.105</td>
<td>0.385</td>
<td>492.8</td>
<td>3</td>
<td>8.9</td>
</tr>
<tr>
<td>6</td>
<td>0.105</td>
<td>0.49</td>
<td>549.8</td>
<td>4</td>
<td>2.9</td>
</tr>
<tr>
<td>7</td>
<td>0.14</td>
<td>0.63</td>
<td>831.7</td>
<td>4</td>
<td>5.7</td>
</tr>
<tr>
<td>8</td>
<td>0.14</td>
<td>0.77</td>
<td>979.4</td>
<td>3</td>
<td>9.8</td>
</tr>
<tr>
<td>9</td>
<td>0.14</td>
<td>0.91</td>
<td>1081.8</td>
<td>3</td>
<td>4.2</td>
</tr>
<tr>
<td>10</td>
<td>0.14</td>
<td>1.05</td>
<td>1093.4</td>
<td>3</td>
<td>5.4</td>
</tr>
<tr>
<td>11</td>
<td>0.175</td>
<td>1.225</td>
<td>1318.8</td>
<td>3</td>
<td>9.5</td>
</tr>
<tr>
<td>12</td>
<td>0.175</td>
<td>1.4</td>
<td>1222.7</td>
<td>3</td>
<td>7.4</td>
</tr>
<tr>
<td>13</td>
<td>0.175</td>
<td>1.575</td>
<td>1127.3</td>
<td>3</td>
<td>5.9</td>
</tr>
</tbody>
</table>

**TABLE 3 LOAD INCREMENTS AND CONVERGENCE FOR MODEL I - LARGE DEFLECTIONS ANALYSIS**
<table>
<thead>
<tr>
<th>$\theta$</th>
<th>Memb.</th>
<th>Bend. (bottom)</th>
<th>Comb. (max.)</th>
<th>Memb.</th>
<th>Bend. (bottom)</th>
<th>Comb. (max)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1</td>
<td>-17.6</td>
<td>16.8</td>
<td>-34.4</td>
<td>-44.9</td>
<td>3.6</td>
<td>-48.5</td>
</tr>
<tr>
<td>18.9</td>
<td>2.2</td>
<td>4.9</td>
<td>7.1</td>
<td>19.9</td>
<td>-3.4</td>
<td>23.3</td>
</tr>
<tr>
<td>29.1</td>
<td>-86.3</td>
<td>14.9</td>
<td>-101.2</td>
<td>35.4</td>
<td>9.2</td>
<td>44.6</td>
</tr>
<tr>
<td>42.9</td>
<td>-80.4</td>
<td>16.9</td>
<td>-97.3</td>
<td>35.9</td>
<td>6.9</td>
<td>42.8</td>
</tr>
<tr>
<td>53.1</td>
<td>15.7</td>
<td>9.4</td>
<td>25.1</td>
<td>13.9</td>
<td>-8.9</td>
<td>22.8</td>
</tr>
<tr>
<td>66.9</td>
<td>13.5</td>
<td>3.8</td>
<td>17.3</td>
<td>-18.1</td>
<td>-7.6</td>
<td>-25.7</td>
</tr>
<tr>
<td>79.6</td>
<td>-0.8</td>
<td>6.9</td>
<td>7.7</td>
<td>2.5</td>
<td>3.0</td>
<td>5.5</td>
</tr>
<tr>
<td>100.4</td>
<td>-1.6</td>
<td>4.3</td>
<td>5.9</td>
<td>3.7</td>
<td>0.2</td>
<td>3.9</td>
</tr>
<tr>
<td>115.6</td>
<td>2.2</td>
<td>8.0</td>
<td>10.2</td>
<td>0.6</td>
<td>3.5</td>
<td>4.1</td>
</tr>
<tr>
<td>136.4</td>
<td>1.0</td>
<td>5.9</td>
<td>6.9</td>
<td>-1.2</td>
<td>1.2</td>
<td>-2.4</td>
</tr>
<tr>
<td>151.6</td>
<td>-0.7</td>
<td>6.3</td>
<td>7.0</td>
<td>-0.3</td>
<td>1.8</td>
<td>-2.1</td>
</tr>
<tr>
<td>172.4</td>
<td>-1.6</td>
<td>3.9</td>
<td>5.5</td>
<td>0.6</td>
<td>0.6</td>
<td>1.2</td>
</tr>
</tbody>
</table>

**Table 4** Hoop Stresses Along the Wind Girder for L.F. = 1.0
<table>
<thead>
<tr>
<th>θ</th>
<th>z</th>
<th>1.83</th>
<th>6.85</th>
<th>10.53</th>
<th>15.57</th>
<th>18.37</th>
<th>20.98</th>
<th>22.16</th>
<th>22.77</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1</td>
<td>-0.6</td>
<td>-1.6</td>
<td>-3.6</td>
<td>-5.1</td>
<td>-4.8</td>
<td>1.0</td>
<td>26.7</td>
<td>-12.8</td>
<td></td>
</tr>
<tr>
<td>18.9</td>
<td>2.4</td>
<td>0.8</td>
<td>-0.1</td>
<td>5.1</td>
<td>-6.2</td>
<td>5.8</td>
<td>-39.2</td>
<td>-31.5</td>
<td></td>
</tr>
<tr>
<td>29.1</td>
<td>2.4</td>
<td>6.6</td>
<td>3.6</td>
<td>-9.7</td>
<td>21.7</td>
<td>-32.1</td>
<td>25.3</td>
<td>52.7</td>
<td></td>
</tr>
<tr>
<td>42.9</td>
<td>15.8</td>
<td>10.2</td>
<td>18.6</td>
<td>5.9</td>
<td>35.3</td>
<td>-22.7</td>
<td>21.8</td>
<td>63.2</td>
<td></td>
</tr>
<tr>
<td>53.1</td>
<td>-15.1</td>
<td>-15.0</td>
<td>-18.9</td>
<td>-16.3</td>
<td>-10.3</td>
<td>-8.0</td>
<td>15.4</td>
<td>-24.5</td>
<td></td>
</tr>
<tr>
<td>66.9</td>
<td>-4.1</td>
<td>-2.6</td>
<td>-0.5</td>
<td>-2.6</td>
<td>2.4</td>
<td>8.0</td>
<td>-4.0</td>
<td>-19.8</td>
<td></td>
</tr>
<tr>
<td>79.6</td>
<td>-1.4</td>
<td>-0.1</td>
<td>0.9</td>
<td>-1.6</td>
<td>6.0</td>
<td>-4.8</td>
<td>6.1</td>
<td>11.1</td>
<td></td>
</tr>
<tr>
<td>100.4</td>
<td>-0.5</td>
<td>-0.6</td>
<td>-0.2</td>
<td>-3.1</td>
<td>4.2</td>
<td>-3.8</td>
<td>1.7</td>
<td>4.8</td>
<td></td>
</tr>
<tr>
<td>115.6</td>
<td>-0.9</td>
<td>-0.4</td>
<td>0.4</td>
<td>-1.5</td>
<td>3.3</td>
<td>-3.4</td>
<td>4.9</td>
<td>0.6</td>
<td></td>
</tr>
<tr>
<td>136.4</td>
<td>0.9</td>
<td>0.8</td>
<td>0.4</td>
<td>-1.3</td>
<td>3.4</td>
<td>-2.4</td>
<td>3.7</td>
<td>1.9</td>
<td></td>
</tr>
<tr>
<td>151.6</td>
<td>0.0</td>
<td>0.4</td>
<td>0.6</td>
<td>-1.3</td>
<td>2.5</td>
<td>-2.9</td>
<td>6.0</td>
<td>1.7</td>
<td></td>
</tr>
<tr>
<td>172.4</td>
<td>-0.8</td>
<td>-0.1</td>
<td>0.0</td>
<td>-0.9</td>
<td>2.7</td>
<td>-1.9</td>
<td>2.0</td>
<td>0.4</td>
<td></td>
</tr>
</tbody>
</table>

**TABLE 5 VERTICAL STRESSES ON THE WINDWARD SIDE OF THE SHELL FOR L.F. = 1.0**
FIGURE 4 Deflection profile for $\theta = 0^\circ$
FIGURE 5  Deflection profiles for the values of z shown
Figure 6a - Maximum combined stress of the outer fibre
FIGURE 6b Maximum combined stress at the outer fibre of the shell \( z = 22.7739 \text{ m} \)
Figure 6c - Maximum combined stress at the outer fibre
FIGURE 7 Maximum combined stress at the bottom fibre of wind girder.
Figure 7a Maximum combined stress at the bottom fibre of wind girder

Wind girder

\[ z = 21.9300 \text{ m} \]

\[ r = 38.3954 \text{ m} \]
FIGURE 8  Deflection profiles for θ = 0°
FIGURE B a - Deflection profiles for the values of $z$ shown
Figure 9a - Maximum combined stress at the outer fibre

$z = 10.5279$
Figure 9b Maximum combined stress at the outer fibre

\[ z = 18.371 \text{ m} \]

\[ \theta \text{ (degrees)} \]

\[ 50 \times 10^3 \]

\[ -50 \times 10^3 \]

\[ -100 \times 10^3 \]
Figure 9c - Maximum combined stress at the outer fibre
Figure 9.d - Maximum combined stress at the outer fibre

\[ z = 22.7739 \]
Figure 9e - Maximum combined stress at the bottom fibre
FIGURE 95: Maximum combined stress at the bottom fibre of the wind girder (Model II)

\[
z = 21.9300 \text{ m} \\
r = 39.4767 \text{ m}
\]
$z = 21.93 \text{ m}, r = 39.873 \text{ m}$

$\theta = 0^\circ$

$\theta = 2^\circ$

$\theta = 5^\circ$

$\theta = 36^\circ$

$\theta = 18^\circ$

FIGURE 10 - Local buckling of the wind girder

Z - Displacement
Figure 11 - Local buckling at the base of the tank
FIGURE 12(a)  First buckling mode (Model II)
FIGURE 12 (b) First buckling mode (Model II)
Figure 13a - Second buckling mode (model II)
Figure 13b - Second buckling mode (model II)
Figure 14. Third buckling mode (model II)
Figure 14b - Third buckling mode (model II)
FIGURE (a) First buckling mode (Model I)
FIGURE (b) First buckling mode (Model I)
(a) Circumferential section for $z = 23$ m

(b) Longitudinal sections for the value of $\theta$ shown

FIGURE 16 - First vibrating mode
Figure 17 - Load deflection curves (model I)
\[ \theta = 72^\circ \]
\[ z = 23.0 \text{ m} \]

\[ \theta = 72^\circ \]
\[ z = 21.93 \text{ m} \]

**Linear solution**

**Load factor**

**y-displacement (cm)**

**y-displacement (cm)**

\[ \theta = 72^\circ \]
\[ z = 17.42 \text{ m} \]

\[ \theta = 72^\circ \]
\[ z = 8.68 \text{ m} \]

**y-displacement (cm)**

**y-displacement (cm)**

**FIGURE 18 - Load-deflection curves (Model I)**
Figure 19 - Load deflection curves (model I)
Figure 20 - Load deflection curves (model 1)
FIGURE 21 - Load-deflection curves (Model II)
FIGURE 22 - Deformation of the upper part of the tank for $\theta = 0^\circ$ (Model II)
FIGURE 25 - Load deflection curves (Model \( \frac{\pi}{2} \))
FIGURE 26 Circumferential deflection profiles (Model III)
CHAPTER VIII - GENERAL CONCLUSIONS

CHAPTER VIII
GENERAL CONCLUSIONS

The work reported in this thesis includes several topics in the field of finite element analysis of arbitrary plates and shells, a number of finite element computer programs having been developed.

Although conclusions are given at the end of each chapter, they will be briefly reviewed here and recommendations for improvement and extension of this work will also be presented.

Chapter I gives a brief introduction to the finite element method leading to the motivation of the work presented in this thesis.

In Chapter II, a plate bending element based on the semiloof shell theory is developed. The applications of several patch tests to the element shows that rigid body motions and states of constant strain are exactly reproduced for straight boundary cases and very nearly so for cases involving curved boundaries. In this case it is shown that a mesh refinement along the curved boundary very much improves the results.

For other more realistic problems several convergence studies are presented and, for both constant and variable thickness plates, a high rate of convergence is observed, displacements and stresses being obtained with a high degree of accuracy.

The element was compared with other plate bending elements and it was shown to be very competitive for both curved and straight boundary plates.

Chapter III presents a matrix formulation of the semiloof shell element. The element is compared with others and proven to yield
extremely accurate results. The same conclusion transpires from the linear analysis included in Chapter VII.

In Chapter IV, the semiloof shell and plate elements have been used for the solution of a wide range of eigenvalue problems. Two finite element computer programs were developed, one including the plate element and able to deal with linear analysis and vibration analysis, the other including the shell element, and capable of dealing with linear analysis, vibration analysis, instability analysis and analysis of the vibrations of initially stressed shells.

The numerical results presented, show the elements to yield very accurate representation of the lower part of the eigenvalue spectrum.

Comparisons were made with other elements which showed the elements to be very competitive, reasonably accurate results being obtained with very coarse meshes.

The eigenvalue solution system developed proved to be very versatile and efficient, keeping the computer core storage requirement at practically the same level as that required for a simple linear solution.

Some improvements could however still be done to make the program more efficient and more general.

The first improvement proposed refers to the "interval of searching". As described in Chapter IV the search for eigenvalues starts at a certain given value (say WLEFT) and the search is carried out until another value is reached (say WRIGHT). For small size problems the initial guess can be very wrong but the program will very quickly find the right eigenvalue. However, if the problem is larger the location of the first eigenvalue can be very time consuming. By making use of the theorem enunciated in reference (I-1) better bounds can be obtained.
This theorem relates the eigenvalues of the structure with the eigenvalues of the elements and can be expressed as follows:

\[ \lambda_{\min}^{i} \text{ (over all elements)} < \lambda_{\min} \text{ (for structure)} < \lambda_{\max} \text{ (for structure)} < \lambda_{\max}^{i} \text{ (over all elements)} \]

which implies that the highest frequency of the structure is smaller than the highest frequency of its smallest element and the smallest frequency of the structure is bigger than the smallest frequency of the biggest element. Therefore, two previous (and simple) eigenvalue analysis allow the determination of bounds for the frequencies of vibration, these being possibly more exact than the ones obtained by a simple guess.

A further improvement of the program could be achieved by introducing an economizer (I-2). This would make possible the solution of big eigenvalue problems at a very modest cost.

Regarding stability problems, another point should perhaps be noted: if a structure is subject to its self-weight and any other load, the program, as it stands now, considers both loads to vary proportionately to the parameter \( \lambda \). In reality, however, this does not happen and the load resulting from the structure's self-weight should be considered independently, resulting in the following eigenvalue problem:

\[
\begin{bmatrix}
[K_E + K_G^w] + \lambda [K_G]
\end{bmatrix} \{x\} = \{0\}
\]

where \([K_G^w]\) represents a stiffness matrix resulting from the structure’s self-weight.
In Chapter V, the semiloof shell element is used for the solution of geometrically non-linear problems. A good agreement is observed between the solutions yielded by the semiloof element and other known solutions.

The computer program developed in this chapter uses an incremental and iterative method that was found to be particularly attractive for the case of the semiloof shell element, this resulting from the particularities of the element formulation.

The buckling analysis described in Chapter IV has been carried out in connection with a linear analysis. It is a well known fact that this type of analysis does not always give an appropriate estimate of the buckling load of the structure. It would however be possible to carry out an eigenvalue analysis in conjunction with the type of non-linear analysis formulated in Chapter V. This, we think, would constitute a worthwhile extension of the work reported in Chapters IV and V, allowing the effects of imperfections and "initial" displacements to be taken into account.

From the computational point of view this would correspond to a merging of the two programs described in these chapters, with the formulation of an eigenvalue problem after each load increment.

In Chapter VI, two finite element computer programs for the elasto-plastic analysis of structures were developed. These include respectively the semiloof plate and shell elements.

The problems solved show a good convergence rate, the solutions obtained with a small number of elements being very similar to the ones where a larger number of elements were used.
CHAPTER VIII - GENERAL CONCLUSIONS

For all the problems solved the solutions obtained lay within all known bounds, and a good comparison with other finite element solutions was also observed.

The initial stress method was used as a method of solution and proved to be very efficient for the elasto-plastic analysis of plates and shells. Its efficiency could however be improved with the introduction of a "resolve" facility in the program.

The work reported in Chapters V and VI could be extended to the geometric and material non-linear analysis of shells by combining the corresponding shell programs. In view of the methods of solution used, this extension could be easily implemented.

In Chapter VII the semiloof shell element, and some of the programs developed in other chapters, have been applied to the analysis of a large reinforced shell structure (storage tank).

In particular linear elastic, elastic buckling, large-deflection and free vibration situations have been considered. No numerical difficulties were experienced in obtaining any of the solutions and for situations where two types of meshes were employed a good correlation of results is generally evident.

It is shown that an eigenvalue analysis results in an extremely low buckling load which does not represent the true load carrying capacity of the structure. On the other hand, a simple linear elastic computation also leads to results considerably different from those obtained by a complete nonlinear large deflection analysis.
The influence of the base boundary conditions on the behaviour of the structure is also considered. It is shown that, if uplift of the base at the windward side of the tank is permitted, a considerable increase in deflections occur.

For the analysis of this structure it would have been helpful if the semiloof beam element (A-1) was included in the programs. The introduction of this beam element in the linear program, and in all the other programs developed, would make the programs more versatile and applicable for a wider range of structures including frame works where it is known that a considerable interaction exists between frames, slabs and shear walls (M-1).
BIBLIOGRAPHY

A-1 Albuquerque, F.C.S.
'A beam element for use with the semiloof shell element'.

I-1 Irons, B.M., and Treharne, G.
'A bound theorem in eigenvalues and its practical applications'
University of Wales, Swansea.

I-2 Irons, B.M.
'Structural eigenvalue problems. Elimination of unwanted
variables'

M-1 Majid, K.I.
'The effect of composite action on the elasto-plastic analysis
of complete building structures'
Proceeding of the 1974 Conference of Finite Element Methods in
Engineering held at the University of New South Wales, Australia.