Equivalence of economic models under different time representations

Catarina Peralta Correia de Almeida

Dissertation
Master in Economics

Supervised by
Sofia Balbina Santos Dias de Castro Gothen
Pedro Rui Mazeda Gil

2018
Acknowledgements

I am very grateful to my two supervisors, Sofia Dias de Castro and Pedro Mazeda Gil, for all their time, commitment and enthusiasm during this stage. Their continuous support, guidance and review were indispensable for this work. Their knowledge and passion, not only during this journey but also in the MSc lectures, were a fundamental part of this dissertation.

To my parents and brother, for all the opportunities and support they have given me. I will always be grateful for their continuous efforts.

Finally, to my best friend and life partner, João, for his patience and kindness during the last 5 years. His support, positive mind and heart and continuous encouragement help me everyday to succeed.

This accomplishment would not have been possible without them. Thank you.
Abstract

In economics, it is commonly believed that models under different time representations are equivalent. The different results shown in the literature prove this wrong. There are different implications for the stability properties of the steady state, brought by the different time representations, that may imply contradicting macroeconomic policy recommendations. In this dissertation, we aim to find sufficient conditions that guarantee an equivalence between the two setups in different economic growth models, such as Solow (1956) and Ramsey-Cass-Koopmans (Ramsey, 1928; Cass, 1965; Koopmans, 1965). For this purpose, we study the robustness of some relevant features of these models under different types of Euler discretization, i.e., backward- and forward-looking discretizations. In the present dissertation we show that robustness of the steady-states exists under discretization and, for reasonable calibrations, robustness of the stability. Additionally, we find that the speed of convergence is not invariant to the discretization step and direction.

JEL code: C02, C61, C62, O41
Keywords: Discrete time; Continuous time; Discretization; Dynamic analysis; Economic growth models
Resumo

Em Economia, é comum pensar-se que modelos com diferentes representações do tempo são equivalentes. Os diferentes resultados encontrados na literatura mostram o contrário. Ao utilizar diferentes representações do tempo, podem surgir diferentes implicações para a estabilidade do ponto de equilíbrio e, por conseguinte, diferentes recomendações para políticas macroeconómicas. A presente dissertação tem como objetivo encontrar condições suficientes que garantam a equivalência entre o tempo contínuo e o tempo discreto em diferentes modelos de crescimento económico, nomeadamente, o modelo de Solow (1956) e o modelo de Ramsey-Cass-Koopmans (Ramsey, 1928; Cass, 1965; Koopmans, 1965). Com este objetivo, é estudada a robustez de algumas características relevantes destes modelos sob diferentes tipos de discretizações de Euler, i.e., discretizações backward- e forward-looking. Os resultados mostram que existe robustez do ponto de equilíbrio ao abrigo de diferentes discretizações e para, calibragens razoáveis, robustez da estabilidade. No entanto, é mostrado que a velocidade de convergência depende do passo e da direção da discretização.

Códigos JEL: C02, C61, C62, O41
Palavras-chave: Tempo discreto; Tempo contínuo; Discretização; Análise dinâmica; Modelos de crescimento económico
# Contents

1 Introduction 1

2 Literature Review 5  
   2.1 General discussion 5  
   2.2 Continuous- versus discrete-time modelling 8  
   2.3 The choice of the period length 10  
   2.4 Time-to-build theory 10  

3 Discretizations 12  
   3.1 The Euler method 13  
   3.2 Cautionary notes 14  

4 Dynamics 16  

5 The Solow model 18  

6 The Ramsey-Cass-Koopmans model 21  
   6.1 Backward-looking discretization 22  
   6.2 Forward-looking discretization 25  
   6.3 Hybrid discretization 26  

7 Forward-looking: further analysis 30  

8 Conclusion 33  

A Mathematical proof 36  
   A.1 Detailed Proof of Proposition 5.1 36  
   A.2 Computing the Jacobian matrix in the Proof of Proposition 6.1 37  
   A.3 The characteristic polynomial in the Proof of Proposition 6.1 39  
   A.4 Computing the Jacobian matrix in the Proof of Proposition 6.4 39
## CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.5 Sign of the Discriminant in Proposition 6.4</td>
<td>42</td>
</tr>
<tr>
<td>A.6 Hybrid discretization with consumption at time $t$</td>
<td>43</td>
</tr>
<tr>
<td><strong>Bibliography</strong></td>
<td>46</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

“During the last two centuries, the way economic science is done has changed radically: it has become a social science based on mathematical models in place of words.” (Morgan, 2012, Abstract)

In order to understand the main characteristics of an economy and properly advise and implement accurate policies, economists formulate models to study particular questions and to help them make the most appropriate decisions based on robust theoretical predictions. Samuelson and Nordhaus (1998) define model as a formal framework for representing the basic features of a complex system by a few central relationships. The choice of the specific characteristics of the model depends on the research questions (and therefore on the specific features of the complex system) economists wish to address. A typical choice is between a static and a dynamic framework.

The theory of dynamics is a theory that determines the behavior through time of the variables of interest possibly given a set of boundary conditions. However, upon formulating a dynamic model, there are typically two distinct alternatives regarding the representation of time that can be used: discrete-time or continuous-time. As Zhang (2006) defines, time is continuous when the time space is the set of real numbers, $\mathbb{R}$. In this framework, the pattern of change of a variable is provided by its derivatives with respect to the (infinitesimal) change of time. Time is discrete when the time space is the set of integer numbers, $\mathbb{Z}$. Then the pattern of change of a variable is also described by discrete differences, as it will change only when the variable time shifts from one integer value to the next.

When addressing this question, very seldom do we find in the literature any explanation for the choice between different representations of time in economic models. Gandolfo (1997) and Turnovsky (1977) are two of the few authors that, to
the best of our knowledge, provide some discussion on this issue. However, when it comes to choice, typically, there is no explanation. The decision is often based upon the methodological needs of the researcher and the mathematical tools the researcher is familiar with \cite{Anagnostopoulos2005}. For instance, often researchers use a continuous-time framework in the growth literature where the main focus is the long-run; whereas in the business cycle literature discrete time may be more helpful for the short-run analysis. However, it is often necessary for continuous-time models to be cast into a discrete-time framework, e.g., when data must be considered \cite{Guzowska2018}. To use discrete time, researchers may adopt a discrete setup from the beginning while building the model itself or just apply a discretization method to the dynamical system derived from a model initially built in a continuous-time setup. In this dissertation, our main focus is on the second approach. We address this issue in more detail, in Chapter 3.

The importance of this discussion lies on the fact that, frequently, we find the results of the two alternative representations of the same model different. It is well-known – see for instance \cite{Krivine2007} – that while discrete dynamical systems in 1-dimension can exhibit chaotic behavior, a phase-space of dimension $d \geq 3$ is required to observe (possibly) chaotic behavior in a continuous dynamical system. Also, \cite{Samuelson1941} shows that a first-order difference equation is richer in solution than the corresponding first-order differential equation. In fact, the stability of the steady-state may differ depending on the model of time. Dynamic indeterminacy may occur in one of the time setups and be absent from the other. When dynamic indeterminacy arises, the perfect-foresight equilibrium is at least locally indeterminate in the sense that there are multiple converging transition paths \cite{Mino2008}. In addition, in 1-dimensional discrete-time, oscillations may be present whereas they are absent from the 1-dimensional continuous-time model.

Furthermore, when we apply a discretization we, must chose the order, step and direction. The step gives the length of the period in discrete time. The order is that of the Taylor expansion applied to the continuous-time model. The direction depends on the backward or forward-looking nature of this Taylor expansion \cite{Bosi2012}. These choices may have implications on the stability properties. For instance, researchers often use a unit step length; however, it seems that this assumption could lead to different results concerning the stability of the steady-state and the speed of convergence. As \cite{Krivine2007} refer, one can be a large number. Additionally, different directions applied to the same model may also generate different results. In fact, in Chapter 7, we discuss the nature of the forward-
looking direction and the indirect implications that this framework might have from the point of view of the discrete-time optimal control problem that underlies the dynamical system.

To fully understand the economic relevance of this question, Anagnostopoulos and Giannitsarou (2005) conclude that these different implications for the stability properties of the steady state, brought by the different time representations, may imply contradicting macroeconomic policy recommendations. Additionally, we may think about the implications of different transitional dynamics to the long-run equilibrium. The information brought by an oscillatory behavior is quite different from a monotonous one. When the path is oscillatory, economists expect cycles to occur during the transition to the steady-state. On the other hand, a monotonic behavior precludes cycles and oscillations.

The different behavior exhibited by discrete- and continuous-time representations and the scarce literature on this issue motivate the study and the need to understand the relationship between the two representations of time. Therefore, this dissertation aims to answer the following research question: What is the relation between the description of an economy achieved using discrete- and continuous-time models? In order to achieve our goal, we aim to find sufficient conditions that guarantee an equivalence between the two descriptions of an economy, in well-known growth models such as Solow (1956) and Ramsey-Cass-Koopmans (Ramsey, 1928; Cass, 1965; Koopmans, 1965). In particular, we are going to study the robustness of some relevant features of these models, under different types of discretizations. For instance, we are going to focus on the existence of equilibria, the stability of equilibria, existence or absence of oscillatory behavior and the speed of convergence to equilibria. We start by showing that robustness of the steady-states exists under discretization. Also, for reasonable calibrations, we prove the robustness of stability. Moreover, we show that there is a dependence of the speed of convergence on the discretization step.

To the best of our knowledge, a systematic approach such as the one described above is lacking from the literature and therein lies our effort to contribute to the economic science.

This dissertation is organized as follows. In Chapter 2, we present the literature review focusing on different aspects of this issue. Chapter 3 presents the discretization method and several cautions to take before its application. In Chapter 4, we provide some essential concepts that are useful for the analysis of the models. Chapter 5 and 6 present, respectively, the results of the discretization of the Solow and
Ramsey-Cass-Koopmans models. Chapter 7 includes an analysis focused on the forward-looking direction. Our conclusions are shown in Chapter 8.
Chapter 2

Literature Review

The issue of modelling time is a matter of interest not only in Physics and Mathematics, but also in Economics. Regarding Economics, the discussion on the decision on whether to use discrete or continuous models is yet little explored. However, it is mainly debated in the branch of Macroeconomics. Typically, the literature that uses continuous-time models focuses on economic growth phenomena, whereas models in the business-cycle literature are, to a large extent, written in discrete time. If the researcher is interested in analysing short-run effects, discrete-time models tend to be more useful. However, for steady-state and stability analyses, continuous-time models are usually more practical (Turnovsky 1977).

The macroeconomic literature does not always recognize the distinction between the two specifications and has occasionally confused them over time (Foley 1975). Frequently, the choice of time in a model is constrained by the mathematical theories and tools that are available to the authors. This often explains why different authors choose different time frames.

2.1 General discussion

As previously mentioned, the different results brought by continuous- and discrete-time boosted the necessity to understand the relationship between the two frameworks. Most of the researchers do not fully clarify which representation of time is more suitable to use or give any explanation on why they chose continuous- over discrete-time, or vice versa. However, concerning this issue, there are a few challenging questions that may arise. For example: what is the true nature of time? Which representation of time describes better the real world? In the branch of Economics, the question that some bring to light is: what is the proper way to treat time in
May (1970) is one of the first authors to bring this question to the literature. He defends that continuous-time is preferable due to two reasons: The principal reason is that the discrete approach may be misleading, i.e., if the time interval used is too long, the state of the economy at given moments within the interval can differ significantly from its state at the beginning or at the end of the interval. Secondly, in discrete-time, all quantity variables have the dimension of stocks, whereas in continuous analysis one is forced to distinguish between stocks and flows. For May, this distinction is very important and should be emphasized when studying Economics, thus it gives a noteworthy advantage to continuous-time.

Simultaneously, both Gandolfo (1997) and Turnovsky (1977) discuss the relevance of the choice of the representation of time in economic models. They describe a number of arguments generically in favour of the continuous version vis-à-vis the discrete version.

The authors explain that, although individual economic decisions are generally made at discrete time intervals, it is difficult to believe that they are coordinated in such perfectly synchronized way. In their opinion, the variables that are usually considered and observed by the economist, in particular at the macro level, are the outcome of a great number of decisions taken by different operators at different points of time. Thus, it seems natural to treat economic phenomena as if they were continuous.

A related difficulty with discrete-time analysis is that usually there is no obvious time interval that can serve as a “natural” unit. Both authors believe that the assumption of a certain fixed period length may unwittingly be the source of misleading conclusions. Some authors attach special importance to this property and argue that the test of the invariance of results with respect to the length of the period is fundamental in order to ascertain whether a discrete model is well defined and consistent. Further, the use of continuous time may allow a more satisfactory treatment of distributed lag processes.

Also, they concur that, from the analytical point of view, differential equations are usually more easily handled than difference systems. However, we must bear in mind that these views were expressed in an era in which computational methods were barely used, or not used at all. Nowadays, the computers allow us to solve, numerically, difference equations. Therefore, it seems easy to use discrete time.

As well as May (1970), Gandolfo (1997) discusses that a specification in continu-
ous time is particularly useful for the formulation of dynamic adjustment processes. A plausible discrete equivalent is often difficult to find when both stock and flow variables are involved.

In what concerns econometrics techniques, Gandolfo (1997) refers that since the estimation of models was based on discrete-time, for many economists, it was natural to formulate the models in discrete time, in order to estimate them. Nowadays, these limitations have been exceeded. In continuous-time models, the estimator of these models is independent of the observation interval, *i.e.*, the availability of a model formulated as a system of differential equations enables its user - once the parameter estimates have been obtained - to get forecasts and simulations for any time interval, and not only for the time unit inherent in the data. Also, with the continuous formulation, it is always possible to obtain asymptotically unbiased estimates of the adjustment speed even for relatively long observation periods.

More recently, Jarrow and Protter (2012) conclude that, for asset pricing theory, continuous time is the preferred modeling approach. They justify this assertion on two grounds. The first is that continuous-time modelling provides more information than discrete-time models. In fact, discrete-time models do not address important economic phenomena related to arbitrage opportunities, large traders, asset price bubbles, and market efficiency. The second is described by the requisite of fixing a time grid in discrete time. Once fixed, the model implies that trading can only take place on the grid. However, this is clearly false, as trading can take place at any point, in continuous time, in actual security markets.

On the other hand, there are some aspects of discrete time that clarify the advantages of using this type of representation. For example, the possibility to specify the direction. Carlstrom and Fuerst (2005) refer that discrete time allows them to examine both current- and forward-looking Taylor rules, whereas, in continuous time, there is no possible distinction in that regard. In Foley’s (1975) perspective discrete-time models are also more attractive. The author refers that economic transactions do not take place continuously; for example, purchases of final goods, capital account transactions, wage payments, and dividend payments all take place at discrete times. In his opinion, only if all agents have the same planning period but their initial planning dates are distributed uniformly over calendar time, would it be more plausible to use continuous time. Additionally, it is crucial to elucidate that whenever we use numerical methods to solve a model, we often transform the model to a discrete-time version in order to introduce it on the computer. Therefore, the computational methods are by far one of the significant reasons to use discrete
time when modelling.

The previous advantages of continuous and discrete time reflect the importance of using the two frameworks. In both versions, we find efficient techniques that are useful to any researcher. Given this, it seems more relevant to find sufficient conditions that make the two setups equivalent, instead of trying to prove which version of time is more suitable for economic modelling. When found, these conditions enable the use of both representations of time, providing the same conclusions and thence, the same consequences to policy-making.

2.2 Continuous- versus discrete-time modelling

While Turnovsky (1977) and Gandolfo (1997) are only concerned in pointing which version is more appropriated to use in economic modelling, there is some literature showing that the results highly depend on the choice of time and that they are indeed different.

The important paper by May (1970) uses Patinkin’s macroeconomic model to show that the additional information gained from continuous analysis leads to a steady-state that differs from the one derived from the discrete analysis. Also, Telser and Graves (1968) analyse continuous and discrete-time approaches to a maximization problem and they show that not only are there surprising differences between the models in the two setups, but also there are differences between the monopoly and investment problems. In spite of being different analysis, one being related to steady-state analysis and other to maximization problems, both had different findings in the two representations. Therefore, it seems that the issue that this dissertation intends to clarify is not just a concern for dynamical analysis.

Focusing on the dynamical analysis, in a model of uncertain lifetime with intertemporal utility maximization, Leung (1995) shows that the continuous and discrete-time versions do not generate the same predictions on the consumption and savings trajectories. Similarly, Li (2003) analyses the determinacy of equilibrium in a discrete-time version of Dupor’s model. The determinacy results obtained contrast with those of Dupor (2001). The author finds that the continuous-time limit proposed by Dupor does not correctly approximate the behavior of a discrete-time model with arbitrarily short periods.

As referred above, typically, economic growth models are written in continuous time and business cycle models in discrete time. However, when specified in different representations, as it is expected, they also show different results. In a working
paper, Anagnostopoulos and Giannitsarou (2005) also show that the results of the two frameworks are different. While investment decisions in the continuous-time setup depend on present rates of return, in the discrete-time setup they depend on expected future rates of return. In their most recent article, Anagnostopoulos and Giannitsarou (2013) analyse a business cycle model with balanced budget rules and show that the continuous-time model gives indeterminacy, whereas the discrete-time model has determinate dynamics. Further, Mino et al. (2008) consider a class of endogenous growth models with social constant returns in the presence of positive externalities. They depart from discrete time and show that conventional propositions obtained in continuous time need not to be valid. This is supported by Mitra and Sorger (2013) who refer that the differences in the predictions of models formulated in the two settings can be significant. In their article, they analyse the continuous-time Ramsey’s model with an identical structure of the economy studied by Becker (1980), the only distinction being indeed the formulation of time. They find their results to stand in stark contrast to results that have been found in the discrete-time version. As a consequence, the authors conclude one has to be very careful with intuitive explanations that do not take into account the way in which time is modeled.

More recently, Gómez (2014) and Brida et al. (2017) compare the two (time) versions of the same model. In his paper, Gómez (2014) analyses an endogenous growth model with durable consumption. The author shows that in the continuous-time version the steady state is locally saddle-path stable with monotonic convergence, whereas in discrete-time the steady state may be unstable or saddle-path stable with monotonic or oscillatory convergence. Despite only analysing the discrete version of the Razin model, Brida et al. (2017) compare their results with the original one. They find the dynamics of the model dramatically different.

Bosi and Ragot (2012) have a different approach from the authors mentioned above. In their article, Bosi and Ragot (2012) apply backward- and forward-looking discretizations on the continuous-time dynamical system of the Ramsey model in order to preserve the original stability properties. They show that the discretized version of the Ramsey model is a hybrid discretization, i.e., a backward-looking budget constraint and a forward-looking Euler equation. In their analysis, the original saddle-path stability is a robust property under the hybrid discretization. Similarly, Guzowska and Michetti (2018) show that the hybrid discrete-time version is a satisfying approximation of the continuous time setup since not only equilibria are unchanged but also their local stability properties coincide.
2.3 The choice of the period length

When carrying out a discretization, the definition of the integration step, the order and the direction is required. The step gives the length of the period in discrete time. The order is that of the Taylor expansion applied to the continuous-time model. The direction depends on the backward or forward-looking nature of this Taylor expansion (Bosi and Ragot, 2012). When the discrete version is selected, researchers must choose the period length and by doing that, they may unintentionally influence the results of the model. Therefore, there are some that focus on this particular aspect of the discretization.

For example, Anagnostopoulos and Giannitsarou (2008) take the view of Turnovský (1977) and focus on the step. They study local stability and how this is affected by shifts in period length. In their working paper, the authors find that in models with endogenous capital accumulation, as the period gets longer, indeterminacy occurs less often. Also, in the case of the New Keynesian model, they show that standard continuous and discrete time versions have entirely different local stability properties due to a discontinuity at zero period length. Furthermore, Hintermaier (2005), in a business cycle model, shows that the local stability properties may depend on the length of a period. Similarly, Franke and Sacht (2014) find the same results in a small-scale New-Keynesian model with a hybrid Phillips curve and IS equation. Finally, Medio (2013) refers that the paths of endogenous variables generated by discrete-time optimal (Ramsey) growth models are optimal under the constraint that decisions can be taken only at certain intervals. Medio (2013) shows that certain qualitatively interesting properties of those paths may disappear when the length of those intervals is reduced, or continuous decision-making is permitted.

2.4 Time-to-build theory

There is also a theory relevant to economic dynamics that should be mentioned in this dissertation. The time delay concept was first introduced in the 30’s. Several researchers apply the delay in models to induce cyclic behavior. Kalecki (1935) was one of the first to formulate this type of problem and his idea of the time for building investment capital appeared in the context of explaining changes in the market cycle.

Concerning the dissertation’s topic, Licandro and Puch (2006) propose time-to-build as a solution for the gap between continuous and discrete time. They show that the discrete-time representation of the standard optimal growth model is
consistent with the continuous-time representation under the additional assumption of time–to–build. Otherwise, Bambi and Gori (2014) explore the differences between models with time-to-build in continuous and discrete time. They propose a new procedure as an alternative approach which allows for a unifying theory of the two classes of economic growth models.
Chapter 3

Discretizations

To successfully achieve the goal of this dissertation and to give an answer to our research question, we intend to follow the approach used by Bosi and Ragot (2012). However, before explaining what this method consists of, it is very important to distinguish two different approaches that researchers may adopt when using discrete time: to build a model in discrete time versus to discretize a dynamical system built in a continuous-time setup.

When we build a model in discrete time, we apply discretizations to the first equations of the model, i.e., for instance in a typical growth model, we use a discrete-time production function and a discrete-time consumer utility function. Therefore, when setting up the corresponding dynamical optimization problem, we face a discrete-time Hamiltonian function subject to a budget constraint also in discrete time, in case there is any. Day (1982), Nourry et al. (2013), Gómez (2014) and Guzowska and Michetti (2018) are authors that apply this type of scheme. In this approach, the first order conditions we find differ from the continuous-time version and often lead to a dynamical system different from the one achieved by the second approach. Differently, in the latter we just apply discretizations to a continuous-time dynamical system that resulted from a model originally built in continuous time. See, for instance, Bosi and Ragot (2012).

When discretizing an existent continuous-time model, researchers have to make decisions related to the discretization process. This often seems straightforward; yet, frequently this is not true. Typically, economists apply this method in order to introduce the model in the computer and simulate it. By being just a piece of their research projects, they do not give much thought to the method itself and sometimes make wrong assumptions. Moreover, this approach is seldom addressed in the literature.
CHAPTER 3. DISCRETIZATIONS

Given this, all our conclusions refer to the discretizations applied to a dynamical system built in a continuous-time setup and may differ from the ones obtained when a discrete-time model is studied.

3.1 The Euler method

As [Bosi and Ragot (2012)], we use discretizations introduced by Euler. We apply a polynomial Taylor expansion to discretize the continuous-time version of the growth models by Solow (1956) and by Ramsey-Cass-Koopmans ([Ramsey 1928] [Cass 1965] [Koopmans 1965]).

The Euler method consists of approximating the instant rate of change, represented by the derivative of the state variable, by a difference between states of the state variable at distinct instances of time. In the same way as [Bosi and Ragot (2012)], let us consider a regular sequence of time values:

\[(t_n)_{n=0}^{\infty} = (nh)_{n=0}^{\infty},\]

where \(h\) is a positive constant (the discretization step), and the associated states:

\[x_n \equiv x(t_n) = x(nh).\]

The Euler-Taylor discretization is a polynomial approximation given by either:

\[x_{i,n+1} - x_{i,n} \approx \sum_{p=0}^{q} \frac{(h - 0)^p}{p!} \phi_i^{(p)}(0) (3.1)\]

or

\[x_{i,n+1} - x_{i,n} \approx \sum_{p=0}^{q} \frac{(0 - h)^p}{p!} \psi_i^{(p)}(h) (3.2)\]

where the subscript \(i\) denotes the \(i\)th component of the vector. Setting the order \(q = 1\), we obtain from (3.1) and (3.2) two first-order discretizations:

\[x_{i,n+1} - x_{i,n} \approx hf_i(x_n) (3.3)\]

and

\[x_{i,n+1} - x_{i,n} \approx hf_i(x_{n+1}) (3.4)\]

Equation (3.3) is called a backward-looking discretization because the variation

\footnote{For further information see [Bosi and Ragot (2012)].}
$x_{i,n+1} - x_{i,n}$ depends on the past value $x_n$ on the left hand of the interval $[x_n, x_{n+1}]$. Equation (3.4) is called a forward-looking discretization since the variation $x_{in+1} - x_{in}$ depends on the future value $x_{n+1}$.

It is common to see equation (3.3) designated as forward Euler method and equation (3.4) as backward Euler method, which can be misunderstood due to the directions (backward- and forward-looking) that we just described. For clarification, we shall distinguish backward-looking (forward-looking) from the backward method (the forward method). When researchers use the term looking, they aim at showing the way decisions are made by the economic agents in the model, i.e., based on the past circumstances (backward-looking) or based on expectations (forward-looking). On the other hand, when the term method is linked with backward or forward, it refers to the position of the point in the interval used in the Taylor approximation. This means that if the approximation is done at the beginning of the interval, as in equation (3.1), the forward method is being used. Otherwise, if the approximation is done at the end of the interval, as in equation (3.2) the backward method is being used.

In this dissertation, we shall use the terms backward and forward as the direction (looking) and not as the method of the discretization.

### 3.2 Cautionary notes

As already noted, when applying a discretization, decisions must be made regarding the order, step and direction of the discretization. Since some of these decisions can influence the results of the models, we shall clarify some aspects that should be taken into account before performing the discretization. “One has to be very careful with intuitive explanations that do not take into account the way time is modeled.” (Mitra and Sorger, 2013, pp. 1975)

The easiest choice to make is related to the order of the discretization. This must be the same as that of the original differential equation.

As we mentioned before, the direction can be forward-looking or backward-looking. This choice should be carefully analysed, since the mathematical equation must be an adequate representation of the features of the real world that are of interest to the researchers. Typically, the direction used, when doing discretizations, is the backward-looking. However, in Economics, a forward-looking behavior is common, i.e., the assumption that economic agents form their decisions based on their expectations of the structure of the changes in the economic environment. It
is crucial to take into account what the model aims to describe before applying a discretization. Researchers and economists should not use a forward-looking direction without checking its theoretical foundations. There must be a reason behind the choice, for example, if expectations take part in a model, it would seem natural to use forward-looking as the direction of the discretization.

Lastly, the discretization step, $h$, is usually assumed to be 1. However, as Krivine et al. (2007) argue, 1 can be a very large step and problems may come with it. As shown in the literature review, this can be the main problem for the different results brought about by the two representations of time.

Given the aspects pointed out, we are going to study the effects of the step size and the direction on relevant characteristics of the dynamics such as existence and location of steady-states, stability of the steady-states, types and speed of convergence.
Chapter 4

Dynamics

In this chapter, we describe some essential concepts for the upcoming analysis of the Solow model and the Ramsey-Cass-Koopmans model. The interested reader may find more detail in Gandolfo (1997, Chapter 21).

A steady-state or equilibrium is a state that does not change over time. As we mentioned, the continuous-time system \( \dot{x} = g(x) \) has a discrete-time approximation \( x_{n+h} \approx x_n + h g(x_n) \). In discrete time, we find the steady-state \( (x^*) \) of the difference equation \( x_{n+h} = x_n + h g(x_n) = f(x) \) by computing \( x = f(x) \), whereas in continuous time the steady-state satisfies \( \dot{x} = g(x) = 0 \).

**Proposition 4.1.** The steady-state is invariant to the step \( (h) \) and direction of the discretization.

**Proof.** Let \( \dot{x} = g(x) \) be a continuous-time system with discretization given by \( x_{n+h} = x_n + h g(x_n) \). Define \( f(x) = x + h g(x) \) where the steady-state is given by, in discrete time, \( x = f(x) \iff 0 = g(x) \). Note that in continuous-time, we find the equilibrium state by computing \( \dot{x} = 0 \iff g(x) = 0 \). Hence, the equilibria coincide in continuous and discrete time. Since the equation for the steady-state is independent of the step \( h \), the steady-state is also invariant to the direction of the discretization (see also Bosi and Ragot, 2012).

To study the stability, we follow a qualitative approach. As Gandolfo (1997) defines, this consists of the analysis of the properties of the solutions to a differential or difference equation without actually knowing the solution itself.

**Definition 4.2.** An equilibrium state \( x^* \) is stable (in the sense of Lyapunov) if for every real number \( \epsilon > 0 \) there exists a real number \( \delta(\epsilon, t_0) \) such that if \( ||x_0 - x^*|| \leq \delta \) holds then \( ||x(t) - x^*|| < \epsilon \) also holds for all \( t > 0 \).
An equilibrium state \( x^* \) is asymptotically stable if it is stable, and if every motion starting sufficiently near \( x^* \) converges to \( x^* \) as \( t \to \infty \). In other words, to every real number \( \mu > 0 \) there corresponds a real number \( T(\mu, x_0, t_0) > 0 \) such that the inequality \( ||x_0 - x^*|| \leq r(t_0) \), where \( r(t_0) > 0 \) is a real constant, implies

\[
\lim_{t \to \infty} x(t, x_0) = x^*.
\]

In order to study the stability of a steady-state we use the following

**Theorem 4.3.** Let \( A \) be the Jacobian matrix of the system defined by \( f(x) = x + hg(x) \) at an equilibrium. Then:

- the equilibrium solution is locally asymptotically stable if all the eigenvalues of \( A \) belong to the interior of the unit disc;
- the equilibrium solution is unstable if at least one eigenvalue of \( A \) is, in absolute value, greater than 1;

Additionally, we study the speed of convergence. This feature of the stability provides important information, since the knowledge that the system tends to equilibrium as \( t \to \infty \) may not be very interesting if, due to the slowness of the motion, the time required for the system to get near the equilibrium is too long [Gandolfo 1997].

**Proposition 4.4.** In discrete time, the convergence speed is given by \( |\lambda_d - 1| \) where \( \lambda_d \) is the largest in absolute value eigenvalue of the discrete-time Jacobian \( J(x^*) \).

**Proof.** In a continuous-time dynamical system, the speed of convergence corresponds to the largest eigenvalue \( (\lambda_c) \) of the Jacobian matrix, i.e., the eigenvalue that ensures a convergence to \( x^* \):

\[
x(t) - x^* = e^{-\lambda_c t}[x(0) - x^*]
\]

while in discrete-time dynamical system, it corresponds to \( \lambda_d \):

\[
x_t - x^* = \lambda_d^t[x_0 - x^*],
\]

where \( \lambda_d \) is the associated eigenvalue of a discrete-time dynamical system.

The correspondence between continuous and discrete time is thus given by \( \lambda_d^t \to e^{\lambda d t} \). For \( t = 1 \), \( \lambda_d \to e^{\lambda_c} \), where \( \lambda_c \approx \lambda_d + 1 \), or equivalently, \( \lambda_c \approx \lambda_d - 1 \). \( \square \)
Chapter 5

The Solow model

Solow (1956) is the seminal paper in the neoclassical economic growth literature. Although this literature has expanded to more complex models, the Solow model still remains a very important piece to understand the economic growth in the real world (e.g. Barro and Sala-i-Martin 2004, Chapter 1).

The Solow model features the parameters $\delta \in (0, 1)$, $n > 0$, $s \in (0, 1)$ and $\alpha \in (0, 1)$, which represent, respectively, the depreciation rate of capital, the population growth rate, the savings rate and the share of income spent on capital, and describes the behavior of the per capita capital variable, $k \geq 0$.

The dynamics of the continuous version of the Solow model, with a Cobb-Douglas production function and without technical progress, is given by the following differential equation:

$$\dot{k}_t = sk^\alpha_t - (\delta + n)k_t = f(k_t). \quad (5.1)$$

The steady-states are the origin and

$$k^* = \left(\frac{s}{n + \delta}\right)^{\frac{1}{1-\alpha}}. \quad (5.2)$$

We are mostly interested in the interior steady-state $k^*$ and the behavior of solutions nearby. Before applying the discretization, let us recall the cautions we should have regarding the order, the direction and the step. As the original differential equation (5.1) has order 1, the corresponding difference equation must have the same order. Additionally, we are considering a 1-dimensional dynamical system $\dot{k}_t = f(k_t)$, thus the direction can be either backward- or forward-looking. As we mentioned, it is important to do a careful analysis of the theory behind the dif-
Chapter 5. The Solow Model

Differential equation, in other words, which characteristics of economic activity the model is aimed at representing. As we are dealing with capital accumulation, \( \dot{k} \), it is nonsensical to expect future values \( k_{t+h} \) to have a part in the variation \( k_{t+h} - k_t \). Therefore, the forward-looking discretization is excluded and a backward-looking is embraced. To properly study the role of the step, we keep it as \( h \) in the discretization process. This strategy will help us define for which values of \( h \) the original dynamic properties are preserved.

Hence, by applying a first-order backward-looking discretization to (5.1), we get

\[ k_{t+h} \approx k_t + h[s k_t^\alpha - (n + \delta)k_t] = g(k_t). \] (5.3)

Given Proposition 4.1, the steady-state is invariant to the step of the discretization (see Section 4). Hence, in the discrete-time version (5.3), the original steady-state (5.2) is maintained.

Let us focus on the stability properties. In continuous time, the equilibrium is stable and approached through monotonic convergence. In the discretized model we have the following

**Proposition 5.1.** The steady-state \( k^* \) is stable for the dynamics in (5.3) if \( h < \frac{2}{(1-\alpha)(n+\delta)} \). Furthermore, convergence to \( k^* \) is monotonic if \( h < \frac{1}{(1-\alpha)(n+\delta)} \).

**Proof.** According to Theorem 4.3, \( k^* \) is stable if \( \frac{dg}{dk}(k^*) \in (-1, 1) \). Computing the derivative of \( g \) and using (5.2) we obtain

\[ \frac{dg}{dk}(k^*) = 1 + h[(\alpha - 1)(n + \delta)] = \lambda_1. \]

Hence, \( \frac{dg(k^*)}{dk} \in (-1, 1) \) if, since \( h > 0 \), \( h < \frac{2}{(1-\alpha)(n+\delta)} = h_s \). Our results are consistent with Krivine et al. (2007). The authors refer that the first-order backward-looking equation \( x_{n+h} = x_n + h f(x_n) \) destabilises in \( h = \frac{2}{|f'(x^*)|} \), where \( f'(x^*) \) is the continuous-time eigenvalue. Furthermore, to reach a monotonic convergence, the eigenvalue \( \lambda_1 \) must be positive, and therefore, \( h < \frac{h_s}{2} \) is necessary. \( \Box \)

For typical calibrations, \( \alpha \approx 0.3 \), \( n \approx 0.01 \) and \( \delta \approx 0.05 \) (e.g. Barro and Sala-i-Martin 2004, Chapter 1). Replacing these values in \( \frac{2}{(1-\alpha)(n+\delta)} \), we obtain \( \frac{2}{0.7 \times 0.06} = \frac{1000}{42} \approx 47.6 \). Therefore, a step length lower than 47.6 ensures the stability of \( k^* \), whereas a step lower than 23.8 is needed to establish a monotonic convergence. Note that \( h < \frac{1}{(1-\alpha)(n+\delta)} \approx \frac{1}{0.7 \times 0.06} = \frac{1000}{42} \approx 23.8 \) for this calibration.

\(^1\)See Appendix A.1 for the detailed calculations in this proof.
However, as we mentioned in the previous chapters, the step length is always a value that belongs to the domain $(0, 1]$. In the Solow model, only large values of $h$ would change the stability properties of the continuous-time version. Therefore, we can conclude that the Solow model is a robust model concerning the existence of steady-states and their stability.

Now, let $\zeta_d$ and $\zeta_c$ denote the speed of convergence in continuous and discrete time, respectively.

**Proposition 5.2.** The discrete-time speed of convergence $\zeta_d$ is equivalent to the continuous version if and only if $h = 1$.

**Proof.** According to Proposition 4.4, the speed of convergence in discrete time is given by $|\lambda_1 - 1|$. Hence, $\zeta_d = (\alpha - 1)(n + \delta)h$ whereas, in continuous time, $\zeta_c = (\alpha - 1)(n + \delta)$ (see Barro and Sala-i-Martin, 2004, Chapter 1). Thus, if and only if $h = 1$, is the speed of convergence maintained. $\square$

The analysis of the stability features of the backward-looking discretization of the Solow model shows that the discretization step has a relevant impact on the convergence speed. Since $h \in (0, 1]$, Proposition 5.1 shows that the stability of the equilibrium stock of capital is preserved by discretization, as is the monotonicity of the dynamics. Proposition 5.2 shows that for $h < 1$, the rate of convergence in the continuous-time original model differs from its discretization.

Due to the way the discrete-time rate of convergence ($\zeta_d$) depends on $h$, we can conclude that the lower the step of discretization, the smaller is the rate $\zeta_d$; and $\zeta_d < \zeta_c$ given any $h < 1$. We know that in continuous time, for typical calibrations, the rate of convergence of the Solow model is, approximately, 4%. However, the existent literature often considers this rate to be too high in light of the empirical evidence. If we assume $h = \frac{1}{2}$, a lower value of $\zeta_i \approx 2.1\%$ can be obtained. In fact, this is a satisfactory value for the convergence speed and consistent with the empirical data (see, for instance, Durlauf et al., 2005).
Chapter 6

The Ramsey-Cass-Koopmans model

As Barro and Sala-i-Martin (2004) refer, a key element in the Ramsey-Cass-Koopmans growth model is the explicit consideration of consumer behavior. Here, infinitely-lived households choose consumption and savings to maximize their dynastic utility, subject to an intertemporal budget constraint. The Ramsey-Cass-Koopmans model was constructed by Ramsey (1928) and refined by Cass (1965) and Koopmans (1965). As this structure is the workhorse of the growth models in the modern growth literature, we find it a relevant model to be analysed.

The dynamical system of the continuous-time version with a Cobb-Douglas production function and without technical progress is (Barro and Sala-i-Martin, 2004, Chapter 2):

\[ \dot{k} = Ak(t)\alpha - c(t) - (\delta + n)k(t) \quad (6.1) \]
\[ \dot{c} = \frac{c(t)}{\theta}(\alpha Ak(t)^{\alpha-1} - \delta - \rho) \quad (6.2) \]

where variables \( k \) and parameters \( \delta, n \) and \( \alpha \) are the same as in Chapter 5. \( A > 0, c \geq 0, \theta > 0 \) and \( \rho \in (0, 1) \) represent, respectively, the technology level, the consumption per capita, the risk aversion, and the intertemporal discount rate. Notice also that, under general equilibrium conditions, the real interest rate, \( r(t) \), equals \( \alpha A k_t^{\alpha-1} - \delta \) in (6.2), which is the marginal productivity of capital. In this model, there are three steady-states: the first corresponds to the origin, the second is \( (k^*, c^*) \) with \( k^* > 0 \) and \( c^* > 0 \) and the third is \( (k^{**}, 0) \) with \( k^{**} > 0 \). As in the Solow model, we are interested in the interior steady-state, which corresponds now to \( (k^*,c^*) \):
\[ c^* = Ak^\alpha - (\delta + n)k^* \]  
(6.3)

\[ k^* = \left( \frac{A\alpha}{\delta + \rho} \right)^{\frac{1}{1-\alpha}} \]  
(6.4)

In the continuous version of the model, the Jacobian matrix is given by

\[
J(k^*, c^*) = \begin{bmatrix}
\rho - n & -1 \\
B & 0 
\end{bmatrix},
\]
where \( B = -\frac{(1-n)(\delta+\rho)}{\alpha \theta} \left[ (1-\alpha)\delta + \rho - \alpha n \right] < 0. \)

The trace, \( \text{Tr} \left( J \right) \), is \( \rho - n \), which is positive by the transversality condition (see Barro and Sala-i-Martin, 2004, Chapter 2) and the determinant, \( \det \left( J \right) \), is \( B \), which is negative also by the transversality condition. The sign of the determinant implies that the two eigenvalues have opposite signs. As a result, the dynamical system is locally saddle-path stable.

As the differential equations (6.1) and (6.2) have order 1, we shall apply first-order discretizations to the Ramsey-Cass-Koopmans model. In this example we are considering a 2-dimensional dynamical system; thus, a hybrid discretization is a possible choice besides backward- and forward-looking. Therefore, we shall apply the different possible discretizations to the original dynamical system of the Ramsey-Cass-Koopmans model and study the different results that may be obtained.

### 6.1 Backward-looking discretization

Applying a first-order backward-looking discretization to (6.1) and (6.2), we obtain

\[
k_{t+h} \approx k_t + h \left[ Ak^\alpha_t - c_t - (\delta + n)k_t \right] = g_1(k, c) \]  
(6.5)

\[
c_{t+h} \approx c_t + h \left[ \frac{c_{t}}{\theta} (\alpha Ak_{t}^{\alpha-1} - \delta - \rho) \right] = g_2(k, c) \]  
(6.6)

Proposition 4.1 guarantees that the steady-state is invariant to the step and direction of the discretization so that \((k^*, c^*)\) in (6.3) and (6.4) is the steady-state of (6.5) and (6.6).

Let us focus now on the stability properties of the backward-looking discretization. As we already mentioned, the continuous-time system is characterized by a saddle-path stability.
Proposition 6.1. The steady-state \((k^*, c^*)\) is saddle-path stable for the dynamics in (6.5) and (6.6) if \(0 < h < \frac{(\rho - n) + \sqrt{(\rho - n)^2 - 4B}}{|B|}\). Furthermore, the dynamics are monotonic if \(h < \rho - n - \sqrt{(\rho - n)^2 - 4B^2}\).

Proof. To study the stability of the equilibrium, we calculate the Jacobian matrix:

\[
J_1(k^*, c^*) = \begin{bmatrix} 1 + h(\rho - n) & -h \\ hB & 1 \end{bmatrix},
\]

where \(\text{Tr}(J_1) = 2 + h(\rho - n)\) and \(\det(J_1) = Bh^2 + h(\rho - n) + 1\). Therefore, the discriminant \(\Delta_1\) is given by

\[
\Delta_1 = [2 + h(\rho - n)]^2 - 4 \left[1 + h(\rho - n) + h^2B\right] = 4 + 4h(\rho - n) + h^2(\rho - n)^2 - 4 - 4h(\rho - n) - 4h^2B = h^2\left((\rho - n)^2 - 4B \right).
\]

Since \(B\) is negative, \(\Delta_1\) is always greater than zero. Thus, the existence of oscillatory behavior brought by complex eigenvalues is excluded.

We evaluate the characteristic polynomial of the Jacobian \(J_1\) to study the signs of the eigenvalues. We have

\[
p(\lambda) = \begin{bmatrix} 1 + h(\rho - n) - \lambda & -h \\ hB & 1 - \lambda \end{bmatrix} \Leftrightarrow
\]

\[
\Leftrightarrow p(\lambda) = \lambda^2 - [2 + h(\rho - n)]\lambda + 1 + h(\rho - n) + h^2B. \tag{6.7}
\]

We have \(p(1) = h^2B < 0\), since \(B < 0\). This means the eigenvalues of \(J_1\) are on opposite sides of 1, in particular one eigenvalue of \(J_1\) is larger than 1. In order to have a saddle path the other eigenvalue of \(J_1\) must be in \((-1, 1)\). This is guaranteed if \(p(-1) = Bh^2 + 2(\rho - n)h + 4 > 0\).

In Appendix A.3, we find that, for \(h \in (0, 1]\), \(p(-1) > 0\) if and only if \(0 < h < \frac{(\rho - n) + \sqrt{(\rho - n)^2 - 4B}}{|B|} = h_+\).

In discrete-time, non-monotonic behavior may arise from a negative eigenvalue of \(J_1\). Since we know that one of the eigenvalues of \(J_1\) is positive, a sufficient condition for monotonic dynamics is that \(\det(J_1) > 0\) so that both eigenvalues have the same sign. Notice that \(\det = \lambda_1 \times \lambda_2\), where \(\lambda_1\) and \(\lambda_2\) are the two eigenvalues of a 2 \times 2

\[1\]See Appendix A.2.
matrix. Therefore, we have
\[ \det(J_1) = 0 \iff Bh^2 + (\rho - n)h + 1 = 0. \]

The zeros are \( \bar{h}_+ = \frac{(\rho - n) + \sqrt{(\rho - n)^2 - 4B}}{2B} < 0 \) and \( \bar{h}_- = \frac{(\rho - n) - \sqrt{(\rho - n)^2 - 4B}}{2B} > 0. \) Since \( \bar{h}_- < h_+ \) for all parameter values, monotonicity of the dynamics further requires that \( 0 < h < \bar{h}_- \).

Using the parameter values \( \alpha \approx 0.3, n \approx 0.01, \delta \approx 0.05, \rho \approx 0.02 \) and \( \theta \approx 2 \) (see Barro and Sala-i-Martin 2004, Chapter 2), we obtain \( B \approx \left( \frac{-49}{3000} \right) \left[ \frac{70}{300} - \frac{6}{100} \right] \approx -0.00425. \) Thus, \( h_+ = \frac{(0.01) + \sqrt{(0.01)^2 - 4B}}{|B|} \), replacing \( B \), we obtain \( h_+ = \frac{(0.01) + \sqrt{0.0171}}{|0.00425|} \). Hence, a stable saddle path occurs for \( 0 < h < 33.12. \) As before, the dynamic properties would be lost only for large values of the period length, that would never be chosen in the process of discretization. Furthermore, only for values of the discretization step below 14.2, is a monotonic convergence achieved.

In any case, for \( h \in (0, 1] \), both stability and monotonicity are preserved.

**Proposition 6.2.** The speed of convergence is not invariant to the step.

**Proof.** Since just one of the eigenvalues belongs to the unit circle \((-1, 1)\), let \( \lambda_2 \) be this eigenvalue. Solving \( p(\lambda) = 0 \) in (6.7) we obtain
\[ \lambda = \frac{2 + h(\rho - n) \pm h\sqrt{(\rho - n)^2 - 4B}}{2}. \]

Since \( B < 0 \), we have
\[ \lambda_2 = \frac{1}{2} \left[ 2 + h(\rho - n) - h\sqrt{(\rho - n)^2 - 4B} \right]. \]

By Proposition 4.4
\[ \zeta_d = |\lambda_2 - 1| = \frac{1}{2} \left| h(\rho - n) - h\sqrt{(\rho - n)^2 - 4B} \right| = \frac{1}{2} h \left| (\rho - n) - \sqrt{(\rho - n)^2 - 4B} \right|. \]

In continuous time, the stable eigenvalue is \( \zeta_c = \frac{1}{2} \left| (\rho - n) + \sqrt{(\rho - n)^2 - 4B} \right| \) (e.g. Barro and Sala-i-Martin 2004, Chapter 2) so that \( \zeta_d = \zeta_c \) only occurs if
\[ h \left| (\rho - n) - \sqrt{(\rho - n)^2 - 4B} \right| = (\rho - n) + \sqrt{(\rho - n)^2 - 4B}. \]
If $\lambda_2 > 0$, in which case the dynamics are monotonic in both continuous and discrete time, we have $\zeta_c > \zeta_d$ for all $h \in (0, 1]$. In fact,

$$\zeta_c - \zeta_d = (1 - h)(\rho - n) + (1 + h)\sqrt{(\rho - n)^2 - 4B} > 0.$$  

We use the above parameter values to give a numerical example of the convergence speed. First, let us set $h = 1$. In this case, the Ramsey-Cass-Koopmans rate of convergence, in our discrete-time version, is given by

$$\zeta_d = \left| \frac{2 + 0.01 - \sqrt{(0.01)^2 - 4B}}{2} - 1 \right| \approx \left| -0.06038 \right| = 0.06038.$$  

whereas, in the continuous-time version, $\zeta_c \approx 0.07$.

However, if we consider $h = \frac{1}{2}$,

$$\zeta_d = \left| \frac{2 + (0.5 \times 0.01) - \sqrt{0.5^2[(0.01)^2 - 4B]} - 1}{2} \right| \approx \left| -0.030192 \right| = 0.030192.$$  

As in Chapter 5, we find that a lower discretization step results in a smaller rate of convergence. In this numerical example, the value of the convergence speed, with $h = 1$, is too high. However, a lower step length allows us to reach a convergence rate consistent with the empirical data of transitional economies (Durlauf et al., 2005).

### 6.2 Forward-looking discretization

Given our cautionary notes, a researcher only should apply a forward-looking discretization if there is a logical explanation for an equation to have this type of structure. Namely, the mathematical equation needs to be an adequate description of the features of real world that are of interest to the researcher.

As we analyse the two equations that comprise the dynamical system of the original continuous-time Ramsey-Cass-Koopmans model, (6.1) and (6.2), we find that only the Euler consumption equation may adopt this direction. Equation (6.2) incorporates an underlying interest rate, $r(t)$, brought by the budget constraint of the households and which is connected to their savings decisions. It can be argued that forward-looking households will then take their savings decisions by considering the (expected) value of the interest rate tomorrow (e.g. Bosi and Ragot, 2012).
CHAPTER 6. THE RAMSEY-CASS-KOOPMANS MODEL

However, there is no plausible reason for (6.2) to have a forward-looking nature. Therefore, we conclude that we should not apply a forward-looking discretization to both (6.1) and (6.2). Instead, we shall adopt a hybrid discretization.

6.3 Hybrid discretization

A hybrid discretization consists of applying a backward-looking and a forward-looking discretization to different equations of the same dynamical system. As we mentioned before, a forward-looking direction may be applied to the Euler consumption equation in the Ramsey-Cass-Koopmans model given the existence of an underlying interest rate and of forward-looking households.

Applying a backward-looking and a standard forward-looking discretization to (6.1) and (6.2) respectively, we obtain:

\[
\begin{align*}
  k_{t+h} &\approx k_t + h \left[ Ak_t^\alpha - c_t - (\delta + n)k_t \right] = g_1(k_t, c_t) \quad (6.8) \\
  c_{t+h} &\approx c_t + h \left[ \frac{c_{t+h}}{\theta} (\alpha Ak_{t+h}^\alpha - \delta - \rho) \right] = g_2(k_{t+h}, c_{t+h}) \quad (6.9)
\end{align*}
\]

Notice that, under this formulation, the interest rate, \( r \), appears in equation (6.9) implicitly and is given by \( r_{t+h} = \alpha Ak_{t+h}^\alpha - \delta \).

The steady-state \((k^*, c^*)\) remains equal to the continuous-time version of the Ramsey-Cass-Koopmans given Proposition 4.1.

**Proposition 6.3.** The steady-state \((k^*, c^*)\) is saddle-path stable for the dynamics of (6.8) and (6.9), under reasonable calibrations. Furthermore, the dynamics are monotonic.

**Proof.** To study the stability of the equilibrium, we calculate the Jacobian matrix:

\[
J_2(k^*, c^*) = \begin{bmatrix}
  1 + h(\rho - n) & -h \\
  h[1-h(\rho - n)]B & 1
\end{bmatrix},
\]

where \( \text{Tr}(J_2) = 2 + h(\rho - n) \) and \( \det(J_2) = [1 + h(\rho - n)] + h^2[1 - h(\rho - n)]B \). The

---

2See Chapter 5
3See Appendix A.4
Further we assume \( \Delta_2 \) is given by

\[
\Delta_2 = [2 + h(\rho - n)]^2 - 4\left[1 + h(\rho - n) + h^2[1 - h(\rho - n)]B\right]
\]

\[
= 4 + 4h(\rho - n) + h^2(\rho - n)^2 - 4 - 4h(\rho - n) - 4h^2[1 - h(\rho - n)]B
\]

\[
= h^2(\rho - n)^2 - 4h^2[1 - h(\rho - n)]B
\]

\[
= h^2\left\{ (\rho - n)^2 - 4[1 - h(\rho - n)]B \right\}.
\]

In Appendix A.5, we show that only for values of \( h < \frac{1}{(\rho - n)} - \frac{(\rho - n)}{4B} \), is the discriminant \( \Delta_2 \) positive. However, with the values of \( \alpha \approx 0.3, n \approx 0.01, \delta \approx 0.05, \rho \approx 0.02 \) and \( \theta \approx 2 \), we find that only for a value of \( h \approx 100 \) or above do we obtain \( \Delta_2 < 0 \). Thus, we shall assume in our calculations that the discriminant is positive.

We study the signs of the eigenvalues through the characteristic polynomial of the Jacobian \( J_2 \):

\[
p(\lambda) = \begin{vmatrix}
1 + h(\rho - n) - \lambda & -h \\
1 - h(\rho - n)B & 1 - \lambda
\end{vmatrix}
\]

\[
\Leftrightarrow p(\lambda) = \lambda^2 - [2 + h(\rho - n)]\lambda + 1 + h(\rho - n) + h^2[1 - h(\rho - n)]B. \tag{6.10}
\]

We want to study the position of the eigenvalues with respect to the stability domain \((-1, 1)\) so that the original dynamics be obtained. A stable saddle path occurs if \( p(1)p(-1) < 0 \).

Since \( B < 0 \) and \( h > 0 \), \( p(1) = h^2[1 - h(\rho - n)]B < 0 \) if \( h < \frac{1}{(\rho - n)} \). Notice that, under typical calibrations \( \rho - n = 0.01 \), thus for \( h < 100 \) we obtain \( p(1) < 0 \). As \( 0 < h \leq 1 \), \( p(1) \) is always negative which means that the eigenvalues are on opposite sides of 1. This is compatible with the occurrence of a stable saddle path if \( p(-1) > 0 \). We have \( \det(J_2) = [1 + h(\rho - n)] + h^2[1 - h(\rho - n)]B \) which can be simplified to

\[
\det(J_2) = [1 + h^2B] [1 - h(\rho - n)] + 2h(\rho - n).
\]

Under reasonable calibrations \( \rho - n < 1 \) and \( |B| < 1 \), \( \det(J_2) > 0 \)\(^4\) Hence, the eigenvalues have the same sign. As \( p(1) < 0 \), one eigenvalue is positive, thus the other one is also positive. Hence, \( p(-1) > 0 \).

Furthermore, since the eigenvalues are both positive, a monotonic behavior is

\(^4\)Note that the parameters \( \rho \) and \( n \) refer to rates, therefore \( \rho \in (0, 1) \) and, usually, \( n \in (0, 1) \). Further we assume \( B = \left[-\frac{(1-\alpha)(\delta + \rho)}{\alpha\delta}\right][1 - \delta(\delta + \rho - \alpha n)] < 1 \).
ensured. □

**Proposition 6.4.** *The speed of convergence is not invariant to the step.*

*Proof.* Given the saddle-path stability, we are mostly interested in the eigenvalue belonging to \((-1, 1)\). Let this be \(\lambda_2\).

Solving \(p(\lambda) = 0\) in (6.10) we obtain

\[
\lambda = \frac{2 + h(\rho - n) \pm h \sqrt{(\rho - n)^2 - 4 \left[1 - h(\rho - n)\right] B}}{2}
\]

and

\[
\lambda_2 = \frac{1}{2} \left[2 + h(\rho - n) - h \sqrt{(\rho - n)^2 - 4 \left[1 - h(\rho - n)\right] B}\right].
\]

Given Proposition 4.4,

\[
\zeta_d = |\lambda_2 - 1| = \frac{1}{2} \left|h(\rho - n) - h \sqrt{(\rho - n)^2 - 4 \left[1 - h(\rho - n)\right] B}\right| = \frac{1}{2} h \left|(\rho - n) - \sqrt{(\rho - n)^2 - 4 \left[1 - h(\rho - n)\right] B}\right|.
\]

Given the original speed of convergence \(\zeta_c = \frac{1}{2} \left[(\rho - n) + \sqrt{(\rho - n)^2 - 4B}\right] \) (e.g. Barro and Sala-i-Martin 2004, Chapter 2), \(\zeta_d = \zeta_c\) only occurs if

\[
h \sqrt{(\rho - n)^2 - 4 \left[1 - h(\rho - n)\right] B} = \rho - n + \sqrt{(\rho - n)^2 - 4B}.
\]

In fact,

\[
\zeta_c - \zeta_d = (1 - h)(\rho - n) + \sqrt{(\rho - n)^2 - 4B} + h \sqrt{(\rho - n)^2 - 4 \left[1 - h(\rho - n)\right] B} > 0.
\]

Therefore, the original \(\zeta_c\) is never recovered. □

We use the same values of the parameters as before to give an numerical example. With \(h = 1\), the hybrid version of the speed of convergence is

\[
\zeta_d = \left|\frac{2 + 0.01 - \sqrt{(0.01)^2 - 4(1 - 0.01)B}}{2} - 1\right| \approx |0.060058| = 0.060058.
\]

And, defining \(h = \frac{1}{2}\) we achieve \(\zeta_d \approx 0.030111\).

In Section 6.1 we find that the speed of convergence in the backward-looking discretization is \(\zeta_d = \frac{1}{2} h \left|(\rho - n) - \sqrt{(\rho - n)^2 - 4B}\right|\), whereas in the hybrid discretization \(\zeta_d = \frac{1}{2} h \left|(\rho - n) - \sqrt{(\rho - n)^2 - 4 \left[1 - h(\rho - n)\right] B}\right|\). Notice that \(1 - h(\rho - n)\) is the only difference between the two discretizations we preformed.
Corollary 6.5. *The speed of convergence in the hybrid discretization is smaller than that of the backward-looking discretization.*

We apply different discretizations to the Ramsey-Cass-Koopmans model to analyse possible implications to the stability properties. Our results suggest that the direction of the discretization process is not innocuous. The two discretizations that we adopt show different Jacobian matrices, and thus, different traces and a determinants. However, the saddle-path stability is a robust feature of the model as in both discretizations we obtained the original dynamics. Our results are consistent with Bosi and Ragot (2012) whose hybrid discretization leads also to a stable saddle-path.

Additionally, our results suggest that different directions generate different rates of convergence, whereas, in both discretizations, a smaller step leads to a smaller rate.
Chapter 7

Forward-looking: further analysis

As we mentioned, a forward-looking discretization could be applied to the Euler consumption equation in the Ramsey-Cass-Koopmans model. The reason relies on the fact that an interest rate underlies this equation. In Economics, this tends to be the reason why some researchers use the forward-looking direction when dealing with models with forward-looking agents. However, when we choose this type of discretization, additional precautions are necessary.

To begin with, we must assert that all variables present in the equation have foundations that justify the adoption of a forward nature. For example, in the Euler consumption equation, it could be interesting to consider a forward behavior in the variable \( k \) as its marginal productivity determines the interest rate in general equilibrium. However, it is not clear whether it is appropriate to have future values of consumption, \( c_{t+h} \), affecting the variation \( c_{t+h} - c_t \).

Furthermore, we have to pay attention to the indirect implications that we may originate by using a variable at time \( t + h \). Even though we are only focusing on the dynamical system of the continuous-time model\(^1\), there are implications from the point of view of the discrete-time optimal control problem that underlies the dynamics of the economy, when we decide to choose one temporal reference over the other. If we follow the standard formula of the forward-looking discretization (3.4), we will consider consumption at time \( t + h \) and capital stock also at time \( t + h \) on the right-hand side of the Euler equation. This procedure means that the underlying budget constraint features an interest rate, \( r \), at time \( t + h \), due to \( k_{t+h} \), while the underlying Hamiltonian function features a shadow price of capital (the co-state variable of the model), \( v \), at time \( t + h \), due to \( c_{t+h} \).

Let us focus first on the co-state variable of the model, \( i.e., v \). In the Hamil-

\(^1\)See Chapter 3.
tonian of the Ramsey-Cass-Koopmans model, it controls for the present value of
an increment of households’ assets occurred at time $t$ in units of utils by measur-
ing the impact of postponing consumption today on tomorrow’s utility. Typically,
researchers use $v_t$ when dealing with similar models in discrete-time. See, for in-
stance, Gómez (2014). Furthermore, it is not clear what could be the meaning of
$v_{t+h}$ given the above interpretation of the co-state variable in the Hamiltonian. To
study the impact of the temporal reference regarding variable $c$, we discretize, as an
alternative, the Ramsey-Cass-Koopmans model with consumption at time $t$ in the
Euler consumption equation. In Appendix A.6 we show that the Jacobian obtained
is exactly the same as in the discretization in Section 6.3. However, we note that,
even though the results are innocuous for the Ramsey-Cass-Koopmans model, other
models might have different outcomes.

As regards the consideration of $k_{t+h}$, although it can be understandable to use
this variable at time $t + h$ in the discretization process as an effort to feature the
household’s forward-looking behavior concerning savings decisions, these choices
must be consistent with the underlying assumptions of the original dynamical model.
We find that the consideration of an (implicit) interest rate, $r_{t+h}$, may be inconsis-
tent with the aggregate budget constraint. To show this, we study the derivation
of the general equilibrium when we consider $r_{t+h}$. Therefore, let the discrete-time
household’s budget constraint be:

$$b_{t+1} - b_t = r_{t+1}b_t + w_t - c_t.$$ 

Under the usual general equilibrium conditions for a closed economy, $b_t = k_t$, $r_{t+1} = f'(k_{t+1})$ (herein, we consider $\delta = 0$ for the sake of simplicity) and $w_t = f(k_t) - k_tf'(k_t)$, the aggregate budget constraint becomes:

$$k_{t+1} - k_t = f'(k_{t+1})k_t + f(k_t) - k_tf'(k_t) - c_t.$$ 

Therefore, the condition of equilibrium in the market of goods and services is:

$$f(k_t) = c_t + (k_{t+1} - k_t) + k_t (f'(k_{t+1}) - f'(k_t)).$$ 

In this case, the residual brought by $k_t (f'(k_{t+1}) - f'(k_t))$ does not let us reach the
usual equilibrium condition, which is $f(k_t) = c_t + (k_{t+1} - k_t)$. The interpretation of
this residual remains unclear.

Acemoglu (2009) provides an approach that is compatible with this equilibrium
condition. The author assumes that $r_{t+1} = f'(k_t)$, that is, individuals rent one unit of capital at time $t$ in terms of date $t + 1$ goods. The budget constraint is given by $b_{t+1} = (1 + r_{t+1})b_t - c_t + w_t$, the timing underlying here being that the individuals rent their capital or asset holdings, $b_t$, to firms to be used as capital at time $t + 1$. Even though this strategy allows Acemoglu (2009) to be consistent with the underlying assumptions, in our analysis the question still remains. Since the assumption that $r_{t+1} = f'(k_t)$ leads to a backward-looking discretization, we are unable to explain what it really means to have a forward-looking variable $k_{t+h}$.

We conclude that the questions that arise pertaining to the forward-looking direction suggest that this approach should be taken with caution.
Chapter 8

Conclusion

In this dissertation, we analyse the effect of time discretization on two well-known growth models, originally built in a continuous-time setup. For that purpose, we follow the view of Bosi and Ragot (2012) and apply Euler discretizations to the dynamical systems of the Solow and Ramsey-Cass-Koopmans model. This framework requires certain choices concerning the discretization step, order and direction, which, often, may lead to different stability properties. Given this, we study the impact of the step and direction of the discretization on the steady-state, stability properties and speed of convergence.

The gap in the literature on this issue motivated the study since, typically, authors do not provide an explanation for the use of continuous or discrete time. Through the existent literature, we find that few authors justify the use of continuous time over discrete time. Often, the choice of time in a model is constrained by the mathematical theories and tools that are familiar to the authors. For instance, it is common to find, in the growth literature, a preference for a continuous-time setup. However, when researchers need a discrete-time framework, they often use discretization methods that may affect the results, e.g., the stability properties of the steady-state may differ implying different recommendations to policy making (Anagnostopoulos and Giannitsarou, 2005).

As Bosi and Ragot (2012), we conclude that the steady-state is invariant to the discretization step and direction. Furthermore, in the analysis of the Solow model we show that it is necessary that \( h < \frac{2}{(\alpha-1)(\eta+\delta)} \) for the stability to be preserved and \( h < \frac{1}{(\alpha-1)(\eta+\delta)} \) to observe a monotonic behavior. However, for typical calibrations, we find that these condition have very high values; thus, we conclude that the stability properties of the steady-state are preserved under \( h \in (0,1] \). Concerning the speed of convergence, the discretization step has a relevant impact since for any \( h < 1 \),
this rate would differ from the one in continuous time. In fact, we conclude that lower values of $h$, lead to a smaller rate of convergence. Given this, a smaller step in the discretization version of the Solow model would help to obtain a lower rate of convergence, often considered too high in the light of the empirical evidence.

The results concerning the Ramsey-Cass-Koopmans are analogous to the Solow model. However, in this model we are able to study different discretizations with different directions. We first apply a backward-looking discretization and we find that, in order to maintain the original saddle-path stability, the following condition is required: $h < \frac{(\rho - n) + \sqrt{(\rho - n)^2 - 4B}}{|B|}$. Furthermore, the dynamics are monotonic if and only if $h < \frac{(\rho - n) - \sqrt{(\rho - n)^2 - 4B}}{2B}$. However, as in the Solow model, these conditions for the step are greater than its domain. Therefore, the stability properties are kept under a backward-looking discretization. Additionally, we find that the continuous-time speed of convergence is never recovered since it is not invariant to the step. As a matter of fact, $\zeta_c > \zeta_d$ whatever $h$. Moreover, the smaller the step, the lower the rate of convergence.

The second discretization we perform is a hybrid discretization. In this analysis, we also achieve a stable saddle path and it seems that it is only necessary to have reasonable calibrations for this to occur, i.e., $(\rho - n) < 1$ and $|B| < 1$. Our results are consistent with Bosi and Ragot (2012) even though in their analysis they study the original Ramsey model (instead of the modern Cass-Koopmans version of the model). Nevertheless, we find the speed of convergence not invariant to the step.

In spite of the stability properties being preserved under both types of discretizations of the Ramsey-Cass-Koopmans model, the different Jacobian matrices and, thus, different traces and determinants suggest that the direction of the discretization is not an innocuous choice. Different outcomes may be achieved in other models; therefore, this choice should be carefully analysed. In particular, our analysis suggests that the forward-looking discretization is not fully understood and may have indirect implications in terms of the economic theory the model aims to describe. As a result, we remain doubtful about its use.

Overall, our results show that the discretization step has an impact on the speed of convergence leading to outcomes different from the continuous-time. Moreover, the different directions may also generate different results as the rate of convergence in the hybrid discretization is smaller than that of the backward-looking discretization. Hence, we conclude that the speed of convergence is not invariant to the step and direction of the discretization.

As future research, it would be interesting to deepen the knowledge of the
forward-looking discretization, and not only regarding the Euler discretization, but also in the context of other methods, since, nowadays, computers might use more sophisticated methods. On the other hand, it would be interesting to apply an approach similar to the one explored in this thesis to more recent, sophisticated, models, e.g., the class of endogenous growth models.
Appendix A

Mathematical proof

A.1 Detailed Proof of Proposition 5.1.

To study the stability of $k^*$, we shall evaluate the eigenvalue, given by:

$$\frac{dg(k)}{dk}(k^*) = 1 + h\{sαk^{α-1} - (n + \delta)\}.$$

Replacing $k^*$, we get

$$\frac{dg}{dk}(k^*) = 1 + h\{sα\left[\left(\frac{s}{n + \delta}\right)^{\frac{1}{1-α}}\right]^{α-1} - (n + \delta)\} = 1 + h[(α - 1)(n + \delta)].$$

According to Theorem 4.3, $k^*$ is stable if $\frac{dg}{dk}(k^*) \in (-1, 1)$. Therefore, the following condition must be valid:

$$-1 < 1 + h[(α - 1)(n + \delta)] < 1.$$

First, since $h > 0$ and $(α - 1)(n + \delta) < 0$, the inequality $1 + h[(α - 1)(n + \delta)] < 1 \Leftrightarrow h[(α - 1)(n + \delta)] < 0$ is always valid. Hence, we just need to verify for which values of $h$ the following inequality is valid:

$$1 + h[(α - 1)(n + \delta)] > -1 \Leftrightarrow$$

$$\Leftrightarrow h[-(1 - α)(n + \delta)] > -2 \Leftrightarrow$$
Thus, we prove that \( k^* \) is stable if \( h < \frac{2}{(1 - \alpha)(n + \delta)} \).

Because, in continuous time, the dynamics properties of this model are brought by a monotonic convergence to the steady state, we shall study for which values of \( h \) the monotonic convergence is obtained. Since, in discrete time, \( \frac{dg}{dk}(k^*) \) must be positive in order to have a monotonic convergence, we shall study the following condition:

\[
1 + h \left[ (\alpha - 1)(n + \delta) \right] > 0 \iff \iff \iff \iff h < \frac{1}{(1 - \alpha)(n + \delta)}.
\]

Therefore, for values of \( h < \frac{1}{(1 - \alpha)(n + \delta)} \), a monotonic convergence is achieved.

### A.2 Computing the Jacobian matrix in the Proof of Proposition 6.1.

To study the stability of the interior steady-state \((k^*, c^*)\), we must evaluate the Jacobian matrix:

\[
J_1 = \begin{bmatrix}
\frac{\partial g_1}{\partial k} & \frac{\partial g_1}{\partial c} \\
\frac{\partial g_2}{\partial k} & \frac{\partial g_2}{\partial c}
\end{bmatrix}_{|(k^*, c^*)},
\]

where \( g_1(k, c) = k + h \left[ Ak^\alpha - c - (\delta + n)k \right] \) and \( g_2(k, c) = c + h \left[ \frac{\delta}{\alpha} (\alpha A k^{\alpha - 1} - \delta - \rho) \right] \).

The inputs to \( J_1 \) are the following. First, we have:

\[
\frac{dg_1}{dk}(k^*, c^*) = 1 + h \left[ A\alpha (k^*)^{\alpha - 1} - (\delta + n) \right]. \tag{A.1}
\]

By replacing \( k^* \) in (A.1), we obtain:
\[
\frac{dg_1}{dk}(k^*, c^*) = 1 + h\left[A\alpha \left(\frac{A\alpha}{\delta + \rho}\right)^{\frac{1}{1-n}} - (\delta + n)\right] = 1 + h\left[A\alpha \left(\frac{A\alpha}{\delta + \rho}\right)^{-1} - (\delta + n)\right] = 1 + h\left[A\alpha \left(\frac{\delta + \rho}{A\alpha}\right) - (\delta + n)\right] = 1 + h(\rho - n).
\]

We have
\[
\frac{dg_1}{dc}(k^*, c^*) = -h
\]

and
\[
\frac{dg_2}{dk}(k^*, c^*) = h\left\{\frac{1}{\theta} A\alpha (\alpha - 1)(k^*)^{\alpha - 2}(c^*)\right\}. \tag{A.2}
\]
Replacing \(k^*\) and \(c^*\) in (A.2), we get:
\[
\frac{dg_2}{dk}(k^*, c^*) = h\left\{\frac{1}{\theta} A\alpha (\alpha - 1)\left[\left(\frac{A\alpha}{\delta + \rho}\right)^{\frac{1}{1-n}}\right]^{\alpha - 2}\right\}\left\{A\left(\frac{A\alpha}{\delta + \rho}\right)^{\frac{1}{1-n}} - (\delta + n)\left(\frac{A\alpha}{\delta + \rho}\right)^{\frac{2}{1-n}}\right\}.
\]

Multiplying the exponents and applying the distributive property we get:
\[
h\left\{\frac{1}{\theta} A\alpha^2 (\alpha - 1)\left(\frac{A\alpha}{\delta + \rho}\right)^{-2}\right\} - h\left\{\frac{1}{\theta} A\alpha (\alpha - 1)(\delta + n)\left(\frac{A\alpha}{\delta + \rho}\right)^{-1}\right\} = h\left\{\frac{1}{\theta} A\alpha^2 (\alpha - 1)\left[\frac{\delta + \rho}{\alpha}\right]^2\right\} - h\left\{\frac{1}{\theta} A\alpha (\alpha - 1)(\delta + n)(\delta + \rho)\right\} = h\left\{-\frac{(1 - \alpha)(\delta + \rho)}{\alpha\theta}\left[(1 - \alpha)\delta + \rho - \alpha n\right]\right\} = hB.
\]
Finally,
\[
\frac{dg_2}{dc}(k^*, c^*) = 1 + h\left[\frac{1}{\theta} (\alpha Ak^{a - 1} - \delta - \rho)\right]. \tag{A.3}
\]
Replacing \(k^*\) in (A.3), we get:
\[
\frac{\partial g_2}{\partial c}(k^*, c^*) = 1 + h\left[\frac{1}{\theta} (\alpha A\left(\frac{A\alpha}{\alpha A}\right)^{\frac{a-1}{1-n}} - \delta - \rho)\right] = 1 + h\left[\frac{1}{\theta} (\alpha A\left(\frac{\delta + \rho}{\alpha A}\right) - \delta - \rho)\right] = 1.
\]

38
Replacing the partial derivatives in the Jacobian matrix,

\[ J_1(k^*, c^*) = \begin{bmatrix} 1 + h(\rho - n) & -h \\ hB & 1 \end{bmatrix}. \]

### A.3 The characteristic polynomial in the Proof of Proposition 6.1.

We determine the characteristic polynomial \((6.7)\) of the Jacobian \(J_1\) to study the signs of the eigenvalues:

\[
p(\lambda) = \lambda^2 - [2 + h(\rho - n)]\lambda + 1 + h(\rho - n) + h^2B.
\]

From Section \(6.1\), we know that \(p(1) < 0\). However, to study the sign of \(p(-1) = Bh^2 + 2(p - n)h + 4\), we need to study for which values of \(h\) the following condition is valid:

\[ Bh^2 + 2(p - n)h + 4 > 0. \]

Since \(B < 0\), the inequality holds for \(h_- < h < h_+\), where \(h_-\) and \(h_+\) are the solutions of \(Bh^2 + 2(p - n)h + 4 = 0\) such that \(h_+ > h_-\). Hence,

\[
h_\pm = \frac{-[2(p - n)] \pm \sqrt{[2(p - n)]^2 - 16B}}{2B} = \frac{(\rho - n) \pm \sqrt{(\rho - n)^2 - 4B}}{|B|},
\]

and \(h_+ = \frac{(\rho - n) + \sqrt{(\rho - n)^2 - 4B}}{|B|} > 0\) and \(h_- = \frac{(\rho - n) - \sqrt{(\rho - n)^2 - 4B}}{|B|} < 0\).

### A.4 Computing the Jacobian matrix in the Proof of Proposition 6.4.

In order to compute the partial derivatives of \(J_2\), the different terms corresponding to variable \(c_{i+h}\) must be on the same side of the equation. Therefore,
APPENDIX A. MATHEMATICAL PROOF

\[ c_{t+h} \approx c_t + h \left[ \frac{c_{t+h}}{\theta} \left( \alpha A k_{t+h}^{\alpha - 1} - \delta - \rho \right) \right] \]

\[ \iff c_{t+h} - h \left[ \frac{c_{t+h}}{\theta} \left( \alpha A k_{t+h}^{\alpha - 1} - \delta - \rho \right) \right] \approx c_t \iff \]

\[ \iff \left[ 1 - \frac{h}{\theta} \left( \alpha A k_{t+h}^{\alpha - 1} - \delta - \rho \right) \right] c_{t+h} \approx c_t \iff \]

\[ \iff c_{t+h} \approx \frac{c_t}{1 - \frac{h}{\theta} \left( \alpha A k_{t+h}^{\alpha - 1} - \delta - \rho \right)} = g_2(k_{t+h}, c_t) \]

and \( g_1(k_t, c_t) = k_t + h \left[ Ak_t^\alpha - c_t - (\delta + n)k_t \right] \).

The inputs of \( J_2 \) are the following.

The calculations for \( \frac{dg_1}{dk}(k^*, c^*) \) and \( \frac{dg_1}{dc}(k^*, c^*) \) are the same as in the previous discretization. See in Appendix A.2.

To compute \( \frac{dg_2}{dk}(k^*, c^*) \), we must replace \( k_{t+h} \) in \( g_2 \).

\[ g_2(k_{t+h}, c_t) = c_t \left\{ \frac{1}{1 - \frac{h}{\theta} \left[ \alpha A \left[ k_t + h[Ak_t^\alpha - c_t - (\delta + n)k_t] \right]^{\alpha - 1} - \delta - \rho \right]} \right\}. \]

Thus,

\[ \frac{dg_2}{dk}(k^*, c^*) = c^* \left\{ -1 \left[ \frac{1}{1 - \frac{h}{\theta} \left[ \alpha A \left[ k_t + h[Ak_t^\alpha - c_t - (\delta + n)k_t] \right]^{\alpha - 1} - \delta - \rho \right]} \right]^2 \right\}. \]

Since we are in the steady-state, \( k_{t+h} = k_t + h[Ak_t^\alpha - c_t - (\delta + n)k_t] = k^* \). Therefore,

\[ \frac{dg_2}{dk}(k^*, c^*) = c^* \left\{ \frac{\frac{h}{\theta} \left[ \alpha A(\alpha - 1)[k^*]^{\alpha - 2} \right] \left[ 1 + h(\alpha A(k^*)^{\alpha - 1} - (\delta + n)) \right]}{1 - \frac{h}{\theta} \left[ \alpha A(k^*)^{\alpha - 1} - \delta - \rho \right]} \right\}. \]
Let us work first on the numerator
\[ c^* \frac{h}{\theta} \left[ \alpha A (\alpha - 1) \right]^{\alpha - 2} \left[ 1 + h (\alpha A (k^*)^{\alpha - 1} - (\delta + n)) \right]. \]

Replacing \( k^* \), we obtain:
\[ \frac{h}{\theta} c^* \left\{ \alpha A (\alpha - 1) \left[ \left( \frac{A \alpha}{\delta + \rho} \right)^{\frac{1}{\alpha - 1}} \right]^{\alpha - 2} \left[ 1 + h \left( \alpha A \left( \frac{A \alpha}{\delta + \rho} \right)^{\frac{1}{\alpha - 1}} - (\delta + n) \right) \right] \right\} =
\[ = h \frac{c^*}{\theta} \left\{ (\alpha - 1) A \alpha \left( \frac{\delta + \rho}{A \alpha} \right) \left( \frac{1}{k^*} \right) \left[ 1 + h \left( (\delta + \rho) - (\delta + n) \right) \right] \right\} =
\[ = h \frac{c^*}{\theta} \left\{ (\alpha - 1)(\delta + \rho) \left( \frac{1}{k^*} \right) \left[ 1 + h (\rho - n) \right] \right\}. \]

Replacing \( c^* = \left[ A (k^*)^{\alpha - 1} - (\delta + n) \right] k^* \), we get
\[ = h \frac{c^*}{\theta} \left[ A (k^*)^{\alpha - 1} - (\delta + n) \right] k^* (\alpha - 1)(\delta + \rho) \left( \frac{1}{k^*} \right) \left[ 1 + h (\rho - n) \right] =
\[ = h \frac{c^*}{\theta} \left[ A (k^*)^{\alpha - 1} - (\delta + n) \right] (\alpha - 1)(\delta + \rho) \left[ 1 + h (\rho - n) \right]. \]

Replacing \( k^* \), we obtain:
\[ = h \frac{c^*}{\theta} \left[ A \left( \frac{A \alpha}{\delta + \rho} \right)^{-1} - (\delta + n) \right] (\alpha - 1)(\delta + \rho) \left[ 1 + h (\rho - n) \right] =
\[ = h \left[ 1 - h (\rho - n) \right] \left\{ - \frac{(1 - \alpha)(\delta + \rho)}{\alpha \theta} \left[ (1 - \alpha) \delta + \rho - \alpha n \right] \right\} = h \left[ 1 - h (\rho - n) \right] B. \]

Replacing \( h \left[ 1 - h (\rho - n) \right] B \) and \( k^* \) in \( \frac{dQ_k}{dk} (k^*, c^*) \), we get
\[
\frac{dg_2}{dk}(k^*, c^*) = \frac{h[1 - h(\rho - n)] B}{1 - h[1 - h(\rho - n)] B} = h[1 - h(\rho - n)] B.
\]

Finally,
\[
\frac{dg_2}{dc}(k^*, c^*) = 1 + h \left[ \frac{1}{\theta} \left( \frac{1}{\frac{\alpha A}{\delta + \rho}} \right)^{-1} - \delta - \rho \right].
\]
Replacing \( k^* \), we get
\[
\frac{dg_2}{dc}(k^*, c^*) = 1 + h \left[ \frac{1}{\theta} \left( \frac{A \alpha}{\delta + \rho} \right)^{-\frac{1}{\alpha}} - \delta - \rho \right] = 1.
\]

Therefore, we obtain the Jacobian \( J_2 \):
\[
J_2(k^*, c^*) = \begin{bmatrix}
1 + h(\rho - n) & -h \\
\frac{h[1 - h(\rho - n)] B}{1} & 0
\end{bmatrix}.
\]

### A.5 Sign of the Discriminant in Proposition 6.4.

To preclude the existence of oscillatory behaviour, we look for \( h \) such that \( \Delta_2 > 0 \).
Since \( h > 0 \), we must evaluate the following inequation:

\[
(\rho - n)^2 - 4[1 - h(\rho - n)] B > 0 \iff
\]
\[
\iff 4[1 - h(\rho - n)] B < (\rho - n)^2 \iff
\]
\[
\iff 1 - h(\rho - n) > \frac{(\rho - n)^2}{4B} \iff
\]
\[
\iff -h(\rho - n) > \frac{(\rho - n)^2}{4B} - 1 \iff
\]
\[
\iff h < \frac{1}{(\rho - n)} - \frac{(\rho - n)}{4B}.
\]
A.6 Hybrid discretization with consumption at time $t$

We want to study if there is any difference between the forward-looking discretization with all variables at time $t+h$ and the forward-looking with just the stock of capital $k$ at time $t+h$. Applying to (6.1) and (6.2) a first order hybrid discretization with consumption at time $t$, we obtain:

\[
k_t + h \left(Ak_t^\alpha - c_t - (\delta + n)k_t \right) = g_1(k_t, c_t) \quad (A.4)
\]

\[
c_t + h \left(c_t + \frac{c_t}{\theta} (\alpha Ak_t^{\alpha-1} - \delta - \rho) \right) = g_2(k_t+h, c_t) \quad (A.5)
\]

The steady-state is the same of that in continuous time, as in the other discretizations. See Section 4. To study the stability of equilibria, we study the Jacobian matrix:

\[
J_3(k^*, c^*) = \begin{bmatrix}
\frac{dg_1}{dk} & \frac{dg_1}{dc} \\
\frac{dg_2}{dk} & \frac{dg_2}{dc}
\end{bmatrix}.
\]

The entries of $J_3$ are the following:

\[
\frac{dg_1}{dk}(k^*, c^*) = 1 + h(\rho - n)
\]

and

\[
\frac{dg_1}{dc}(k^*, c^*) = -h.
\]

In order to compute $\frac{dg_2}{dk}(k^*, c^*)$, we must replace $k_{t+h}$ in (A.5). Thus,

\[
g_2(k_t, c_t) = c_t + h \left(\frac{c_t}{\theta} \left(\alpha A \left[ k_t + h \left(Ak_t^\alpha - c_t - (\delta + n)k_t \right) \right]^{\alpha-1} - \delta - \rho \right) \right)
\]

and

\[
\frac{dg_2}{dk}(k^*, c^*) = h \theta \left(\frac{\alpha A (\alpha-1) \left[ k_t + h \left(Ak_t^\alpha - c_t - (\delta + n)k_t \right) \right]^{\alpha-2}}{\theta} \right) \left[ 1+h \left(\alpha Ak_t^{\alpha-1} - (\delta + n) \right) \right].
\]

\footnote{See in Appendix A.2}
Replacing the variables by their steady-state values, we get

\[
\frac{dg_2}{dk}(k^*, c^*) = \frac{h}{\theta} c^* \left\{ \alpha A (\alpha - 1) (k^*)^{-2} \left[ 1 + h \left( \alpha A (k^*)^{-1} - (\delta + n) \right) \right] \right\}.
\]

Replacing \( k^* \), we obtain

\[
\frac{\partial g_2}{\partial k}(k^*, c^*) = \frac{h}{\theta} c^* \left\{ \alpha A (\alpha - 1) \left[ \alpha \left( \frac{A \alpha}{\delta + \rho} \right)^{1-\alpha} \right]^{-2} \left[ 1 + h \left( \alpha A \left( \frac{A \alpha}{\delta + \rho} \right)^{1-\alpha} - (\delta + n) \right) \right] \right\}.
\]

Notice that \( \frac{\alpha-2}{1-\alpha} = -1 - \frac{1}{1-\alpha} \); thus,

\[
\frac{\partial g_2}{\partial k}(k^*, c^*) = \frac{h}{\theta} c^* \left\{ (\alpha - 1) A \alpha \left( \frac{\alpha - 1}{k^*} \right) \left[ 1 + h \left( \alpha - 1 \right) (\delta + \rho) \right] \left[ 1 + h (\rho - n) \right] \right\}.
\]

Replacing \( c^* = A (k^*)^{\alpha-1} - (\delta + n) \), we get

\[
= \frac{h}{\theta} \left[ A (k^*)^{\alpha-1} - (\delta + n) \right] k^* (\alpha - 1) (\delta + \rho) \left( \frac{1}{k^*} \right) \left[ 1 + h (\rho - n) \right] = \frac{h}{\theta} \left[ A (k^*)^{\alpha-1} - (\delta + n) \right] (\alpha - 1) (\delta + \rho) \left[ 1 + h (\rho - n) \right]
\]

and replacing \( k^* \), we obtain:

\[
= \frac{h}{\theta} \left[ A \left( \frac{A \alpha}{\delta + \rho} \right)^{-1} - (\delta + n) \right] (\alpha - 1) (\delta + \rho) \left[ 1 + h (\rho - n) \right] = h \left[ 1 - h (\rho - n) \right] B.
\]

Finally,

\[
\frac{dg_2}{dc}(k, c) = 1 + h \left[ \frac{1}{\theta} (\alpha A k_i^{\alpha-1} - \delta - \rho) \right]
\]
At steady-state, we obtain
\[
1 + h \left[ \frac{1}{\vartheta} \left( A \frac{\alpha}{\delta + \rho} \right)^{\frac{\alpha-1}{\alpha-\frac{1}{n}}} - \delta - \rho \right] = 1.
\]
Replacing the partial derivatives in the $J_3$, we obtain:
\[
J_3(k^*, c^*) = \begin{bmatrix}
1 + h(\rho - n) & -h \\
h[1 - h(\rho - n)] B & 1
\end{bmatrix},
\]
where $\text{Tr}(J_3) = 2 + h(\rho - n)$ and $\text{det}(J_3) = [1 + h(\rho - n)] + h^2[1 - h(\rho - n)] B$. 
Bibliography


