Nominated Barrier Options: An Approach to Valuing American Options on a Lattice

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Abstract

American style options are of considerable importance in the financial markets. However, to value them requires numerical methods or approximate formulae since in general no explicit formulae exist. Typically convergence to the true option value is slow resulting in practice in inaccurate prices.

In this paper we value on a lattice with novel probabilities an instrument, a nominated barrier option, whose value converges to that of an American option as the number of time steps increases. We find that the value of the nominated barrier option on the lattice seems to converge to the value of the underlying American option faster than existing methods, but with greater initial bias.

1 Introduction

American style options are an important part of the options markets. Unlike European options, American options rarely have explicit solutions. There is no known formula even for an American put on an asset following geometric Brownian motion. Consequentially approximate formula or numerical methods, such as PDE or simulation methods, need to be used.

A number of analytic approximations have been suggested in the literature. Geske and Johnson (1984) use Richardson's extrapolation on the valuation formula of a portfolio of Bermudan options with different exercise dates. This method was later modified by Bunch and Johnson (1992), Ho, Stapleton and Subrahmanyam (1994) and Huang, Subrahmanyam and Yu (1996) proposing various methods to value Bermudan options. MacMillan (1986) used a quadratic approximation approach to a PDE, extended by Baroni-Adesi and Whaley (1987). Along the same lines but with increased accuracy, specially for long term options, is the approximation by Ju and Zhong (1999). Another approach (Fu *et al.* (2001)) is to approximate the value function by a piecewise linear interpolation, so that the continuation value can be expressed as a summation of European call option values.

To get numerical solutions of the Black-Scholes PDE, subject to American option boundary conditions, several methods have been used, among others: finite difference methods (Brennan and Schwartz (1977), Courtadon (1982), Dempster and Hutton (1997), and Wu and Kwok (1997)); finite element methods (Wilmott *et al.*(1993), and Zvan, Forsyth and Vetzal (2001)); the method of lines (Meyer and van der Hoek (1997)); and the differential quadrature method (Wu and Ding (2002)). Mallier and Alobaidi (2000) applied a Laplace transform to obtain an integral equation that can be solved numerically.

Simulation methods include both Monte Carlo and lattice methods. There has been significant progress in using Monte Carlo methods to value American options, applied mainly for path-dependent payoffs. For instance, Longstaff and Schwartz (2001) and Van Roy and Tsitsiklis (2001) use regression based models to estimate continuation values from simulated paths. The stochastic mesh method, originally introduced by Broadie and Glasserman (1997a) and also used by Avramidis and Matzinger (2002), solves a randomly sampled dynamic programming problem to approximate the price of an American option. The main distinction between the stochastic mesh method and the random tree method (Broadie and Glasserman (1997b)) is that in valuing the option at a specific node the former uses values from all the nodes at the subsequent time step whereas the latter uses only the values of the successor nodes from that node. Haugh and Kogan (2001) and Rogers (2002) establish a dual approach in which the American option price is represented through a minimization problem as opposed to the most common formulation of maximization over stopping times.

Ever since lattice methods were introduced for the valuation of American options (Cox, Ross and Rubinstein (1979)) they have received much attention in the literature. Generalizations of the binomial method include the multinomial methods of Boyle (1988b), and Kamrad and Ritchken (1991). Broadie and Detemple (1996) found a modification of the binomial method that uses the Black-Scholes value at the time step just before maturity to compensate for convergence problems in the standard method. Figlewski and Gao (1999) obtain considerable gains of accuracy and efficiency using an adaptive mesh method. Alford and Webber (2001) achieved very high rates of convergence using lattices with high-order branching. The convergence of lattice methods has been established, for instance, by Amin and Khanna (1994), Broadie and Detemple (1996), Lamberton (1993), and Schmidt (1997).

A problem with using numerical methods for valuing American options is their slow rate of convergence. Only relatively inaccurate option values can be found in a short time.

This paper presents a lattice method based upon valuing an instrument, which we call a nominated barrier option, exercisable between reset dates, whose value converges to that of an American option as the number of reset dates increases. To value this instrument we exploit an idea of Kuan and Webber (2002) who used a probability correction term in a lattice for barrier options. Two alternative forms of correction are used. In the first the correction term is based upon the distribution of the maximum of a Brownian bridge connecting nodes in the lattice. In the second the hitting time distribution is calculated, and branching probabilities found by matching moments to those of the associated defective probability distribution.

Our exposition focuses on the example of the American put, but it is clear that the method generalizes to other types of American option.

The next section discusses the nominated barrier option. Section three describes the Dirichlet lattice method and how it can be applied to valuing nominated barrier options. In section four we describe a modified Dirichlet lattice and how it may be applied. We refer to this lattice as the American Dirichlet lattice. Section five gives numerical results and section six concludes.

2 The Nominated Barrier Option

A nominated barrier option is an option where at each reset date the option holder nominates a barrier level to apply until the next reset date. If the value of an underlying asset hits the barrier level the option expires but the holder receives a payoff. If the option survives until maturity, the holder again receives a payoff.

Suppose the option is created at time $t_0 = 0$ and has a final maturity time T. We suppose there are $N + 1$ equally spaced reset dates, $t_j = t_0 + j\Delta t$, $j = 0, \ldots, N$, where $\Delta t = \frac{T}{N}$. Let the underlying asset be a stochastic process $S = (S_t)_{t \geq 0}$. At each reset date t_j the holder nominates a barrier level u_j. Here, with the example of the American put in mind, we consider only down-barriers so that we require $u_j \in \left[0, S_{t_j}\right)$. Let $\tau_{u_j} = \inf_{t \geq t_j} \{S_t \leq u_j\}$ be the first hitting time of the underlying asset to the barrier. If $\tau_{u_j} \in [t_j, t_j + 1)$ the option expires at the hitting time τ_{u_j} and the option holder receives a payoff of $H\left(\tau_{u_j}, u_j\right)$ at time τ_{u_j}. At time T, if the option is still alive, the holder receives a payoff of $G(S_T)$.

For given payoff functions H and G, write c_N for the value of a nominated barrier option with $N+1$ reset dates. As $N \to \infty$ the option value tends to a limit, $c_N \to c_\infty$. c_∞ is the value of an American option with payoff $G(S_T)$ at time T and payoff $H(t, S_t)$ if exercised early.

Specifically, we now assume that $H\left(\tau_{u_j}, u_j\right) = X - u_j$ and $G(S_T) = X - S_T$. The function

$$U_N : [0, T] \to \mathbb{R}^+, \tag{1}$$

$$U_N(t) = u_j, \text{ for } t \in [t_j, t_{j+1}), \tag{2}$$

determines an exercise policy for an American put. As $N \to \infty$ we have $U_N \to \hat{U}_t$, the optimal exercise policy for an American put, so that c_∞ is the value of an American put with strike X.

In the numerical work below, where N is large, we shall consider a version of the nominated barrier option in which, if the barrier u_j is hit, the payoff is deferred until time t_{j+1}. We call this the deferred nominated barrier option. Writing d_N for its value we note that $d_N \to c_\infty$ as $N \to \infty$.

From now on we assume that the asset S_t follows a geometric Brownian motion with constant volatility σ and constant riskless rate r. We use the accumulator numeraire so that under the equivalent martingale measure the asset process is

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}z_t. \tag{3}$$

3 Nominated Barrier Options and the Dirichlet Lattice

First we describe the Dirichlet lattice of Kuan and Webber (2002), then we show how it may be applied to value (deferred) nominated barrier options.

3.1 The Dirichlet Lattice

Let c_t be the value at time t of an American option maturing at time T with exercise price X. The value of the option at time t_0 is $c_0 = \max_\omega \mathbb{E}_0\left[\mathrm{e}^{-r\tau(\omega)} H(\omega)\right]$ where the maximum is taken over all exercise policies ω; $H(\omega)$ is the payoff received at time $\tau(\omega)$ under policy ω.

Time is discretised into N steps, $0 = t_0 < \ldots < t_N = T$, with constant time step $\Delta t = t_j - t_{j-1}$. We construct a d-nomial lattice for the underlying Wiener process z_t where $d = 2b + 1$ for an integer $b \geq 1$. Nodes on the lattice are labelled (j, i), $j = 0, \ldots, N$, $i = -bj, \ldots, bj$.[1] Set $\Delta z = \sqrt{\kappa \Delta t}$ for some constant κ. The value of the discretised Wiener process $\widehat{z}_t$ at node (j, i) is $i\Delta z$.

[1] In practical implementations the lattice is truncated at high and low levels.

The corresponding value of the asset is

$$S_{j,i} = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right) j\Delta t + \sigma z_{j,i}\right). \tag{4}$$

Branching from node (j,i) is to nodes $(j+1, i+k)$, $k \in \mathcal{B} = \{-b, \ldots, b\}$ with corresponding probabilities p_k.[2] For our numerical work we specialize down to a trinomial lattice with $d = 3$, so that $\mathcal{B} = \{-1, 0, 1\}$. In this case we use the standard values, $\kappa = 3$, $p_{-1} = p_1 = \frac{1}{6}$, $p_0 = \frac{2}{3}$ so that the first five moments of z_t are matched.

Consider an option with a deterministic time varying barrier u_t such that when the asset value hits the barrier the option expires and some rebate value is paid. We assume that $S_0 > u_0$, and the option is a down-and-out with a rebate value $H(\tau, u_\tau)$ paid when the barrier is hit at time $\tau = \min_t \{S_t \leq u_t\}$. At time T, if $\tau > T$, the option expires with value $H(T, S_t)$.

If we set $H(\tau, u_\tau) = X - u_t$ for some constant X, and set u_t to be the early exercise frontier of an American put then the option is equivalent to an American put with strike X.

This option may be valued on a standard (plain) lattice by backwards induction as follows. At time t_N set $c_{N,i} = H(t_N, S_{N,i})$, $i = -bN, \ldots, bN$. Suppose that for time t_{j+1} the values $c_{j+1,i}$ have been found. We now compute the values $c_{j,i}$. Set $\mathcal{B}_j = \left\{k \in \mathcal{B} \,|S_{j+1,i+k} < u_{t_{j+1}}\right\}$. Then set

$$c_{j,i} = \mathrm{e}^{-r\Delta t}\left[\sum_{k \in \mathcal{B}\setminus\mathcal{B}_j} p_k \widehat{c}_{j+1,i+k} + \sum_{k \in \mathcal{B}_j} p_k H_{j+1,i+k}\left(t_{j+1}, u_{t_{j+1}}\right)\right] \tag{5}$$

The value of the option on the plain lattice is $c_{0,0}$. We write p_N for the option value found on a plain lattice with N time steps.

Fast lattice solutions for knock-out options of this sort were given by Kuan and Webber (2002). They showed[3] how these options could be valued on a Dirichlet lattice by both forwards and backwards induction. Here we are concerned only with backwards induction.

Let

$$m_{j,j+1}^{i,i+k} = \min_{t \in [t_j, t_{j+1})} \left\{S_t \mid S_{t_j} = S_{j,i},\ S_{t_{j+1}} = S_{j+1,i+k}\right\} \tag{6}$$

be the minimum of S_t in the range $t \in [t_j, t_{j+1})$ conditional on its end points and let

$$F_{j,j+1}^{i,i+k}(u) = \Pr\left[m_{j,j+1}^{i,i+k} \leq u \mid S_{t_j} = S_{j,i},\ S_{t_{j+1}} = S_{j+1,i+k}\right] \tag{7}$$

be the distribution function of $m_{j,j+1}^{i,i+k}$.

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[2] Optimal values for κ and for the branching probabilities p_k were determined by Alford and Webber (2001).

[3] Their examples are for up-and-in and up-and-out options, but their methods also apply to down-and-in and down-and-out options.

When S_t follows a geometric Brownian motion $F_{j,j+1}^{i,i+k}(u)$ is known. For $u \leq \min\{S_{j,i}, S_{j+1,i+k}\}$ let $\hat{u} = \ln\left(\frac{u}{S_0}\right)$ and $w_{j,i} = \left(r - \frac{1}{2}\sigma^2\right) j\Delta t + \sigma i \Delta z = \ln\left(\frac{S_{j,i}}{S_0}\right)$. Then $F_{j,j+1}^{i,i+k}$ is given by

$$\begin{aligned} F_{j,j+1}^{i,i+k}(u) &= \exp\left(-\frac{2}{\sigma^2 \Delta t}(\hat{u} - w_{j,i})(\hat{u} - w_{j+1,i+k})\right) \\ &= \exp\left(-\frac{2}{\sigma^2 \Delta t}\ln\left(\frac{u}{S_{j,i}}\right)\ln\left(\frac{u}{S_{j+1,i+k}}\right)\right), \end{aligned} \tag{8}$$

(see El Babsiri and Noel (1998)). For $\widehat{u} > \min\{w_{j,i}, w_{j+1,i+k}\}$, so that $u > \min\{S_{j,i}, S_{j+1,i+k}\}$, set $F_{j,j+1}^{i,i+k}(u) = 1$.

At a node (j,i) above the barrier let $p_{j,i,k}^{\mathrm{O}}$ be the probability of branching from node (j,i) to node $(j+1, i+k)$ without hitting the barrier level u_j and let $p_{j,i,k}^{\mathrm{I}}$ be the probability of of branching to node $(j+1, i+k)$ and hitting the barrier. Then for all $k \in \mathcal{B}$ we set

$$p_{j,i,k}^{\mathrm{O}} = p_k\left(1 - F_{j,j+1}^{i,i+k}(u_j)\right), \tag{9}$$

$$p_{j,i,k}^{\mathrm{I}} = p_k F_{j,j+1}^{i,i+k}(u_j), \tag{10}$$

where $F_{j,j+1}^{i,i+k}(u_j) = 1$ if $S_{j+1,i+k} \leq u_j$.

For $S_{j,i} > u_j$, option values $\widehat{c}_{j,i}$ are given by backwards induction as

$$\begin{aligned} \widehat{c}_{j,i} &= \mathrm{e}^{-r\Delta t} \sum_{k \in \mathcal{B}} \left(p_{j,i,k}^{\mathrm{O}} \widehat{c}_{j+1,i+k} + p_{j,i,k}^{\mathrm{I}} H_{j+1,i+k}(u_j)\right) \\ &= \mathrm{e}^{-r\Delta t} \sum_{k \in \mathcal{B}} p_k \left(\left(1 - F_{j,j+1}^{i,i+k}(u_j)\right)\widehat{c}_{j+1,i+k} + F_{j,j+1}^{i,i+k}(u_j) H_{j+1,i+k}(u_j)\right) \end{aligned} \tag{11}$$

for $i \geq m_j$, where $H_{j+1,i+k}(u_j)$ is the payoff to the option at time t_{j+1} if it is knocked out between times t_j and t_{j+1}. The option value is $\widehat{c}_{0,0}$.

Note that in the plain lattice one is effectively setting $F_{j,j+1}^{i,i+k}(u_j)$ to be zero if $S_{j+1,i+k} > u_{t_{j+1}}$ and to be one if $S_{j+1,i+k} \leq u_{t_{j+1}}$.

3.2 Valuing Nominated Barrier Options

We value the nominated barrier option c_N on a Dirichlet lattice with N time steps.[4] At each node on the lattice we allow the option holder to nominate a constant barrier level for the subsequent time step, so that the option is exercised if the barrier level is hit. As $\Delta t \to 0$ the value of this option converges to that of an American option.

Write $\widehat{c}_{j,i}$ for the value of this approximate American option at node (j,i) on the lattice. Over a small time step Δt, conditional on the current state $S_{j,i}$, write

[4] In fact, for simplicity, we value on the lattice the deferred nominated barrier option d_N.

$\widehat{u}_{j,i}$ for the optimal exercise level. At each time step we find an approximation $\widetilde{u}_{j,i}$ to the optimal exercise policy $\widehat{u}_{j,i}$ for the holder of the option.

From node (j,i), if the exercise policy were to exercise at a level u, the (deferred) option would have value

$$\begin{aligned}\widehat{c}_{j,i}(u) &= \mathrm{e}^{-r\Delta t}\left(\left(\sum_{k\in\mathcal{B}} p_k F^{i,i+k}_{j,j+1}(u)\right) H(u) + \sum_{k\in\mathcal{B}} p_k\left(1 - F^{i,i+k}_{j,j+1}(u)\right)\widehat{c}_{j+1,i+k}\right)\\ &= \mathrm{e}^{-r\Delta t}\sum_{k\in\mathcal{B}} p_k\left(F^{i,i+k}_{j,j+1}(u)\,H(u) + \left(1 - F^{i,i+k}_{j,j+1}(u)\right)\widehat{c}_{j+1,i+k}\right). \end{aligned} \tag{12}$$

With our assumption, the optimal exercise policy on the lattice is to exercise at a level u that maximizes $\widehat{c}_{j,i}(u)$; $\widehat{u}_{j,i} = \arg\max_u \widehat{c}_{j,i}(u)$ and $\widehat{c}_{j,i}(u) = \widehat{c}_{j,i}(\widehat{u}_{j,i})$.

If it exists, the maximum occurs when $\frac{\partial \widehat{c}_{j,i}(u)}{\partial u} = 0$. The second order condition is $\frac{\partial^2 \widehat{c}_{j,i}(u)}{\partial u^2} < 0$. We then have

$$\begin{aligned}\mathrm{e}^{r\Delta t}\frac{\partial \widehat{c}_{j,i}(u)}{\partial u} &= \sum_{k\in\mathcal{B}} p_k F^{i,i+k}_{j,j+1}(u)\frac{\partial H(u)}{\partial u} + \sum_{k\in\mathcal{S}_{j,i}} p_k \frac{\partial F^{i,i+k}_{j,j+1}(u)}{\partial u}\left(H(u) - \widehat{c}_{j+1,i+k}\right)\\ &= \sum_{k\in\mathcal{B}} p_k F^{i,i+k}_{j,j+1}(u)\left(\frac{\partial H(u)}{\partial u} + \frac{\partial f^{i,i+k}_{j,j+1}(u)}{\partial u}\left(H(u) - \widehat{c}_{j+1,i+k}\right)\right)\end{aligned} \tag{13}$$

where

$$f^{i,i+k}_{j,j+1}(u) = \ln F^{i,i+k}_{j,j+1}(u) = -\frac{2}{\sigma^2\Delta t}\ln\left(\frac{u}{S_{j,i}}\right)\ln\left(\frac{u}{S_{j+1,i+k}}\right) \tag{14}$$

and

$$\frac{\partial f^{i,i+k}_{j,j+1}(u)}{\partial u} = -\frac{2}{u\sigma^2\Delta t}\ln\left(\frac{u^2}{S_{j,i}S_{j+1,i+k}}\right). \tag{15}$$

To determine the optimal exercise policy the first order condition requires that equation (13) is zero, so that

$$\sum_{k\in\mathcal{B}} p_k F^{i,i+k}_{j,j+1}(u)\frac{\partial f^{i,i+k}_{j,j+1}(u)}{\partial u}\left(H(u) - \widehat{c}_{j+1,i+k}\right) = -\frac{\partial H(u)}{\partial u}\sum_{k\in\mathcal{B}} p_k F^{i,i+k}_{j,j+1}(u) \tag{16}$$

for the optimal barrier level u. Note that for $b \geq 1$ we have $S_{j,i} > S_{j+1,i-b}$. Also, for N large enough so that $\Delta t < \sigma^2\kappa$, we have $S_{j+1,i-1} < S_{j,i} < S_{j+1,i}$. With this assumption there are three possibilities for u: that $u \geq S_{j,i}$, that $u < S_{j+1,i-b}$, and that there exists an index $l \in \{-1,\ldots-b\}$ such that $u \in [S_{j+1,i+l}, S_{j+1,i+l+1})$.

In each case we try to find a candidate value for u that could be the optimal level. Having found a set of candidate values we determine which one maximizes the value of $\widehat{c}_{j,i}(u)$.

3.2.1 The case $u \geq S_{j,i}$

The first case is straightforward. If $u \geq S_{j,i}$ one exercises immediately with value $H(u)$. Hence the optimal value u is given by

$$\widehat{u}_{j,i} = \arg\max_{u} H(u). \tag{17}$$

For a vanilla American put with $H(u) = X - u$ it is optimal to set the candidate value to be $u = \min\{u \geq S_{j,i}\} = S_{j,i}$. When the payoff is received immediately this is the standard exercise condition tested against the continuation value of the option. By including two additional cases, corresponding to two additional continuation values, we obtain a better approximation to the American option value.

3.2.2 The case $u < S_{j+1,i-b}$

Now suppose that $u < S_{j+1,i-b}$. We cannot solve explicitly for $\widehat{u}_{j,i}$ in this case, but we obtain an approximation as the solution to a cubic equation.

Set

$$G_{j,i,k}(u) = \frac{p_k F^{i,i+k}_{j,j+1}(u)}{\sum_{l\in\mathcal{B}} p_l F^{i,i+l}_{j,j+1}(u)} = \frac{1}{\sum_{l\in\mathcal{B}} \frac{p_l F^{i,i+l}_{j,j+1}(u)}{p_k F^{i,i+k}_{j,j+1}(u)}} \tag{18}$$

and rewrite the first order condition at a maximum (16) as

$$\sum_{k\in\mathcal{B}} G_{j,i,k}(u) \frac{\partial f^{i,i+k}_{j,j+1}(u)}{\partial u} \left(H(u) - \widehat{c}_{j+1,i+k}\right) = -\frac{\partial H(u)}{\partial u}. \tag{19}$$

Note that the ratio of distribution functions in $G_{j,i,k}(u)$ is given by

$$\begin{aligned}
\frac{F^{i,i+l}_{j,j+1}(u)}{F^{i,i+k}_{j,j+1}(u)} &= \exp\left(-\frac{2}{\sigma^2\Delta t}\ln\left(\frac{u}{S_{j,i}}\right)\ln\left(\frac{u}{S_{j+1,i+l}}\right)\right) \\
&\quad \times \exp\left(\frac{2}{\sigma^2\Delta t}\ln\left(\frac{u}{S_{j,i}}\right)\ln\left(\frac{u}{S_{j+1,i+k}}\right)\right) \\
&= \exp\left(-\frac{2}{\sigma^2\Delta t}\ln\left(\frac{u}{S_{j,i}}\right)\ln\left(\frac{S_{j+1,i+k}}{S_{j+1,i+l}}\right)\right) \\
&= \exp\left(-\frac{2}{\sigma^2\Delta t}\ln\left(\frac{u}{S_{j,i}}\right)(k-l)\,\sigma\sqrt{\kappa\Delta t}\right) \\
&= \left(\frac{u}{S_{j,i}}\right)^{-\frac{2(k-l)\sqrt{\kappa}}{\sigma\sqrt{\Delta t}}}.
\end{aligned} \tag{20}$$

Since in this case $u < S_{j,i}$, we may conclude that as $\Delta t \to 0$, keeping $S_{j,i}$ fixed,

$$\frac{F^{i,i+l}_{j,j+1}(u)}{F^{i,i+k}_{j,j+1}(u)} \to \begin{cases} \infty, & k > l, \\ 1 & k = l, \\ 0 & k < l. \end{cases} \tag{21}$$

and hence as $\Delta t \to 0$, the limit of $G_{j,i,k}(u)$ is

$$G_{j,i,k}(u) \to \begin{cases} 1, & k = -b, \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

Furthermore, $G_{j,i,k}(u)$ approaches its limit at power speed. As $\Delta t \to 0$ every term on the left hand side of equation (19) goes to zero except the $-b$th term, so

$$\begin{aligned} -\frac{\partial H(u)}{\partial u} &= \sum_{k \in \mathcal{B}} G_{j,i,k}(u) \frac{\partial f_{j,j+1}^{i,i+k}(u)}{\partial u} \left(H(u) - \widehat{c}_{j+1,i+k}\right) \\ &\to \frac{\partial f_{j,j+1}^{i,i-b}(u)}{\partial u} \left(H(u) - \widehat{c}_{j+1,i-b}\right). \end{aligned} \tag{23}$$

For an American put with strike X we have $H_{j+1,i}(u) = X - u$ and $\frac{\partial H_{j+1,i}}{\partial u} = -1$. Hence, substituting in for $\frac{\partial f_{j,j+1}^{i,i-b}(u)}{\partial u}$ and $H(u)$, at a maximum we require, in the limit,

$$1 = -\frac{2}{u\sigma^2 \Delta t} \ln\left(\frac{u^2}{S_{j,i}S_{j+1,i-b}}\right)\left(X - u - \widehat{c}_{j+1,i-b}\right) \tag{24}$$

This result is intuitive. As $\Delta t \to 0$ the branch that has the dominating affect on the value of the exercise level is the lowest branch, closest to the exercise level.

When Δt is small the probability of hitting the exercise boundary is small and, unless $S_{j+1,i-b}$ is near u, can be ignored. When u is close to $S_{j+1,i-b}$ we make a series approximation to $\ln\left(\frac{u^2}{S_{j,i}S_{j+1,i-b}}\right)$ and truncating at first order obtain

$$\left(\frac{u^2}{S_{j,i}S_{j+1,i-b}} - 1\right)\left(X - u - \widehat{c}_{j+1,i-b}\right) = -\frac{u\sigma^2 \Delta t}{2}. \tag{25}$$

Hence to find u we solve the cubic equation

$$u^3 - u^2\left(X - \widehat{c}_{j+1,i-b}\right) - uS_{j,i}S_{j+1,i-b}\left(1 + \frac{\sigma^2 \Delta t}{2}\right) + S_{j,i}S_{j+1,i-b}\left(X - \widehat{c}_{j+1,i-b}\right) = 0. \tag{26}$$

Since the right-hand side of (25) is close to zero for reasonable values of N, the cubic has three real roots, two of which are close to $\pm\sqrt{S_{j,i}S_{j+1,i-b}}$ and the third is close to $X - \widehat{c}_{j+1,i-b}$. Since $\widehat{c}_{j+1,i-b} > X - S_{j+1,i-b}$ the third root is close to but less than $S_{j+1,i-b}$. The second order condition on the maximum implies that we require the middle of the three roots. Thus the third root becomes our candidate value in this case.

Equation (26) can be solved explicitly by standard methods. Since the polynomial discriminant is negative the acos method works well in this case.[5]

[5] The calculation of $\text{acos}(x)$ is accurate for typical values of x encountered in our applications.

3.2.3 The case $u \in [S_{j+1,i-b}, S_{j,i})$

The final case is when $u \in [S_{j+1,i+l}, S_{j+1,i+l+1})$ for some $l \in \{-1, \ldots -b\}$.[6] Set $\mathcal{B}_l = \{k \in \mathcal{B} \mid k > l\}$. Since for $k \in \mathcal{B} \setminus \mathcal{B}_l$ we have $F^{i,i+k}_{j,j+1}(u) = 1$ and $\frac{\partial f^{i,i+k}_{j,j+1}(u)}{\partial u} = 0$, the option value at node (j,i) (equation 12) is

$$\begin{aligned}\widehat{c}_{j,i}(u) &= \mathrm{e}^{-r\Delta t}\sum_{k\in\mathcal{B}} p_k\left(F^{i,i+k}_{j,j+1}(u)\,H(u) + \left(1 - F^{i,i+k}_{j,j+1}(u)\right)\widehat{c}_{j+1,i+k}\right)\\ &= \mathrm{e}^{-r\Delta t}\sum_{k\in\mathcal{B}_l} p_k\left(F^{i,i+k}_{j,j+1}(u)\,H(u) + \left(1 - F^{i,i+k}_{j,j+1}(u)\right)\widehat{c}_{j+1,i+k}\right)\\ &\quad + \mathrm{e}^{-r\Delta t} H(u)\sum_{k\in\mathcal{B}\setminus\mathcal{B}_l} p_k \qquad (27)\end{aligned}$$

and the first order condition (equation (16)) becomes

$$\sum_{k\in\mathcal{B}_l} p_k F^{i,i+k}_{j,j+1}(u)\,\frac{\partial f^{i,i+k}_{j,j+1}(u)}{\partial u}\left(H(u) - \widehat{c}_{j+1,i+k}\right) = -\frac{\partial H(u)}{\partial u}\left(\sum_{k\in\mathcal{B}_l} p_k F^{i,i+k}_{j,j+1}(u) + \sum_{k\in\mathcal{B}\setminus\mathcal{B}_l} p_k\right). \qquad (28)$$

In the appendix we describe an optimization procedure that can be used to solve equation (28) for u. In practice we found it computationally more efficient to compute $\widehat{c}_{j,i}(u)$ directly from equation (12) for a small set of regularly spaced values of u in the interval $[S_{j+1,i-b}, S_{j,i})$.

3.3 The Algorithm

We restrict ourselves to trinomial branching. Iterating backwards, at each node on the lattice at a particular time step we find three values for u, conditional on whether $u \geq S_{j,i}$, $u < S_{j+1,i-1}$, or $u \in [S_{j+1,i-1}, S_{j,i})$. In the last two cases for each candidate value $\tilde{u}_{j,i}$ we compute the option value $\widetilde{c}_{j,i}(\tilde{u}_{j,i})$

$$\widetilde{c}_{j,i}(\tilde{u}_{j,i}) = \mathrm{e}^{-r\Delta t}\sum_{k\in\mathcal{B}} p_k\left(F^{i,i+k}_{j,j+1}(\tilde{u}_{j,i})\,H(\tilde{u}_{j,i}) + \left(1 - F^{i,i+k}_{j,j+1}(\tilde{u}_{j,i})\right)\widehat{c}_{j+1,i+k}\right). \qquad (29)$$

In the first case we set $\widetilde{c}_{j,i}(\tilde{u}_{j,i}) = X - S_{j,i}$. Set the option value to be the maximum of these values.

If at some node $S_{j,i}$ the first case applies and it is optimal to exercise immediately, then at all nodes (j,l) for $l < i$ it will also be optimal to exercise immediately. Similarly if at (j,i) the second case applies then at all nodes (j,l) for $l > i$ the second case will also apply. If at (j,i) the third case holds for some index k then at node $(j, i+1)$ either the second case applies or else the third case applies for some $k' \leq k$.

In practice it is unnecessary to compute the barrier level at every node. For nodes away from the barrier the probability of hitting the barrier is very small.

[6] For $l = -1$ we have $u \in [S_{j+1,i-1}, S_{j,i})$.

Kuan and Webber (2002) found it unnecessary to compute hitting probabilities at a distance of more than ten steps away from the barrier. Here we found it necessary only to compute case three for up to three steps away from the exercise barrier.

4 The American Dirichlet Lattice

For a geometric Brownian motion $S = (S_t)_{t\geq 0}$ its hitting distribution to a constant barrier is known, as is its defective density conditional upon not hitting the barrier. We use these to construct branching probabilities to obtain a modified Dirichlet lattice that we refer to as the American Dirichlet lattice.

As before we construct a lattice for a Wiener process z_t. Suppose we are given a barrier level u. At a node (j,i) with $z_{j,i} > u$, we compute probabilities $p_h(u)$, the probability of hitting the barrier, and $p_k(u)$, $k \in \mathcal{B}$, the defective branching probabilities. We then set

$$c_{j,i}(u) = \mathrm{e}^{-r\Delta t}\left[p_h(u) H(u) + \sum_{k\in\mathcal{B}} p_k(u)\,\widehat{c}_{j+1,i+k}\right]. \tag{30}$$

The optimal barrier level is $\hat{u}_j = \arg\max_{u\in[0,z_{j,i})} c_{j,i}(u)$ and the option value is $\hat{c}_{j,i} = c_{j,i}(\hat{u}_j)$.

At each node, instead of finding the u that maximizes $c_{j,i}$ by optimization, we compute directly values of $c_{j,i}$ for a set of pre-determined values u, selecting for u that value that gives the greatest value of $c_{j,i}$. We chose $u \in U = \{-i\Delta u\}_{i=1,\ldots,N_u}$ for $\Delta u = \frac{1}{2}\Delta z$ and $N_u = 8$.

Branching probabilities are computed by moment matching. For a geometric Brownian motion $S = (S_t)_{t\geq 0}$ with barrier level $u < S_t$ and for $s \geq t$ set $W_s = \ln\left(\frac{S_s}{S_t}\right)$ so that $W_s = \left(r - \frac{1}{2}\sigma^2\right)(s-t) + \sigma z_{s-t}$ for a Wiener process z. W_s is a Brownian motion with drift $\mu = r - \frac{1}{2}\sigma^2$ and volatility σ. Let $f(w, t_j + \Delta t \mid w_{j,i}, t_j; u)$ be the defective density of a Brownian motion with drift μ, with a barrier u so that

$$\Pr\left[W_T \leq w,\ \tau_u^W > u \mid W_t = 0\right] = \int_u^w f(w, t_j + \Delta t \mid 0, t_j; u)\,\mathrm{d}w \tag{31}$$

(for $z_t = z_{j,i}$ we have $w_{j,i} = 0$). A formula for f is given in the appendix.

We have

$$\Pr\left[S_T \leq S,\ \tau_u > T \mid S_t\right] = \Pr\left[W_T \leq w,\ \tau_u^W > \hat{u} \mid W_t = 0\right] \tag{32}$$

where $w = \ln\left(\frac{S}{S_t}\right)$, $\hat{u} = \ln\left(\frac{u}{S_t}\right)$ and τ_u^W is the hitting time of W_s to $\hat{u}$. The first three defective moments for $z_{t_j+\Delta t}$ are given by

$$e(u) = \mathbb{E}\left[z_{t_j+\Delta t} 1_{\{\tau_u^z > t_j+\Delta t\}} \mid z_{j,i}\right] = \sigma \int_{u_z}^{\infty} z f\left(\sigma z + \mu\Delta t, t_j + \Delta t \mid w_{j,i}, t_j; u\right) \mathrm{d}z, \tag{33}$$

$$v(u) = \mathbb{E}\left[z_{t_j+\Delta t}^2 1_{\{\tau_u^z > t_j+\Delta t\}} \mid z_{j,i}\right] = \sigma \int_{u_z}^{\infty} z^2 f\left(\sigma z + \mu\Delta t, t_j + \Delta t \mid w_{j,i}, t_j; u\right) \mathrm{d}z, \tag{34}$$

$$s(u) = \mathbb{E}\left[z_{t_j+\Delta t}^2 1_{\{\tau_u^z > t_j+\Delta t\}} \mid z_{j,i}\right] = \sigma \int_{u_z}^{\infty} z^3 f\left(\sigma z + \mu\Delta t, t_j + \Delta t \mid w_{j,i}, t_j; u\right) \mathrm{d}z. \tag{35}$$

where $u_z = \frac{u-\mu\Delta t}{\sigma}$.

$e(u)$, $v(u)$ and $s(u)$ are found by numerical integration of equations (33), (34) and (35). We found that a straightforward Simpson's rule integration with 501 intervals over the range $[u, z_{\max}]$, for $z_{\max} = 8\Delta z$, gave sufficient accuracy. Note that $e(u)$, $v(u)$ and $s(u)$ can be pre-computed. Their values depend only on the relative position of u with respect to $z_{j,i}$, so their values are the same for every node in the lattice.

Equation (30) values a deferred nominated barrier option. In fact the value of an option that gives a rebate at the moment the barrier is hit is known (a formula is given in the appendix). However substituting this into equation (30), replacing the $p_h(u) H(u)$ term, leads to an insignificant improvement in accuracy, at the expense of a sizable increase in computation time.

4.1 Computing probabilities: matching two moments

At each barrier level there are four probabilities to compute. For a given (relative) barrier level, every node has the same branching probabilities, which can therefore be pre-computed.

For a given barrier level $u < 0$ the four probabilities are the probability of hitting the barrier, $p_h(u)$, and the up, middle and down defective probabilities, $p_1(u)$, $p_0(u)$, $p_{-1}(u)$.

We immediately set

$$p_h(u) = N\left(\frac{u-\mu\Delta t}{\sigma\sqrt{\Delta t}}\right) + \exp\left(\frac{2u\mu}{\sigma^2}\right) N\left(\frac{u+\mu\Delta t}{\sigma\sqrt{\Delta t}}\right). \tag{36}$$

We find the other probabilities by moment matching. When $u < -\Delta z$ we require them to satisfy

$$\begin{array}{lllllll} p_1(u) & + & p_0(u) & + & p_{-1}(u) & = & 1 - p_h(u), \\ p_1(u)\Delta z & + & & & p_{-1}(u)(-\Delta z) & = & e(u), \\ p_1(u)(\Delta z)^2 & + & & & p_{-1}(u)(-\Delta z)^2 & = & v(u). \end{array} \tag{37}$$

Set $e'(u) = \frac{e(u)}{\Delta z w}$ and $v'(u) = \frac{v(u)}{(\Delta z)^2}$. Then

$$p_1(u) = \frac{1}{2}\left(v'(u) + e'(u)\right) \tag{38}$$

$$p_0(u) = 1 - v'(u) - p_h(u) \tag{39}$$

$$p_{-1}(u) = \frac{1}{2}\left(v'(u) - e'(u)\right) \tag{40}$$

When $0 > u > -\Delta z$ then set $p_{-1}(u) = 0$ and

$$\begin{array}{lclcl} p_1(u) & + & p_0(u) & = & 1 - p_h(u), \\ p_1(u)\,\Delta z & & & = & e(u), \end{array} \tag{41}$$

so that

$$p_1(u) = e'(u), \tag{42}$$

$$p_0(u) = 1 - e'(u) - p_h(u). \tag{43}$$

4.2 Incorporating a third moment

Heston and Zhou (2000) determined that if branching probabilities on a lattice matches the first q moments of the underlying stochastic process then the lattice may converge at a rate up to order $-\frac{q-1}{2}$. For our numerical work we match to three moments where possible.

To match the third moment we require a fourth branch, from z_0 to $z_0 + 2\Delta z$. Write $p_2(u)$ for the probability of this branching. Then when $u < -\Delta w$ we choose $p_2(u)$, $p_1(u)$, $p_0(u)$, and $p_{-1}(u)$ to satisfy

$$\begin{array}{lclclclcl} p_2(u) & + & p_1(u) & + & p_0(u) & + & p_{-1}(u) & = & 1 - p^h(u), \\ p_2(u)\,2\Delta z & + & p_1(u)\,\Delta z & & & + & p_{-1}(u)(-\Delta z) & = & e(u), \\ p_2(u)(2\Delta z)^2 & + & p_1(u)(\Delta z)^2 & & & + & p_{-1}(u)(-\Delta z)^2 & = & v(u), \\ p_2(u)(2\Delta z)^3 & + & p_1(u)(\Delta z)^3 & & & + & p_{-1}(u)(-\Delta z)^3 & = & s(u). \end{array} \tag{44}$$

Set $s'(u) = \frac{s(u)}{(\Delta z)^3}$. Then we solve to find

$$p_2(u) = \frac{1}{6}\left(s'(u) - e'(u)\right), \tag{45}$$

$$p_1(u) = \frac{1}{2}\left(2e'(u) + v'(u) - s'(u)\right), \tag{46}$$

$$p_0(u) = 1 - p_h(u) - p_2(u) - p_1(u) - p_{-1}(u), \tag{47}$$

$$p_{-1}(u) = \frac{1}{6}\left(3v'(u) - 2e'(u) - s'(u)\right). \tag{48}$$

When $0 > u > -\Delta z$ we set $p_{-1}(u) = 0$ and solve

$$\begin{array}{lclclcl} p_2(u) & + & p_1(u) & + & p_0(u) & = & 1 - p_h(u), \\ p_2(u)\,2\Delta z & + & p_1(u)\,\Delta z & & & = & e(u), \\ p_2(u)(2\Delta z)^2 & + & p_1(u)(\Delta z)^2 & & & = & v(u), \end{array} \tag{49}$$

with solution

$$p_2(u) = \frac{1}{2}\left(v'(u) - e'(u)\right), \tag{50}$$

$$p_1(u) = 2e'(u) - v'(u), \tag{51}$$

$$p_0(u) = 1 - p_h(u) - p_2(u) - p_1(u), \tag{52}$$

$$p_{-1}(u) = 0. \tag{53}$$

In the second case, if it is not possible to find valid probabilities that match two moments we instead revert to binomial branching, setting $p_2(u) = 0$ and solving

$$\begin{array}{lcccl} p_1(u) & + & p_0(u) & = & 1 - p_h(u), \\ p_1(u)\Delta z & & & = & e(u), \end{array} \tag{54}$$

as before.

We note that probabilities found for the American Dirichlet lattice are similar to those computed for the Dirichlet approach described in section 3.1. However the American Dirichlet probabilities exactly match the moments of the defective distribution and hitting probabilities.

5 Numerical Results

We use the American Dirichlet lattice to vanilla American put values. We consider two underlyings and two sets of options. The first underlying has high volatility with $\sigma = 0.4$; the second underlying has medium volatility with $\sigma = 0.2$. In both cases we set $S_0 = 100$. The first set of options matures in half a year with strikes of 98, 100, and 102, and we use $r = 0.06$; the second set of options matures in one year with strikes of 95, 100, and 105, and we set $r = 0.05$. We shall write, for instance, (o3, u2) to denote the case of the third option valued with the the second underlying.

Benchmark values were obtained in two ways: the first using a PSOR Crank-Nicolson finite difference routine with 500 time steps and $5,000$ space steps, with the space boundaries set at 2 and $5,000$; the second using a heptanomial lattice with $10,000$ time steps, truncated at 8 standard deviations from the mean. Table 1 gives the benchmark values. The top number is the lattice value, the bottom number the Crank-Nicolson value. These values are accurate to at most four decimal places.[7]

The appendix gives tables and figures showing convergence for both the plain and American Dirichlet lattices. The tables give option values as the number of time steps increases from 100 to $5,000$. The figures plot $\ln(c_N)$ and $\ln(p_N)$ (the log of the plain lattice value) against $\ln(N)$. Slopes of these plots can be interpreted as rates of convergence.

[7] Values found on the heptanomial lattice tend to be greater than the values found by Crank-Nicolson, but took about one tenth the time to compute. Figures in the appendix show the heptanomial lattice values.

Benchmark values		Underlyings	
Top value: lattice Bottom value: PDE		Underlying 1, $\sigma = 0.4$	Underlying 2, $\sigma = 0.2$
Option 1: $r = 0.05$	$T = 0.5$ $X = 98$	9.12288 9.12275	3.75928 3.75920
Option 2: $r = 0.05$	$T = 0.5$ $X = 100$	10.14141 10.14127	4.65564 4.65554
Option 3: $r = 0.05$	$T = 0.5$ $X = 102$	11.21794 11.21786	5.67524 5.67520
Option 4: $r = 0.06$	$T = 1$ $X = 95$	10.81207 10.81200	3.77635 3.77634
Option 5: $r = 0.06$	$T = 1$ $X = 100$	13.29563 13.29564	5.79887 5.79882
Option 6: $r = 0.06$	$T = 1$ $X = 105$	16.04444 16.04441	8.41660 8.41661

Table 1: Benchmark values for vanilla American options

Times are not reported. The American Dirichlet lattice runs more slowly than the plain lattice, but not significantly so. This is firstly because branching probabilities on the American Dirichlet lattice can be pre-computed and so do not contribute to individual run times, and secondly because Dirichlet branching need only be used close to the exercise boundary. Away from the boundary the option is either surely exercised (with u_j set to $S_{j,i}$) or never exercised (with u_j set to 0). We found that, for reasonable values of N, Dirichlet branching had a significant influence only within three space steps of the exercise boundary (a not unexpected result).[8]

For both the plain and American Dirichlet lattices option values converge non-uniformly toward the benchmark values, sometimes with rapid oscillations ((o1, u2), (o4, u2)) and sometimes with slow ((o1, u1), (o3, u2)).

In cases where oscillation is slow, and convergence is initially uniform for small values of N, one may investigate rates of convergence for the methods. If c_N is converging uniformly towards c_∞ at a rate M then $c_N - c_\infty$ is $O\left(N^{-M}\right)$, so that $\frac{\mathrm{d}\ln(c_N - c_\infty)}{\mathrm{d}\ln(N)} \sim -M$. This corresponds to the slope of the $\ln - \ln$ plots given in the figures. We see from the figures, for instance for cases (o2, u1), (o3, u2) and (o4, u1), that the American Dirichlet lattice is converging faster than the plain lattice, although usually with greater error.

The American Dirichlet value is always greater than the plain value. In cases where the plain value is biased low, tending to converge upwards towards the benchmark value, the advantage of the American Dirichlet lattice is clear ((o5, u2), (o6, u2)). In other cases both methods appear to have similar convergence properties, with the American Dirichlet value biased upwards by as much as the

[8] Generally, with N taking sizes reported in the tables, the plain and American Dirichlet lattices exercised at the same levels in the lattice.

Error analysis Top value: ADL Bottom value: plain		Underlyings	
		Underlying 1: $\sigma = 0.4$	Underlying 2: $\sigma = 0.2$
Option 1: $r = 0.05$	$T = 0.5$ $X = 98$	0.0141 **0.0053**	0.0037 0.0035
Option 2: $r = 0.05$	$T = 0.5$ $X = 100$	**0.0087** 0.0104	0.0040 0.0040
Option 3: $r = 0.05$	$T = 0.5$ $X = 102$	0.0099 **0.0076**	0.0029 0.0027
Option 4: $r = 0.06$	$T = 1$ $X = 95$	0.0095 0.0093	0.0041 **0.0033**
Option 5: $r = 0.06$	$T = 1$ $X = 100$	**0.0097** 0.0159	**0.0039** 0.0065
Option 6: $r = 0.06$	$T = 1$ $X = 105$	**0.0088** 0.0103	**0.0034** 0.0109

Table 2: Error analysis. Bold indicates a significant difference.

plain values are biased downwards ((o2, u2), (o4, u1)). In other cases the plain lattice is converging faster ((o1, u1), (o4, u2)).

We make this more formal. Table 2 displays some error analysis results. It computes the summed absolute error for time steps $N = 1,000$ to $N = 5,000$, for values given in the tables; $error = \sum_{N=1000}^{N=5000} |c_N - b|$, where c_N is the tabulated value and b is the benchmark value. In the table, numbers in bold are significantly smaller than their paired counterparts.

In five cases ((o2, u1), (o5, u1), (o6, u1), (o5, u2), (o6, u2)) the American Dirichlet lattice has significantly smaller error that the plain lattice. In four cases the there is little to distinguish the two methods. In three cases ((o1, u1), (o3, u1), (o4, u2)) the plain method seems to be converging with less error. In three-quarters of the cases the American Dirichlet lattice is doing no worse than the plain lattice.

Although by no means conclusive, this analysis indicates that the American Dirichlet lattice may have better convergence properties than the plain lattice.

6 Conclusions

We have presented a new lattice method to value American options, the American Dirichlet lattice. It is based on the valuation of a nominated barrier option. The method is slightly slower than a plain lattice, but not significantly so. It has greater initial bias than the plain lattice, but may have superior convergence properties once the number of time steps is sufficient for the bias to have been removed.

We applied the American Dirichlet lattice to value vanilla American puts, but it is more generally applicable. In other cases its convergence advantage over the plain lattice method may be more clear cut.

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7 Appendix: Tables and Figures

	Underlying 1, Option 1		Underlying 2, Option 1	
N_steps	Dirichlet	Plain	Dirichlet	Plain
100	9.11452	9.10846	3.76533	3.76094
200	9.12033	9.11796	3.75709	3.75373
300	9.12224	9.12069	3.76151	3.76002
400	9.12309	9.12186	3.75952	3.75790
500	9.12352	9.12245	3.75945	3.75866
600	9.12375	9.12280	3.76044	3.75967
700	9.12387	9.12301	3.75977	3.75891
800	9.12394	9.12313	3.75886	3.75815
900	9.12397	9.12322	3.75973	3.75928
1000	9.12398	9.12327	3.75999	3.75952
1100	9.12398	9.12330	3.75969	3.75916
1200	9.12396	9.12331	3.75900	3.75840
1300	9.12395	9.12332	3.75929	3.75899
1400	9.12392	9.12332	3.75969	3.75939
1500	9.12390	9.12332	3.75976	3.75944
1600	9.12388	9.12331	3.75960	3.75924
1700	9.12385	9.12330	3.75924	3.75884
1800	9.12382	9.12329	3.75915	3.75889
1900	9.12379	9.12327	3.75945	3.75924
2000	9.12377	9.12326	3.75962	3.75940
2500	9.12364	9.12318	3.75931	3.75916
3000	9.12352	9.12311	3.75935	3.75913
3500	9.12342	9.12304	3.75951	3.75937
4000	9.12333	9.12297	3.75924	3.75912
4500	9.12325	9.12291	3.75946	3.75934
5000	9.12318	9.12286	3.75930	3.75922

Table 3: Dirichlet and plain values for option 1

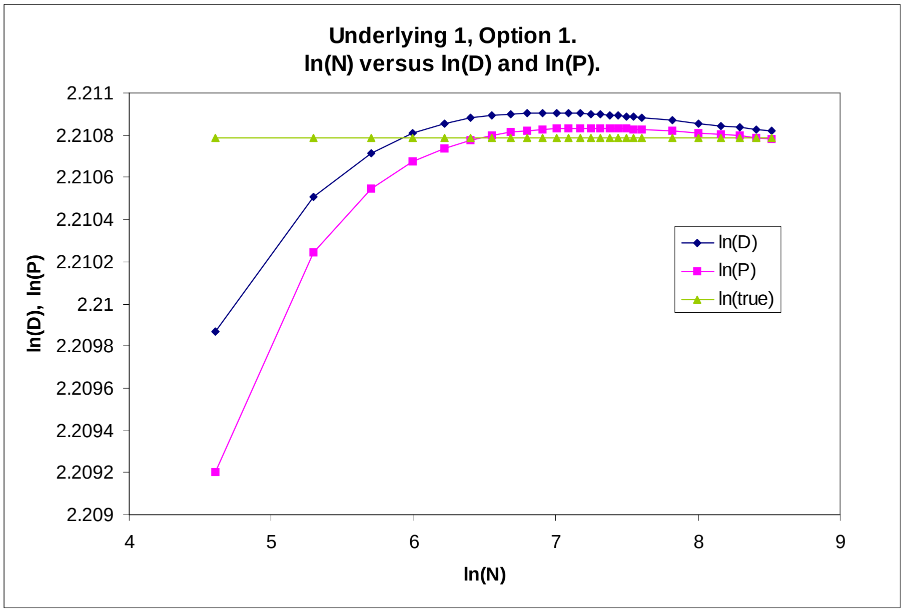

Figure 1: Option 1, underlying 1: log-log convergence

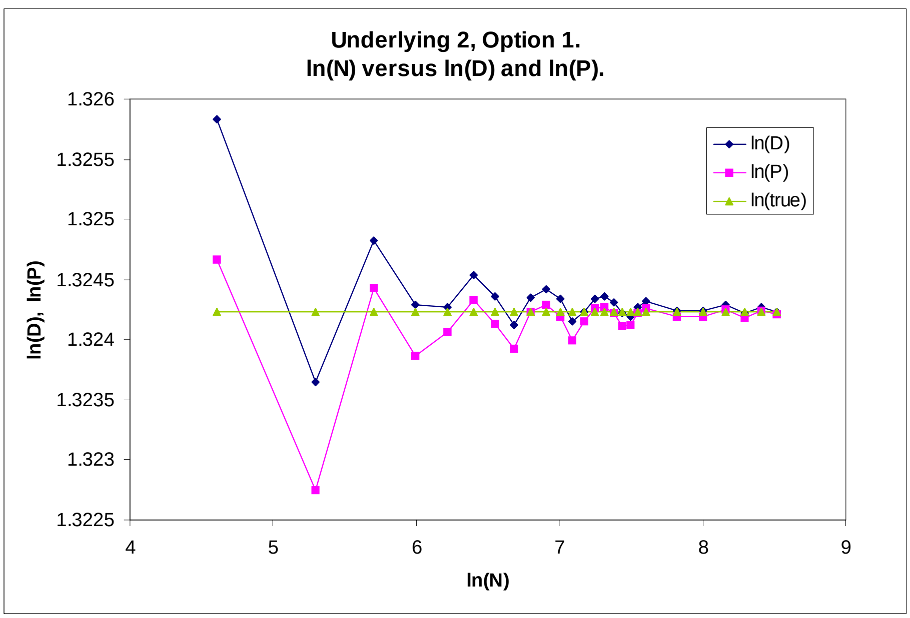

Figure 2: Option 1, underlying 2: log-log convergence

	Underlying 1, Option 2		Underlying 2, Option 2	
N_steps	Dirichlet	Plain	Dirichlet	Plain
100	10.15371	10.14521	4.662402	4.656831
200	10.14877	10.14531	4.656263	4.652746
300	10.14608	10.14403	4.654377	4.652236
400	10.14436	10.14288	4.655918	4.654811
500	10.14311	10.14194	4.656674	4.655754
600	10.14215	10.14117	4.656786	4.655952
700	10.14138	10.14053	4.656602	4.65583
800	10.14078	10.13999	4.65629	4.655535
900	10.14056	10.13953	4.655902	4.655162
1000	10.14046	10.13915	4.655488	4.654758
1100	10.14059	10.13922	4.655287	4.654643
1200	10.14111	10.13988	4.655456	4.655083
1300	10.14148	10.14039	4.655722	4.655393
1400	10.14176	10.14077	4.655911	4.655594
1500	10.14196	10.14105	4.65602	4.655714
1600	10.1421	10.14127	4.656069	4.65577
1700	10.14219	10.14143	4.656074	4.655779
1800	10.14225	10.14155	4.656045	4.655749
1900	10.14229	10.14163	4.655992	4.655692
2000	10.14229	10.14169	4.655917	4.655612
2500	10.14216	10.14171	4.655536	4.655293
3000	10.14187	10.14151	4.655841	4.65569
3500	10.14154	10.14123	4.655865	4.655709
4000	10.14132	10.14094	4.655701	4.655529
4500	10.1414	10.14098	4.65562	4.655517
5000	10.14161	10.14126	4.655768	4.655679

Table 4: Dirichlet and plain values for option 2

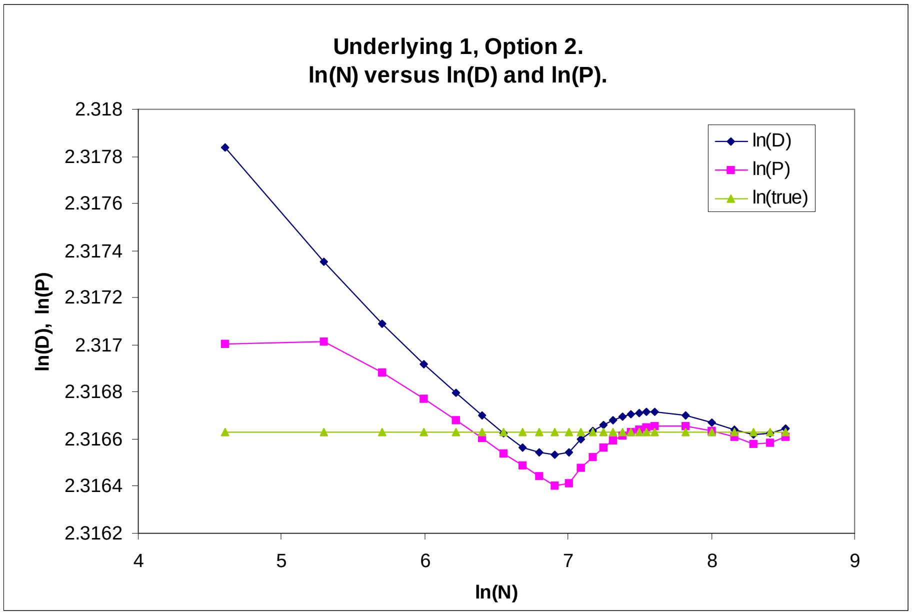

Figure 3: Option 2, underlying 1: log-log convergence

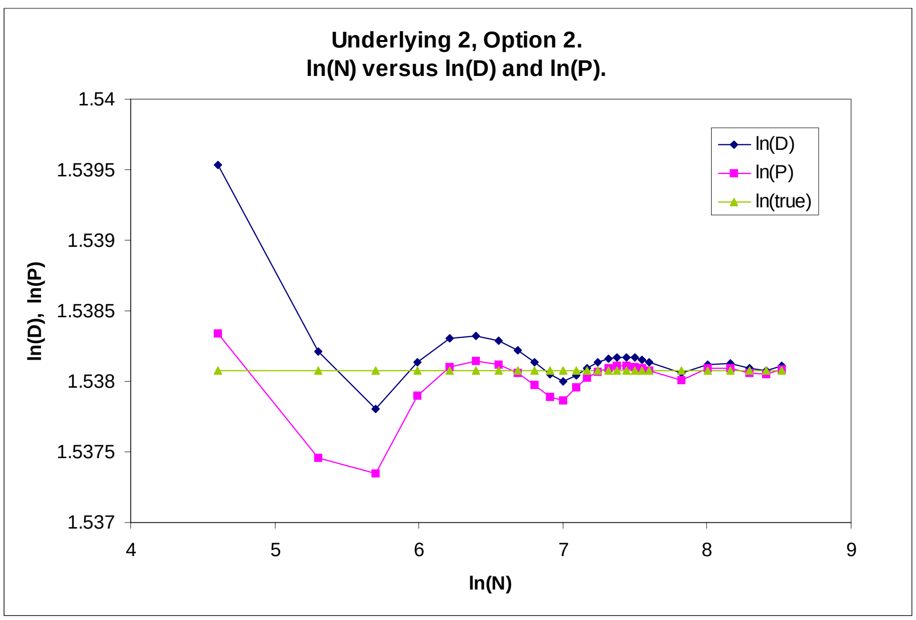

Figure 4: Option 2, underlying 2: log-log convergence

	Underlying 1, Option 3		Underlying 2, Option 3	
N_steps	Dirichlet	Plain	Dirichlet	Plain
100	11.22368	11.21868	5.678409	5.671596
200	11.21141	11.20495	5.677809	5.674559
300	11.22064	11.2174	5.677256	5.675237
400	11.22164	11.21974	5.676857	5.675444
500	11.22055	11.21924	5.676582	5.675464
600	11.21882	11.21781	5.676355	5.675458
700	11.21702	11.21611	5.676169	5.675429
800	11.21664	11.21481	5.67602	5.675384
900	11.21824	11.21679	5.675892	5.675335
1000	11.21907	11.21789	5.675781	5.675287
1100	11.21939	11.21843	5.675684	5.675241
1200	11.2194	11.21859	5.675597	5.675197
1300	11.2192	11.2185	5.67552	5.675155
1400	11.21886	11.21823	5.67545	5.675115
1500	11.21843	11.21786	5.675387	5.675077
1600	11.21793	11.2174	5.675329	5.675042
1700	11.21762	11.2169	5.675277	5.675009
1800	11.21755	11.2166	5.675231	5.674978
1900	11.21805	11.21721	5.675188	5.674947
2000	11.2184	11.21765	5.675149	5.674919
2500	11.21871	11.21825	5.675106	5.674803
3000	11.21789	11.2175	5.675238	5.674989
3500	11.21832	11.21784	5.675337	5.675139
4000	11.2185	11.21815	5.675388	5.675226
4500	11.21818	11.21791	5.675409	5.675274
5000	11.21791	11.2175	5.675412	5.675298

Table 5: Dirichlet and plain values for option 3

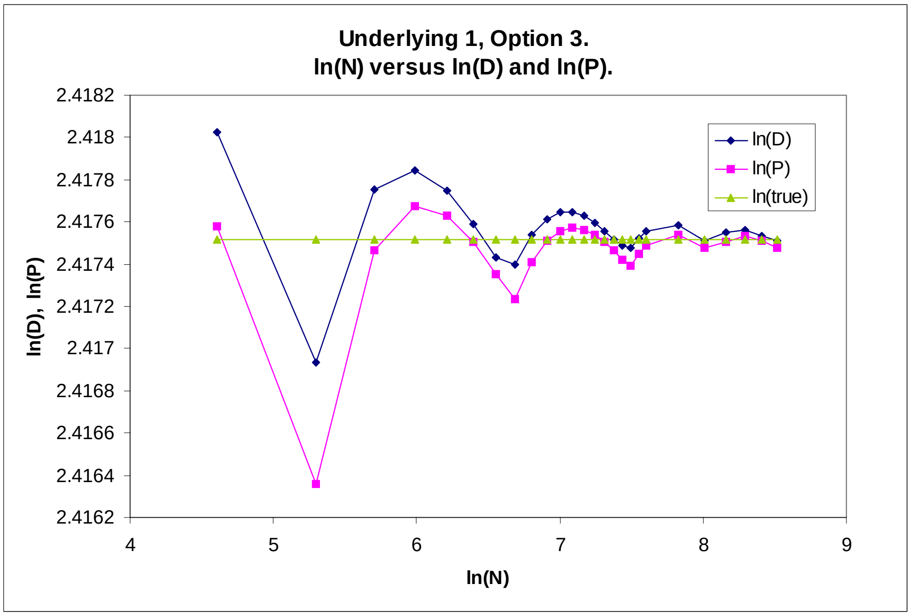

Figure 5: Option 3, underlying 1: log-log convergence

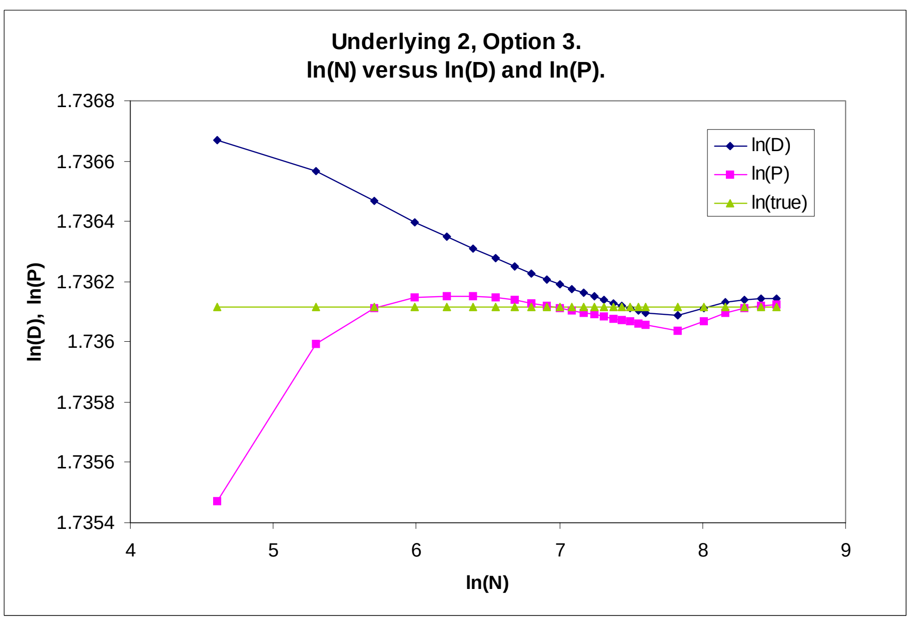

Figure 6: Option 3, underlying 2: log-log convergence

	Underlying 1, Option 4		Underlying 2, Option 4	
N_steps	Dirichlet	Plain	Dirichlet	Plain
100	10.82497	10.81776	3.782549	3.776322
200	10.81887	10.81415	3.778882	3.775091
300	10.8141	10.81025	3.778531	3.776551
400	10.81065	10.80726	3.776829	3.775443
500	10.80846	10.80549	3.776364	3.77449
600	10.80993	10.80871	3.777378	3.776414
700	10.81142	10.81062	3.775819	3.774425
800	10.81247	10.81172	3.777159	3.776347
900	10.81303	10.81231	3.776295	3.775352
1000	10.81327	10.81258	3.776784	3.776215
1100	10.81332	10.81264	3.77682	3.776143
1200	10.81324	10.81256	3.776146	3.775663
1300	10.81307	10.81238	3.776907	3.776416
1400	10.81285	10.81215	3.776472	3.775885
1500	10.81259	10.81187	3.776407	3.776031
1600	10.8123	10.81156	3.776824	3.77642
1700	10.81198	10.81124	3.776452	3.775959
1800	10.81166	10.81091	3.776363	3.776027
1900	10.81133	10.81057	3.776753	3.776399
2000	10.81121	10.81051	3.776584	3.776181
2500	10.81215	10.81192	3.776373	3.776144
3000	10.81251	10.81227	3.776614	3.776399
3500	10.81241	10.81213	3.776604	3.776403
4000	10.81205	10.81174	3.776564	3.776379
4500	10.81171	10.81142	3.776544	3.776382
5000	10.81199	10.81189	3.776543	3.776399

Table 6: Dirichlet and plain values for option 4

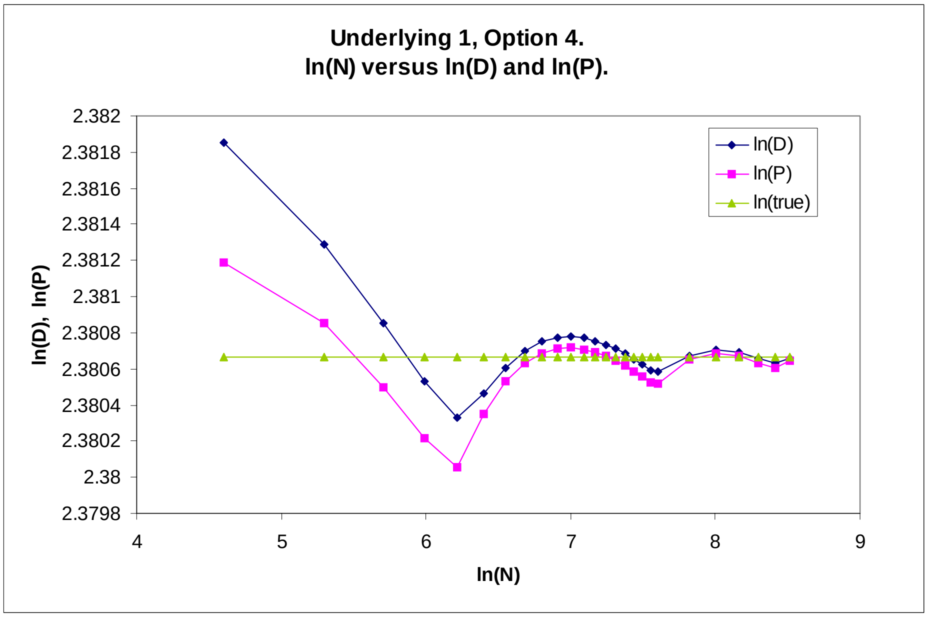

Figure 7: Option 4, underlying 1: log-log convergence

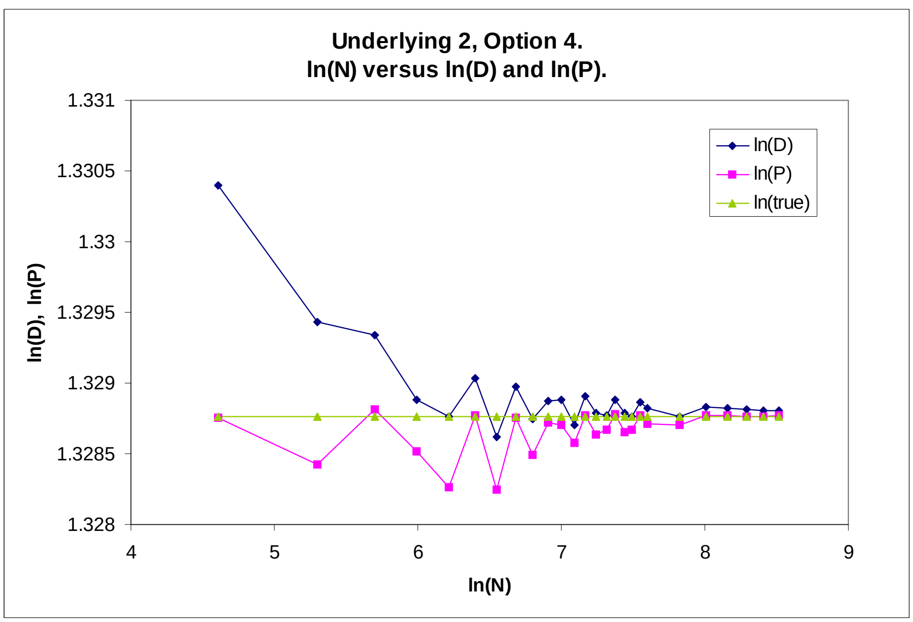

Figure 8: Option 4, underlying 2: log-log convergence

	Underlying 1, Option 5		Underlying 2, Option 5	
N_steps	Dirichlet	Plain	Dirichlet	Plain
100	13.30621	13.29486	5.795916	5.785021
200	13.30254	13.298	5.801767	5.797432
300	13.30013	13.29748	5.796924	5.792434
400	13.29847	13.29667	5.799426	5.797947
500	13.29724	13.29591	5.800164	5.798351
600	13.29627	13.29523	5.799251	5.797335
700	13.29549	13.29465	5.798181	5.796636
800	13.29487	13.29415	5.799098	5.798049
900	13.29437	13.2937	5.799576	5.798578
1000	13.29422	13.29333	5.799516	5.798515
1100	13.29414	13.29301	5.79907	5.798118
1200	13.29405	13.29277	5.79837	5.797513
1300	13.29463	13.29351	5.798732	5.798218
1400	13.29507	13.29408	5.799144	5.798587
1500	13.29541	13.29453	5.799327	5.798734
1600	13.29566	13.29488	5.799283	5.798789
1700	13.29586	13.29515	5.799131	5.798598
1800	13.296	13.29537	5.798866	5.798249
1900	13.29611	13.29554	5.798625	5.798111
2000	13.29618	13.29566	5.798816	5.798448
2500	13.29625	13.29593	5.799055	5.798675
3000	13.29609	13.29586	5.799045	5.798774
3500	13.29583	13.29566	5.798968	5.798695
4000	13.29554	13.29541	5.798998	5.798818
4500	13.29537	13.29514	5.799018	5.798808
5000	13.29545	13.29516	5.798892	5.798741

Table 7: Dirichlet and plain values for option 5

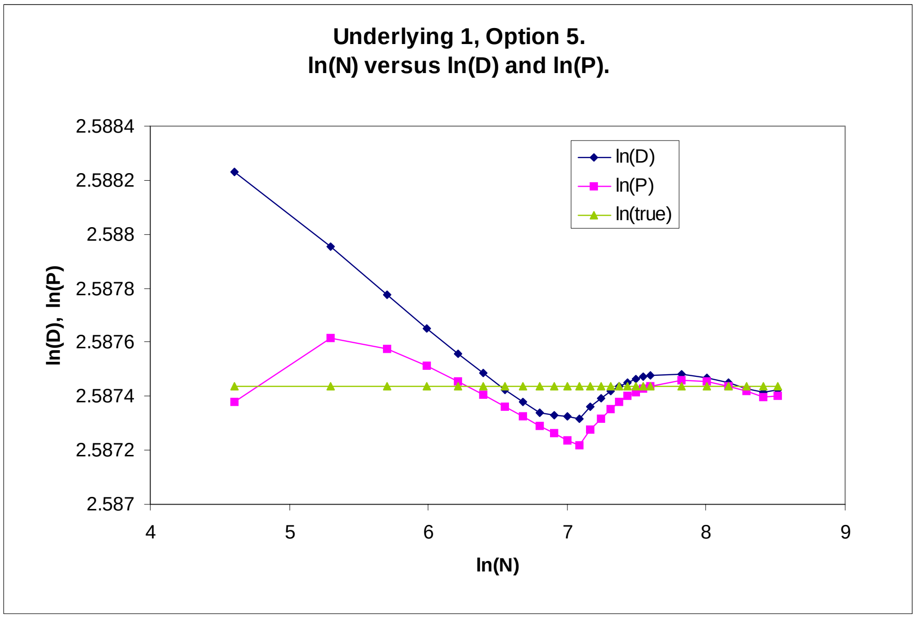

Figure 9: Option 5, underlying 1: log-log convergence

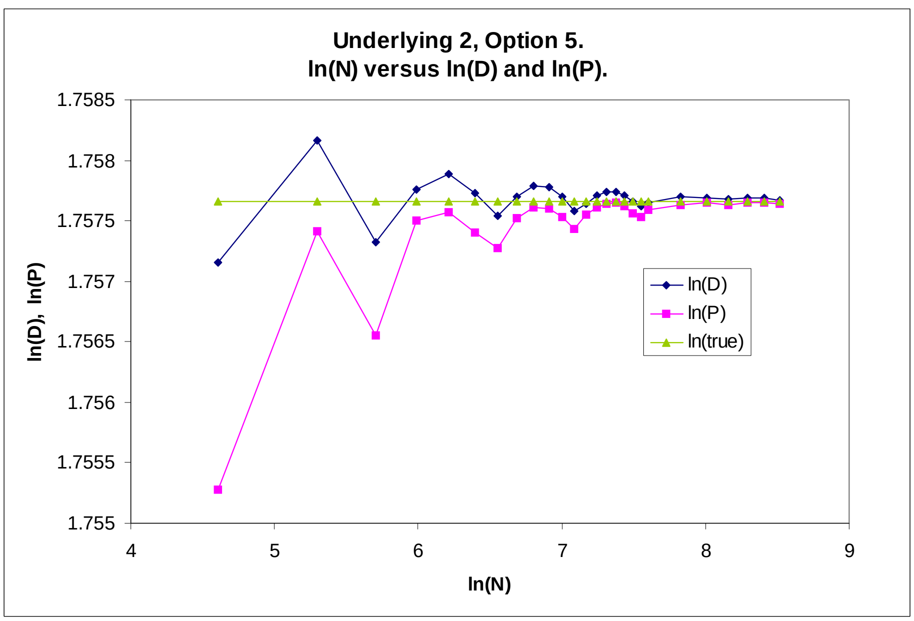

Figure 10: Option 5, underlying 2: log-log convergence

	Underlying 1, Option 6		Underlying 2, Option 6	
N_steps	Dirichlet	Plain	Dirichlet	Plain
100	16.02369	16.00772	8.418093	8.412704
200	16.05057	16.0459	8.418529	8.414394
300	16.04469	16.04257	8.417839	8.415852
400	16.03931	16.03557	8.417698	8.415519
500	16.04518	16.04285	8.41736	8.415717
600	16.04644	16.04505	8.417016	8.415992
700	16.04551	16.04454	8.416885	8.41595
800	16.04355	16.04283	8.41674	8.415582
900	16.04225	16.04065	8.416697	8.415342
1000	16.04409	16.04281	8.416411	8.415786
1100	16.04518	16.04425	8.416359	8.415574
1200	16.04545	16.04476	8.416333	8.415393
1300	16.04519	16.04466	8.416249	8.415643
1400	16.04458	16.04415	8.416315	8.415343
1500	16.04373	16.04336	8.416208	8.415576
1600	16.04326	16.04242	8.416319	8.415473
1700	16.04395	16.04316	8.416382	8.415805
1800	16.04462	16.04398	8.416515	8.415802
1900	16.04496	16.04445	8.41655	8.416054
2000	16.04506	16.04465	8.416654	8.416012
2500	16.04371	16.04321	8.416791	8.416396
3000	16.04486	16.04459	8.416826	8.416517
3500	16.04408	16.04392	8.416815	8.416525
4000	16.04465	16.04439	8.416777	8.41652
4500	16.0446	16.04446	8.41671	8.416541
5000	16.04411	16.04382	8.416665	8.416462

Table 8: Dirichlet and plain values for option 6

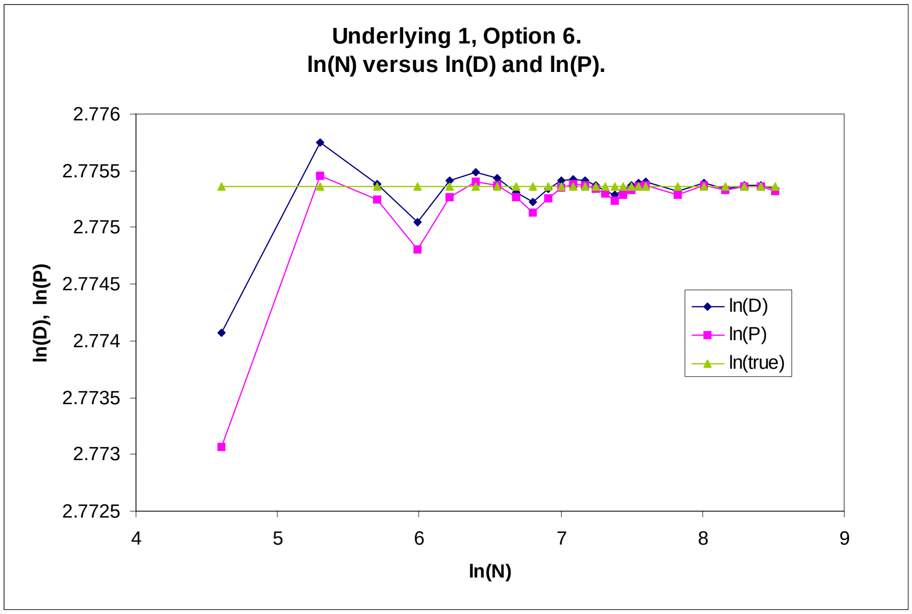

Figure 11: Option 6, underlying 1: log-log convergence

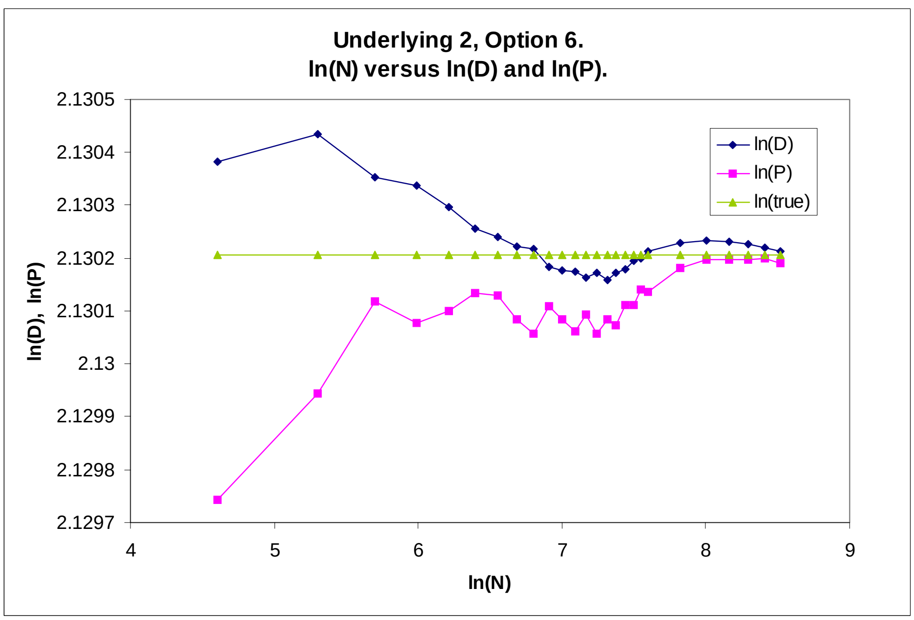

Figure 12: Option 6, underlying 2: log-log convergence

8 Appendix: An optimization procedure for u

To proceed set $u = S_{j+1,i+l} \exp\left(\sigma\varepsilon\sqrt{\kappa\Delta t}\right)$ for some $0 \leq \varepsilon < 1$. Then for $k \in \mathcal{B}_l$ the distribution function of the conditional minimum can be rewritten as

$$F_{j,j+1}^{i,i+k}(u) = \exp\left(-\frac{2}{\sigma^2\Delta t}\ln\left(\frac{u}{S_{j,i}}\right)\ln\left(\frac{u}{S_{j+1,i+k}}\right)\right) = \left(\frac{u}{S_{j,i}}\right)^{-\frac{2\kappa}{\sigma\sqrt{\kappa\Delta t}}(l-k+\varepsilon)} \tag{55}$$

and the ratio of distribution functions in equation (20) as

$$\frac{F_{j,j+1}^{i,i+q}(u)}{F_{j,j+1}^{i,i+k}(u)} = \left(\frac{u}{S_{j,i}}\right)^{-\frac{2\kappa}{\sigma\sqrt{\kappa\Delta t}}(k-q)} \tag{56}$$

Set $\upsilon = \left(\frac{u}{S_{j,i}}\right)^{-\frac{2\kappa}{\sigma\sqrt{\kappa\Delta t}}}$ then $F_{j,j+1}^{i,i+k}(u) = \upsilon^{l-k+\varepsilon}$ and $\frac{F_{j,j+1}^{i,i+q}(u)}{F_{j,j+1}^{i,i+k}(u)} = \upsilon^{k-q}$.

Also

$$\frac{\partial f_{j,j+1}^{i,i+k}(u)}{\partial u} = -\frac{2}{u\sigma^2\Delta t}\left(\left(r-\frac{1}{2}\sigma^2\right)\Delta t + \sigma(2l-k+2\varepsilon)\sqrt{\kappa\Delta t}\right) = \frac{A}{u} + \frac{B}{u}(k-2\varepsilon) \tag{57}$$

where

$$A = -\frac{2}{\sigma^2\Delta t}\left(\left(r-\frac{1}{2}\sigma^2\right)\Delta t + 2\sigma l\sqrt{\kappa\Delta t}\right) \tag{58}$$

$$B = \frac{2}{\sigma\Delta t}\sqrt{\kappa\Delta t}. \tag{59}$$

Restricting ourselves to trinomial branching, so $l = -1$, $\kappa = 3$ and $k = 0, 1$, and given $\frac{\partial H(u)}{\partial u} = -1$ equation (28) becomes

$$\begin{aligned} & p_1 F_{j,j+1}^{i,i+1}(u)\frac{\partial f_{j,j+1}^{i,i+1}(u)}{\partial u}\left(H(u) - \widehat{c}_{j+1,i+1}\right) \\ & + p_0 F_{j,j+1}^{i,i+0}(u)\frac{\partial f_{j,j+1}^{i,i+0}(u)}{\partial u}\left(H(u) - \widehat{c}_{j+1,i+0}\right) \\ & = p_1 F_{j,j+1}^{i,i+1}(u) + p_0 F_{j,j+1}^{i,i+0}(u) + p_{-1}. \end{aligned} \tag{60}$$

Hence, substituting in for υ, $F_{j,j+1}^{i,i+k}(u)$, $\frac{F_{j,j+1}^{i,i+q}(u)}{F_{j,j+1}^{i,i+k}(u)}$, and $\frac{\partial f_{j,j+1}^{i,i+k}(u)}{\partial u}$ we get

$$u = \frac{(A + B(1-2\varepsilon))\left(H(u) - \widehat{c}_{j+1,i+1}\right)}{1 + \frac{p_0}{p_1}\upsilon + \frac{p_{-1}}{p_1}\upsilon^{2-\varepsilon}} + \frac{(A - 2B\varepsilon)\left(H(u) - \widehat{c}_{j+1,i+0}\right)}{\frac{p_1}{p_0}\upsilon^{-1} + 1 + \frac{p_{-1}}{p_0}\upsilon^{1-\varepsilon}} \tag{61}$$

But $p_1 = p_{-1}$, $u = S_{j,i}\upsilon^{-\lambda}$ where $\lambda = \frac{\sigma\sqrt{\kappa\Delta t}}{2\kappa}$ and $H(u) = X - u$ (note that $\lambda = 1/B$) so

$$S_{j,i}\upsilon^{-\lambda}\left(1+\frac{p_0}{p_1}\upsilon+\upsilon^{2-\varepsilon}\right)=\left(A+B\left(1-2\varepsilon\right)\right)\left(X-S_{j,i}\upsilon^{-\lambda}-\widehat{c}_{j+1,i+1}\right)$$
$$+\frac{p_0}{p_1}\upsilon\left(A-2B\varepsilon\right)\left(X-S_{j,i}\upsilon^{-\lambda}-\widehat{c}_{j+1,i+0}\right) \quad (62)$$

Moreover

$$\begin{aligned}\varepsilon &= \frac{1}{\sigma\sqrt{\kappa\Delta t}}\ln\left(\frac{S_{j,i}}{S_{j+1,i+l}}\right)-\frac{1}{2\kappa}\ln\upsilon \\ &= C-\frac{1}{2\kappa}\ln\upsilon\end{aligned}$$

where $C=\frac{1}{\sigma\sqrt{\kappa\Delta t}}\ln\left(\frac{S_{j,i}}{S_{j+1,i+l}}\right)$. Putting this result into equation (62) we get a nonlinear equation in $\upsilon=\left(\frac{u}{S_{j,i}}\right)^{-\frac{2\kappa}{\sigma\sqrt{\kappa\Delta t}}}$ that can be solved, for instance, using Newton's method. This obtains possible values for u that satisfy the first order condition. To be an optimal exercise policy u must also satisfy the second order condition and be within the range of allowed values of u. Any such u becomes a candidate for the optimal exercise level.

9 Appendix: Hitting times and defective densities

The distribution of hitting times of a geometric Brownian motion Given its value S_t at time t, the probability of the process hitting a barrier at level $u < S_t$ before time T is

$$\Pr\left[\tau_u \leq T\right] = N\left(-h_2^{'}\right) + \left(\frac{u}{S_t}\right)^{\frac{2}{\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)} N\left(h_2\right) \tag{63}$$

where

$$h_2^{'} = \frac{1}{\sigma\sqrt{T-t}} \ln\left(\frac{S_t}{u\mathrm{e}^{-r(T-t)}}\right) - \frac{1}{2}\sigma\sqrt{T-t}, \tag{64}$$

$$h_2 = \frac{1}{\sigma\sqrt{T-t}} \ln\left(\frac{u}{S_t\mathrm{e}^{-r(T-t)}}\right) - \frac{1}{2}\sigma\sqrt{T-t}. \tag{65}$$

Hence the value $R(u)$ at time t of a payoff of $(X-u)_+$ received at time T if the barrier is hit is

$$R(u) = \mathrm{e}^{-r(T-t)}\left(X-u\right)_+ \left(N\left(-h_2^{'}\right) + \left(\frac{u}{S_t}\right)^{\frac{2}{\sigma^2}\left(r-\frac{1}{2}\sigma^2\right)} N\left(h_2\right)\right). \tag{66}$$

We use this formula to compute payoffs for the deferred nominated barrier option. For the nominated barrier option, not deferred, there is also an exact formula. The value $\hat{R}(u)$ at time t of a payoff of $(X-u)_+$ received at the time the barrier is hit is (Suo and Wang, p20)

$$\hat{R}(u) = (X-u)_+ \int_t^T \mathrm{e}^{-rs}\mathrm{d}\Pr\left[T_s \leq T\right] \tag{67}$$

$$= (X-u)_+ \left[\left(\frac{u}{S_t}\right)^{\frac{\mu+\nu}{\sigma^2}} N\left(e_2^{'}\right) + \left(\frac{u}{S_t}\right)^{\frac{\mu-\nu}{\sigma^2}} N\left(e_2\right)\right] \tag{68}$$

where N is the standard normal distribution function and

$$e_2 = \frac{1}{\sigma\sqrt{T-t}} \ln\left(\frac{u}{S_t}\right) + \frac{\nu}{\sigma}\sqrt{T-t} \tag{69}$$

$$e_2^{'} = e_2 - 2\frac{\nu}{\sigma}\sqrt{T-t} \tag{70}$$

with $\mu = r - \frac{1}{2}\sigma^2$ and $\nu = \sqrt{\left(r-\frac{1}{2}\sigma^2\right)^2 + 2r\sigma}$.

The defective density of a geometric Brownian motion $\Pr\left[S_T \leq S,\ \tau_u > T \mid S_t\right]$ is the defective density at time $T > t$ for a geometric Brownian motion $S =$

$(S_t)_{t\geq 0}$ and barrier level $u < S_t$. For $s \geq t$ set $W_s = \ln\left(\frac{S_s}{S_t}\right)$ so that $W_s = \left(r - \frac{1}{2}\sigma^2\right)s + \sigma z_s$ for a Wiener process z_s. We have

$$\Pr\left[S_T \leq S,\ \tau_u > T \mid S_t\right] = \Pr\left[W_T \leq w,\ \tau_u^W > \hat{u} \mid W_t\right] \tag{71}$$

where $w = \ln\left(\frac{S}{S_t}\right)$, $\hat{u} = \ln\left(\frac{u}{S_t}\right)$ and τ_u^W is the hitting of W_s to $\hat{u}$. The defective density $f(w, T \mid w_t, t; \hat{u})$ of a Brownian motion with drift is known. Set $\mu = r - \frac{1}{2}\sigma^2$. Then

$$f(w, T \mid w_t, t; \hat{u}) = \mathrm{d}\Pr\left[W_T \leq w,\ \tau_u^W > \hat{u} \mid W_t\right] \tag{72}$$

$$= \frac{1}{\sigma\sqrt{T-t}}\left\{n(e_1) - \exp\left(-\frac{2(w_t - \hat{u})\mu}{\sigma^2}\right)n(e_2)\right\} \tag{73}$$

where n is the standard normal density function and

$$e_1 = \frac{w - w_t - \mu(T-t)}{\sigma\sqrt{T-t}}, \tag{74}$$

$$e_2 = \frac{w + w_t - 2\hat{u} - \mu(T-t)}{\sigma\sqrt{T-t}}. \tag{75}$$