Nominated Barrier Options: An Approach to Valuing American Options on a Lattice

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February 12, 2006

Abstract

American style options are of considerable importance in the financial markets. However, to value them requires numerical methods or approximate formulae since in general no explicit formulae exist. Typically convergence to the true option value is slow resulting in practice in inaccurate prices.

In this paper we value on a lattice with novel probabilities an instrument, a nominated barrier option, whose value converges to that of an American option as the number of time steps increases. We find that the value of the nominated barrier option on the lattice seems to converge to the value of the underlying American option faster than existing methods, but with greater initial bias.
1 Introduction

American style options are an important part of the options markets. Unlike European options, American options rarely have explicit solutions. There is no known formula even for an American put on an asset following geometric Brownian motion. Consequentially approximate formula or numerical methods, such as PDE or simulation methods, need to be used.

A number of analytic approximations have been suggested in the literature. Geske and Johnson (1984) use Richardson’s extrapolation on the valuation formula of a portfolio of Bermudan options with different exercise dates. This method was later modified by Bunch and Johnson (1992), Ho, Stapleton and Subrahmanyam (1994) and Huang, Subrahmanyam and Yu (1996) proposing various methods to value Bermudan options. MacMillan (1986) used a quadratic approximation approach to a PDE, extended by Baroni-Adesi and Whaley (1987). Along the same lines but with increased accuracy, specially for long term options, is the approximation by Ju and Zhong (1999). Another approach (Fu et al. (2001)) is to approximate the value function by a piecewise linear interpolation, so that the continuation value can be expressed as a summation of European call option values.

To get numerical solutions of the Black-Scholes PDE, subject to American option boundary conditions, several methods have been used, among others: finite difference methods (Brennan and Schwartz (1977), Courtadon (1982), Dempster and Hutton (1997), and Wu and Kwok (1997)); finite element methods (Wilmott et al. (1993), and Zvan, Forsyth and Vetzal (2001)); the method of lines (Meyer and van der Hoek (1997)); and the differential quadrature method (Wu and Ding (2002)). Mallier and Alobaidi (2000) applied a Laplace transform to obtain an integral equation that can be solved numerically.

Simulation methods include both Monte Carlo and lattice methods. There has been significant progress in using Monte Carlo methods to value American options, applied mainly for path-dependent payoffs. For instance, Longstaff and Schwartz (2001) and Van Roy and Tsitsiklis (2001) use regression based models to estimate continuation values from simulated paths. The stochastic mesh method, originally introduced by Broadie and Glasserman (1997a) and also used by Avramidis and Matzinger (2002), solves a randomly sampled dynamic programming problem to approximate the price of an American option. The main distinction between the stochastic mesh method and the random tree method (Broadie and Glasserman (1997b)) is that in valuing the option at a specific node the former uses values from all the nodes at the subsequent time step whereas the latter uses only the values of the successor nodes from that node. Haugh and Kogan (2001) and Rogers (2002) establish a dual approach in which the American option price is represented through a minimization problem as opposed to the most common formulation of maximization over stopping times.

Ever since lattice methods were introduced for the valuation of American options (Cox, Ross and Rubinstein (1979)) they have received much attention in the literature. Generalizations of the binomial method include the multino-
and Detemple (1996) found a modification of the binomial method that uses
the Black-Scholes value at the time step just before maturity to compensate for
convergence problems in the standard method. Figlewski and Gao (1999) obtain
considerable gains of accuracy and efficiency using an adaptive mesh method.
Alford and Webber (2001) achieved very high rates of convergence using lattices
with high-order branching. The convergence of lattice methods has been
established, for instance, by Amin and Khanna (1994), Broadie and Detemple

A problem with using numerical methods for valuing American options is
their slow rate of convergence. Only relatively inaccurate option values can be
found in a short time.

This paper presents a lattice method based upon valuing an instrument,
which we call a nominated barrier option, exercisable between reset dates, whose
value converges to that of an American option as the number of reset dates
increases. To value this instrument we exploit an idea of Kuan and Webber
(2002) who used a probability correction term in a lattice for barrier options.
Two alternative forms of correction are used. In the first the correction term is
based upon the distribution of the maximum of a Brownian bridge connecting
nodes in the lattice. In the second the hitting time distribution is calculated, and
branching probabilities found by matching moments to those of the associated
defective probability distribution.

Our exposition focuses on the example of the American put, but it is clear
that the method generalizes to other types of American option.

The next section discusses the nominated barrier option. Section three
describes the Dirichlet lattice method and how it can be applied to valuing nom-
ominated barrier options. In section four we describe a modified Dirichlet
lattice and how it may be applied. We refer to this lattice as the American Dirichlet
lattice. Section five gives numerical results and section six concludes.

2 The Nominated Barrier Option

A nominated barrier option is an option where at each reset date the option
holder nominates a barrier level to apply until the next reset date. If the value
of an underlying asset hits the barrier level the option expires but the holder
receives a payoff. If the option survives until maturity, the holder again receives
a payoff.

Suppose the option is created at time $t_0 = 0$ and has a final maturity time $T$. We suppose there are $N + 1$ equally spaced reset dates, $t_j = t_0 + j\Delta t,$ $j = 0, \ldots, N$, where $\Delta t = \frac{T}{N}$. Let the underlying asset be a stochastic process $S = (S_t)_{t \geq 0}$. At each reset date $t_j$ the holder nominates a barrier level $u_j$. Here, with the example of the American put in mind, we consider only down-barriers so that we require $u_j \in [0, S(t_j)]$. Let $\tau_{u_j} = \inf_{t \geq t_j} \{ S_t \leq u_j \}$ be the first hitting time of the underlying asset to the barrier. If $\tau_{u_j} \in [t_j, t_{j+1})$ the option expires at the hitting time $\tau_{u_j}$ and the option holder receives a payoff of $H(\tau_{u_j}, u_j)$ at
time $\tau_{u_j}$. At time $T$, if the option is still alive, the holder receives a payoff of $G(S_T)$.

For given payoff functions $H$ and $G$, write $c_N$ for the value of a nominated barrier option with $N + 1$ reset dates. As $N \to \infty$ the option value tends to a limit, $c_N \to c_\infty$. $c_\infty$ is the value of an American option with payoff $G(S_T)$ at time $T$ and payoff $H(t,S_t)$ if exercised early.

Specifically, we now assume that $H(\tau_{u_j},u_j) = X - u_j$ and $G(S_T) = X - S_T$. The function

$$U_N : [0,T] \to \mathbb{R}^+,$$

$$U_N (t) = u_j, \text{ for } t \in [t_j,t_{j+1}),$$

determines an exercise policy for an American put. As $N \to \infty$ we have $U_N \to \hat{U}_t$, the optimal exercise policy for an American put, so that $c_\infty$ is the value of an American put with strike $X$.

In the numerical work below, where $N$ is large, we shall consider a version of the nominated barrier option in which, if the barrier $u_j$ is hit, the payoff is deferred until time $t_{j+1}$. We call this the deferred nominated barrier option. Writing $d_N$ for its value we note that $d_N \to c_\infty$ as $N \to \infty$.

From now on we assume that the asset $S_t$ follows a geometric Brownian motion with constant volatility $\sigma$ and constant riskless rate $r$. We use the accumulator numeraire so that under the equivalent martingale measure the asset process is

$$dS_t = rS_t dt + \sigma S_t dz_t.$$  

3 Nominated Barrier Options and the Dirichlet Lattice

First we describe the Dirichlet lattice of Kuan and Webber (2002), then we show how it may be applied to value (deferred) nominated barrier options.

3.1 The Dirichlet Lattice

Let $c_t$ be the value at time $t$ of an American option maturing at time $T$ with exercise price $X$. The value of the option at time $t_0$ is $c_0 = \max_\omega E_0 [e^{-r\tau(\omega)} H(\omega)]$ where the maximum is taken over all exercise policies $\omega$; $H(\omega)$ is the payoff received at time $\tau(\omega)$ under policy $\omega$.

Time is discretised into $N$ steps, $0 = t_0 < \ldots < t_N = T$, with constant time step $\Delta t = t_j - t_{j-1}$. We construct a $d$-nominal lattice for the underlying Wiener process $z_t$ where $d = 2b + 1$ for an integer $b \geq 1$. Nodes on the lattice are labelled $(j,i)$, $j = 0,\ldots,N$, $i = -bj,\ldots,bj$.\footnote{In practical implementations the lattice is truncated at high and low levels.} Set $\Delta z = \sqrt{\kappa \Delta t}$ for some constant $\kappa$. The value of the discretised Wiener process $\hat{z}_t$ at node $(j,i)$ is $i\Delta z$. 

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The corresponding value of the asset is

\[ S_{j,i} = S_0 \exp \left( \left( r - \frac{1}{2}\sigma^2 \right) j \Delta t + \sigma z_{j,i} \right). \]  

Branching from node \((j,i)\) is to nodes \((j+1,i+k)\), \(k \in B = \{-b, \ldots, b\}\) with corresponding probabilities \(p_k\). For our numerical work we specialize down to a trinomial lattice with \(d = 3\), so that \(B = \{-1, 0, 1\}\). In this case we use the standard values, \(\kappa = 3\), \(p_{-1} = p_1 = \frac{1}{6}\), \(p_0 = \frac{2}{3}\) so that the first five moments of \(z_t\) are matched.

Consider an option with a deterministic time varying barrier \(u_t\) such that when the asset value hits the barrier the option expires and some rebate value is paid. We assume that \(S_0 > u_0\), and the option is a down-and-out with a rebate value \(H(\tau, u_\tau)\) paid when the barrier is hit at time \(\tau = \min\{t_i \leq u_t\}\). At time \(T\), if \(\tau > T\), the option expires with value \(H(T, S_t)\).

If we set \(H(\tau, u_\tau) = X - u_t\) for some constant \(X\), and set \(u_t\) to be the early exercise frontier of an American put then the option is equivalent to an American put.

This option may be valued on a standard (plain) lattice by backwards induction as follows. At time \(t_N\) set \(c_{N,i} = H(t_N, S_{N,i})\), \(i = -bN, \ldots, bN\). Suppose that for time \(t_{j+1}\) the values \(c_{j+1,i}\) have been found. We now compute the values \(c_{j,i}\). Set \(B_j = \{k \in B \mid S_{j+1,i+k} < u_{t_{j+1}}\}\). Then set

\[ c_{j,i} = e^{-r\Delta t} \left[ \sum_{k \in B \setminus B_j} p_k \bar{c}_{j+1,i+k} + \sum_{k \in B_j} p_k H_{j+1,i+k}(t_{j+1}, u_{t_{j+1}}) \right]. \]

The value of the option on the plain lattice is \(c_{0,0}\). We write \(p_N\) for the option value found on a plain lattice with \(N\) time steps.

Fast lattice solutions for knock-out options of this sort were given by Kuan and Webber (2002). They showed\(^3\) how these options could be valued on a Dirichlet lattice by both forwards and backwards induction. Here we are concerned only with backwards induction.

Let

\[ m_{j,j+1}^{i,i+k} = \min_{t \in [t_j, t_{j+1}]} \{ S_t \mid S_{t_j} = S_{j,i}, S_{t_{j+1}} = S_{j+1,i+k} \} \]

be the minimum of \(S_t\) in the range \(t \in [t_j, t_{j+1}]\) conditional on its end points and let

\[ F_{j,j+1}^{i,i+k}(u) = \Pr \left[ m_{j,j+1}^{i,i+k} \leq u \mid S_{t_j} = S_{j,i}, S_{t_{j+1}} = S_{j+1,i+k} \right] \]

be the distribution function of \(m_{j,j+1}^{i,i+k}\).

\(^2\)Optimal values for \(\kappa\) and for the branching probabilities \(p_k\) were determined by Alford and Webber (2001).

\(^3\)Their examples are for up-and-in and up-and-out options, but their methods also apply to down-and-in and down-and-out options.
When $S_t$ follows a geometric Brownian motion $F_{j,j+1}^{i,i+k}(u)$ is known. For $u \leq \min \{S_{j,i}, S_{j+1,i+k}\}$ let $\hat{u} = \ln \left( \frac{u}{S_0} \right)$ and $w_{j,i} = (r - \frac{1}{2}\sigma^2) j \Delta t + \sigma i \Delta z = \ln \left( \frac{S_j}{S_0} \right)$. Then $F_{j,j+1}^{i,i+k}$ is given by

$$F_{j,j+1}^{i,i+k}(u) = \exp \left( -\frac{2}{\sigma^2 \Delta t} \left( \hat{u} - w_{j,i} \right) \left( \hat{u} - w_{j+1,i+k} \right) \right)$$

$$= \exp \left( -\frac{2}{\sigma^2 \Delta t} \ln \left( \frac{u}{S_{j,i}} \right) \ln \left( \frac{u}{S_{j+1,i+k}} \right) \right), \quad (8)$$

(see El Babsiri and Noel (1998)). For $\hat{u} > \min \{w_{j,i}, w_{j+1,i+k}\}$, so that $u > \min \{S_{j,i}, S_{j+1,i+k}\}$, set $F_{j,j+1}^{i,i+k}(u) = 1$.

At a node $(j, i)$ above the barrier let $p_{j,i,k}^O$ be the probability of branching from node $(j, i)$ to node $(j + 1, i + k)$ without hitting the barrier level $u_j$ and let $p_{j,i,k}^1$ be the probability of branching to node $(j + 1, i + k)$ and hitting the barrier. Then for all $k \in B$ we set

$$p_{j,i,k}^O = p_k \left( 1 - F_{j,j+1}^{i,i+k}(u_j) \right), \quad (9)$$

$$p_{j,i,k}^1 = p_k F_{j,j+1}^{i,i+k}(u_j), \quad (10)$$

where $F_{j,j+1}^{i,i+k}(u_j) = 1$ if $S_{j+1,i+k} \leq u_j$.

For $S_{j,i} > u_j$, option values $\hat{c}_{j,i}$ are given by backwards induction as

$$\hat{c}_{j,i} = e^{-r \Delta t} \sum_{k \in B} \left( p_{j,i,k}^O \hat{c}_{j+1,i+k} + p_{j,i,k}^1 H_{j+1,i+k}(u_j) \right)$$

$$= e^{-r \Delta t} \sum_{k \in B} p_k \left( \left( 1 - F_{j,j+1}^{i,i+k}(u_j) \right) \hat{c}_{j+1,i+k} + F_{j,j+1}^{i,i+k}(u_j) H_{j+1,i+k}(u_j) \right) \quad (11)$$

for $i \geq m_j$, where $H_{j+1,i+k}(u_j)$ is the payoff to the option at time $t_{j+1}$ if it is knocked out between times $t_j$ and $t_{j+1}$. The option value is $\hat{c}_{0,0}$.

Note that in the plain lattice one is effectively setting $F_{j,j+1}^{i,i+k}(u_j)$ to be zero if $S_{j+1,i+k} > u_{t_{j+1}}$ and to be one if $S_{j+1,i+k} \leq u_{t_{j+1}}$.

### 3.2 Valuing Nominated Barrier Options

We value the nominated barrier option $c_N$ on a Dirichlet lattice with $N$ time steps.\footnote{In fact, for simplicity, we value on the lattice the deferred nominated barrier option $d_N$.} At each node on the lattice we allow the option holder to nominate a constant barrier level for the subsequent time step, so that the option is exercised if the barrier level is hit. As $\Delta t \to 0$ the value of this option converges to that of an American option.

Write $\hat{c}_{j,i}$ for the value of this approximate American option at node $(j, i)$ on the lattice. Over a small time step $\Delta t$, conditional on the current state $S_{j,i}$, write
\( \hat{u}_{j,i} \) for the optimal exercise level. At each time step we find an approximation \( \hat{u}_{j,i} \) to the optimal exercise policy \( \hat{u}_{j,i} \) for the holder of the option.

From node \((j, i)\), if the exercise policy were to exercise at a level \(u\), the (deferred) option would have value

\[
\hat{c}_{j,i}(u) = e^{-r\Delta t} \left( \sum_{k \in \mathcal{B}} \sum_{k \in \mathcal{B}} p_k F_{j,j+1}^{i,i+k} (u) H(u) \right) + \sum_{k \in \mathcal{B}} p_k \left( 1 - F_{j,j+1}^{i,i+k} (u) \right) \hat{c}_{j+1,i+k} = e^{-r\Delta t} \sum_{k \in \mathcal{B}} p_k \left( F_{j,j+1}^{i,i+k} (u) H(u) \right) + \left( 1 - F_{j,j+1}^{i,i+k} (u) \right) \hat{c}_{j+1,i+k}. \tag{12}
\]

With our assumption, the optimal exercise policy on the lattice is to exercise at a level \(u\) that maximizes \( \hat{c}_{j,i}(u) \); \( \hat{u}_{j,i} = \arg \max_u \hat{c}_{j,i}(u) \) and \( \hat{c}_{j,i}(u) = \hat{c}_{j,i}(\hat{u}_{j,i}) \).

If it exists, the maximum occurs when \( \frac{\partial^2 \hat{c}_{j,i}(u)}{\partial u^2} < 0 \). We then have

\[
e^{r\Delta t} \frac{\partial \hat{c}_{j,i}(u)}{\partial u} = \sum_{k \in \mathcal{B}} p_k F_{j,j+1}^{i,i+k} (u) \frac{\partial H(u)}{\partial u} + \sum_{k \in \mathcal{B}} p_k \frac{\partial F_{j,j+1}^{i,i+k} (u)}{\partial u} \left( H(u) - \hat{c}_{j+1,i+k} \right)
\]

\[
= \sum_{k \in \mathcal{B}} p_k F_{j,j+1}^{i,i+k} (u) \left( \frac{\partial H(u)}{\partial u} + \frac{\partial f_{j,j+1}^{i,i+k} (u)}{\partial u} (H(u) - \hat{c}_{j+1,i+k}) \right). \tag{13}
\]

where

\[
f_{j,j+1}^{i,i+k} (u) = \ln F_{j,j+1}^{i,i+k} (u) = -\frac{2}{\sigma^2 \Delta t} \ln \left( \frac{u}{S_{j,i}} \right) \ln \left( \frac{u}{S_{j+1,i+k}} \right). \tag{14}
\]

and

\[
\frac{\partial f_{j,j+1}^{i,i+k} (u)}{\partial u} = -\frac{2}{\sigma^2 \Delta t} \ln \left( \frac{u^2}{S_{j,i}S_{j+1,i+k}} \right). \tag{15}
\]

To determine the optimal exercise policy the first order condition requires that equation (13) is zero, so that

\[
\sum_{k \in \mathcal{B}} p_k F_{j,j+1}^{i,i+k} (u) \frac{\partial f_{j,j+1}^{i,i+k} (u)}{\partial u} \left( H(u) - \hat{c}_{j+1,i+k} \right) = -\frac{\partial H(u)}{\partial u} \sum_{k \in \mathcal{B}} p_k F_{j,j+1}^{i,i+k} (u)
\]

for the optimal barrier level \(u\). Note that for \(b \geq 1\) we have \(S_{j,i} > S_{j+1,i-b}\). Also, for \(N\) large enough so that \(\Delta t < \sigma^2 \kappa\), we have \(S_{j+1,i-1} < S_{j,i} < S_{j+1,i}\). With this assumption there are three possibilities for \(u\): that \(u \geq S_{j,i}\), that \(u < S_{j+1,i-b}\), and that there exists an index \(l \in \{-1, \ldots, -b\}\) such that \(u \in [S_{j+1,i+l}, S_{j+1,i+l+1}]\).

In each case we try to find a candidate value for \(u\) that could be the optimal level. Having found a set of candidate values we determine which one maximizes the value of \(\hat{c}_{j,i}(u)\).
3.2.1 The case \( u \geq S_{j,i} \)

The first case is straightforward. If \( u \geq S_{j,i} \) one exercises immediately with value \( H(u) \). Hence the optimal value \( u \) is given by

\[
\hat{u}_{j,i} = \arg\max_u H(u) .
\]  

(17)

For a vanilla American put with \( H(u) = X - u \) it is optimal to set the candidate value to be \( u = \min\{u \geq S_{j,i}\} = S_{j,i} \). When the payoff is received immediately this is the standard exercise condition tested against the continuation value of the option. By including two additional cases, corresponding to two additional continuation values, we obtain a better approximation to the American option value.

3.2.2 The case \( u < S_{j+1,i-b} \)

Now suppose that \( u < S_{j+1,i-b} \). We cannot solve explicitly for \( \hat{u}_{j,i} \) in this case, but we obtain an approximation as the solution to a cubic equation.

Set

\[
G_{j,i,k}(u) = \frac{\sum_{l \in B} p_l F_{j+1,i+l}^{j,i+k}(u)}{p_k F_{j+1,i+1}^{j,i}(u)} = \frac{1}{\sum_{l \in B} p_l F_{j+1,i+1}^{j,i+k}(u)}
\]  

(18)

and rewrite the first order condition at a maximum (16) as

\[
\sum_{k \in B} G_{j,i,k}(u) \frac{\partial f_{j,i}^{j,i+k}(u)}{\partial u} (H(u) - \hat{c}_{j+1,i+k}) = -\frac{\partial H(u)}{\partial u} .
\]  

(19)

Note that the ratio of distribution functions in \( G_{j,i,k}(u) \) is given by

\[
\frac{F_{j+1,i+k}^{j,i+l}(u)}{F_{j,i+1}^{j,i+1}(u)} = \exp\left(-\frac{2}{\sigma^2 \Delta t} \ln\left(\frac{u}{S_{j,i}}\right) \ln\left(\frac{S_{j+1,i+k}}{S_{j,i}}\right)\right) \\
\times \exp\left(-\frac{2}{\sigma^2 \Delta t} \ln\left(\frac{u}{S_{j,i+1,i+k}}\right) \ln\left(\frac{S_{j+1,i+1}}{S_{j+1,i+k}}\right)\right) \\
= \exp\left(-\frac{2}{\sigma^2 \Delta t} \ln\left(\frac{u}{S_{j,i+1,i+k}}\right) (k - l) \sigma \sqrt{\Delta t}\right) \\
= \left(\frac{u}{S_{j,i+1,i+k}}\right)^{-2(k-l)\sqrt{\Delta t}} .
\]  

(20)

Since in this case \( u < S_{j,i} \), we may conclude that as \( \Delta t \to 0 \), keeping \( S_{j,i} \) fixed,

\[
\frac{F_{j+1,i+k}^{j,i+l}(u)}{F_{j,i+1}^{j,i+1}(u)} \to \begin{cases} 
\infty & k > l, \\
1 & k = l, \\
0 & k < l.
\end{cases}
\]  

(21)
and hence as $\Delta t \to 0$, the limit of $G_{j,i,k}(u)$ is

$$G_{j,i,k}(u) \to \begin{cases} 1, & k = -b, \\ 0, & \text{otherwise}. \end{cases} \quad (22)$$

Furthermore, $G_{j,i,k}(u)$ approaches its limit at power speed. As $\Delta t \to 0$ every term on the left hand side of equation (19) goes to zero except the $-b$th term, so

$$-\frac{\partial H(u)}{\partial u} = \sum_{k \in B} G_{j,i,k}(u) \frac{\partial f_{i,i}^{j,i+k}(u)}{\partial u} (H(u) - \tilde{c}_{j+1,i+k})$$

$$\to \frac{\partial f_{i,i}^{j,i-b}(u)}{\partial u} (H(u) - \tilde{c}_{j+1,i-b}). \quad (23)$$

For an American put with strike $X$ we have $H_{j+1,i}(u) = X - u$ and $\frac{\partial H_{j+1,i}}{\partial u} = -1$. Hence, substituting in for $\frac{\partial f_{i,i}^{j,i-b}(u)}{\partial u}$ and $H(u)$, at a maximum we require, in the limit,

$$1 = -\frac{2}{u \sigma^2 \Delta t} \ln \left( \frac{u^2}{S_{j,i}S_{j+1,i-b}} \right) (X - u - \tilde{c}_{j+1,i-b}) \quad (24)$$

This result is intuitive. As $\Delta t \to 0$ the branch that has the dominating affect on the value of the exercise level is the lowest branch, closest to the exercise level.

When $\Delta t$ is small the probability of hitting the exercise boundary is small and, unless $S_{j+1,i-b}$ is near $u$, can be ignored. When $u$ is close to $S_{j+1,i-b}$ we make a series approximation to $\ln \left( \frac{u^2}{S_{j,i}S_{j+1,i-b}} \right)$ and truncating at first order obtain

$$\left( \frac{u^2}{S_{j,i}S_{j+1,i-b}} - 1 \right) (X - u - \tilde{c}_{j+1,i-b}) = -\frac{u \sigma^2 \Delta t}{2} \quad (25)$$

Hence to find $u$ we solve the cubic equation

$$u^3 - u^2 (X - \tilde{c}_{j+1,i-b}) - u S_{j,i} S_{j+1,i-b} \left( 1 + \frac{\sigma^2 \Delta t}{2} \right) + S_{j,i} S_{j+1,i-b} (X - \tilde{c}_{j+1,i-b}) = 0. \quad (26)$$

Since the right-hand side of (25) is close to zero for reasonable values of $N$, the cubic has three real roots, two of which are close to $\pm \sqrt{S_{j,i} S_{j+1,i-b}}$ and the third is close to $X - \tilde{c}_{j+1,i-b}$. Since $\tilde{c}_{j+1,i-b} > X - S_{j+1,i-b}$ the third root is close to but less than $S_{j+1,i-b}$. The second order condition on the maximum implies that we require the middle of the three roots. Thus the third root becomes our candidate value in this case.

Equation (26) can be solved explicitly by standard methods. Since the polynomial discriminant is negative the acos method works well in this case.$^5$

$^5$The calculation of $\text{acos}(x)$ is accurate for typical values of $x$ encountered in our applications.
3.2.3 The case \( u \in [S_{j+1,i-b}, S_{j,i}] \)

The final case is when \( u \in [S_{j+1,i+1}, S_{j+1,i+i+1}] \) for some \( l \in \{-1, \ldots, b\} \).

Set \( B_l = \{ k \in B \mid k > l \} \). Since for \( k \in B \setminus B_l \) we have \( F_{j,j+1}^{i,i+k} (u) = 1 \) and \( \frac{\partial F_{j,j+1}^{i,i+k} (u)}{\partial u} = 0 \), the option value at node \((j,i)\) (equation 12) is

\[
\hat{c}_{j,i} (u) = e^{-r \Delta t} \sum_{k \in B_l} k \left( F_{j,j+1}^{i,i+k} (u) H (u) + \left( 1 - F_{j,j+1}^{i,i+k} (u) \right) \hat{c}_{j+1,i+k} \right)
\]

\[= e^{-r \Delta t} \sum_{k \in B_l} k \left( F_{j,j+1}^{i,i+k} (u) H (u) + \left( 1 - F_{j,j+1}^{i,i+k} (u) \right) \hat{c}_{j+1,i+k} \right)
+ e^{-r \Delta t} H (u) \sum_{k \in B \setminus B_l} k \]

and the first order condition (equation (16)) becomes

\[
\sum_{k \in B_l} k \left( F_{j,j+1}^{i,i+k} (u) \frac{\partial F_{j,j+1}^{i,i+k} (u)}{\partial u} (H (u) - \hat{c}_{j+1,i+k}) = - \frac{\partial H (u)}{\partial u} \left( \sum_{k \in B_l} k F_{j,j+1}^{i,i+k} (u) + \sum_{k \in B \setminus B_l} k \right) \right.
\]

\[
(28)
\]

In the appendix we describe an optimization procedure that can be used to solve equation (28) for \( u \). In practice we found it computationally more efficient to compute \( \hat{c}_{j,i} (u) \) directly from equation (12) for a small set of regularly spaced values of \( u \) in the interval \([S_{j+1,i-b}, S_{j,i}]\).

3.3 The Algorithm

We restrict ourselves to trinomial branching. Iterating backwards, at each node on the lattice at a particular time step we find three values for \( u \), conditional on whether \( u \geq S_{j,i} \), \( u < S_{j+1,i-1} \), or \( u \in [S_{j+1,i-1}, S_{j,i}] \). In the last two cases for each candidate value \( \tilde{u}_{j,i} \) we compute the option value \( \check{c}_{j,i} (\tilde{u}_{j,i}) \)

\[
\check{c}_{j,i} (\tilde{u}_{j,i}) = e^{-r \Delta t} \sum_{k \in B} k \left( F_{j,j+1}^{i,i+k} (\tilde{u}_{j,i}) H (\tilde{u}_{j,i}) + \left( 1 - F_{j,j+1}^{i,i+k} (\tilde{u}_{j,i}) \right) \hat{c}_{j+1,i+k} \right).
\]

(29)

In the first case we set \( \check{c}_{j,i} (\tilde{u}_{j,i}) = X - S_{j,i} \). Set the option value to be the maximum of these values.

If at some node \( S_{j,i} \) the first case applies and it is optimal to exercise immediately, then at all nodes \((j,l)\) for \( l < i \) it will also be optimal to exercise immediately. Similarly if at \((j,i)\) the second case applies then at all nodes \((j,l)\) for \( l > i \) the second case will also apply. If at \((j,i)\) the third case holds for some index \( k \) then at node \((j,i+1)\) either the second case applies or else the third case applies for some \( k \leq k \).

In practice it is unnecessary to compute the barrier level at every node. For nodes away from the barrier the probability of hitting the barrier is very small.
Kuan and Webber (2002) found it unnecessary to compute hitting probabilities at a distance of more than ten steps away from the barrier. Here we found it necessary only to compute case three for up to three steps away from the exercise barrier.

4 The American Dirichlet Lattice

For a geometric Brownian motion $S = (S_t)_{t \geq 0}$ its hitting distribution to a constant barrier is known, as is its defective density conditional upon not hitting the barrier. We use these to construct branching probabilities to obtain a modified Dirichlet lattice that we refer to as the American Dirichlet lattice.

As before we construct a lattice for a Wiener process $z_t$. Suppose we are given a barrier level $u$. At a node $(j, i)$ with $z_{j,i} > u$, we compute probabilities $p_h(u)$, the probability of hitting the barrier, and $p_k(u)$, $k \in B$, the defective branching probabilities. We then set

$$c_{j,i}(u) = e^{-r \Delta t} \left[ p_h(u) H(u) + \sum_{k \in B} p_k(u) \tilde{c}_{j+1,i+k} \right].$$

(30)

The optimal barrier level is $\hat{u}_j = \arg \max_{u \in [0, z_{j,i})} c_{j,i}(u)$ and the option value is $\hat{c}_{j,i} = c_{j,i}(\hat{u}_j)$.

At each node, instead of finding the $u$ that maximizes $c_{j,i}$ by optimization, we compute directly values of $c_{j,i}$ for a set of pre-determined values $u$, selecting for $u$ that value that gives the greatest value of $c_{j,i}$. We chose $u \in U = \{-i \Delta u\}_{i=1}^{N_u}$ for $\Delta u = \frac{1}{2} \Delta z$ and $N_u = 8$.

Branching probabilities are computed by moment matching. For a geometric Brownian motion $S = (S_t)_{t \geq 0}$ with barrier level $u < S_t$ and for $s \geq t$ set $W_s = \ln \left( \frac{S_s}{S_t} \right)$ so that $W_s = (r - \frac{1}{2} \sigma^2) (s - t) + \sigma z_{s-t}$ for a Wiener process $z$. $W_s$ is a Brownian motion with drift $\mu = r - \frac{1}{2} \sigma^2$ and volatility $\sigma$. Let $f(w, t_j + \Delta t \mid w_{j,i}, t_j; u)$ be the defective density of a Brownian motion with drift $\mu$, with a barrier $u$ so that

$$\Pr \left[ W_T \leq w, \tau_u^W > u \mid W_t = 0 \right] = \int_0^w f(w, t_j + \Delta t \mid 0, t_j; u) \, dw$$

(31)

(for $z_t = z_{j,i}$ we have $w_{j,i} = 0$). A formula for $f$ is given in the appendix.

We have

$$\Pr \left[ S_T \leq S, \tau_u > T \mid S_t \right] = \Pr \left[ W_T \leq w, \tau_u^W > \hat{u} \mid W_t = 0 \right]$$

(32)

where $w = \ln \left( \frac{S}{S_t} \right)$, $\hat{u} = \ln \left( \frac{u}{S_t} \right)$ and $\tau_u^W$ is the hitting time of $W_s$ to $\hat{u}$. The
first three defective moments for \( z_{t,j + \Delta t} \) are given by

\[
e(u) = \mathbb{E} \left[ z_{t,j + \Delta t} 1_{\{z_{t,j + \Delta t} > z_{j,i}\}} \mid z_{j,i} \right] = \sigma \int_{u_z}^{\infty} z f (\sigma z + \mu \Delta t, t_j + \Delta t \mid w_{j,i}, t_j; u) \, dz,
\]

(33)

\[
v(u) = \mathbb{E} \left[ z_{t,j + \Delta t}^2 1_{\{z_{t,j + \Delta t} > z_{j,i}\}} \mid z_{j,i} \right] = \sigma \int_{u_z}^{\infty} z^2 f (\sigma z + \mu \Delta t, t_j + \Delta t \mid w_{j,i}, t_j; u) \, dz,
\]

(34)

\[
s(u) = \mathbb{E} \left[ z_{t,j + \Delta t}^3 1_{\{z_{t,j + \Delta t} > z_{j,i}\}} \mid z_{j,i} \right] = \sigma \int_{u_z}^{\infty} z^3 f (\sigma z + \mu \Delta t, t_j + \Delta t \mid w_{j,i}, t_j; u) \, dz.
\]

(35)

where \( u_z = \frac{u - \mu \Delta t}{\sigma} \).

\( e(u), v(u) \) and \( s(u) \) are found by numerical integration of equations (33), (34) and (35). We found that a straightforward Simpson’s rule integration with 501 intervals over the range \([u, z_{\text{max}}]\), for \( z_{\text{max}} = 8 \Delta z \), gave sufficient accuracy. Note that \( e(u), v(u) \) and \( s(u) \) can be pre-computed. Their values depend only on the relative position of \( u \) with respect to \( z_{j,i} \), so their values are the same for every node in the lattice.

Equation (30) values a deferred nominated barrier option. In fact the value of an option that gives a rebate at the moment the barrier is hit is known (a formula is given in the appendix). However substituting this into equation (30), replacing the \( p_h(u) H(u) \) term, leads to an insignificant improvement in accuracy, at the expense of a sizable increase in computation time.

### 4.1 Computing probabilities: matching two moments

At each barrier level there are four probabilities to compute. For a given (relative) barrier level, every node has the same branching probabilities, which can therefore be pre-computed.

For a given barrier level \( u < 0 \) the four probabilities are the probability of hitting the barrier, \( p_h(u) \), and the up, middle and down defective probabilities, \( p_1(u), p_0(u), p_{-1}(u) \).

We immediately set

\[
p_h(u) = N \left( \frac{u - \mu \Delta t}{\sigma \sqrt{\Delta t}} \right) + \exp \left( \frac{2u\mu}{\sigma^2} \right) N \left( \frac{u + \mu \Delta t}{\sigma \sqrt{\Delta t}} \right).
\]

(36)

We find the other probabilities by moment matching. When \( u < -\Delta z \) we require them to satisfy

\[
\begin{align*}
p_1(u) + p_0(u) + p_{-1}(u) &= 1 - p_h(u), \\
p_1(u) \Delta z + p_{-1}(u) (-\Delta z) &= e(u), \\
p_1(u) (\Delta z)^2 + p_{-1}(u) (-\Delta z)^2 &= v(u).
\end{align*}
\]

(37)
Set $e'(u) = \frac{e(u)}{\Delta z^2}$ and $v'(u) = \frac{v(u)}{(\Delta z)^2}$. Then

\[ p_1(u) = \frac{1}{2} (v'(u) + e'(u)) \quad (38) \]
\[ p_0(u) = 1 - v'(u) - p_h(u) \quad (39) \]
\[ p_{-1}(u) = \frac{1}{2} (v'(u) - e'(u)) \quad (40) \]

When $0 > u > -\Delta z$ then set $p_{-1}(u) = 0$ and

\[ \frac{p_1(u)}{p_1(u)\Delta z} + p_0(u) = 1 - p_h(u), \quad (41) \]

so that

\[ p_1(u) = e'(u), \quad (42) \]
\[ p_0(u) = 1 - e'(u) - p_h(u). \quad (43) \]

### 4.2 Incorporating a third moment

Heston and Zhou (2000) determined that if branching probabilities on a lattice matches the first $q$ moments of the underlying stochastic process then the lattice may converge at a rate up to order $-\frac{2}{q+1}$. For our numerical work we match to three moments where possible.

To match the third moment we require a fourth branch, from $z_0$ to $z_0 + 2\Delta z$. Write $p_2(u)$ for the probability of this branching. Then when $u < -\Delta w$ we choose $p_2(u)$, $p_1(u)$, $p_0(u)$, and $p_{-1}(u)$ to satisfy

\[
\begin{align*}
\frac{p_2(u)}{p_2(u)2\Delta z} + \frac{p_1(u)}{p_1(u)\Delta z} + p_0(u) + p_{-1}(u) &= 1 - p_h(u), \\
\frac{p_2(u)(2\Delta z)^2}{p_2(u)(2\Delta z)^3} + \frac{p_1(u)(\Delta z)^2}{p_1(u)(\Delta z)^3} + p_0(u) &= v(u), \\
\frac{p_2(u)(2\Delta z)^3}{p_2(u)(2\Delta z)^3} + \frac{p_1(u)(\Delta z)^3}{p_1(u)(\Delta z)^3} + p_{-1}(u) &= s(u).
\end{align*}
\]

Set $s'(u) = \frac{s(u)}{(\Delta z)^3}$. Then we solve to find

\[ p_2(u) = \frac{1}{6} (s'(u) - e'(u)), \quad (45) \]
\[ p_1(u) = \frac{1}{2} (2e'(u) + v'(u) - s'(u)) \quad (46) \]
\[ p_0(u) = 1 - p_h(u) - p_2(u) - p_1(u) - p_{-1}(u), \quad (47) \]
\[ p_{-1}(u) = \frac{1}{6} (3v'(u) - 2e'(u) - s'(u)). \quad (48) \]

When $0 > u > -\Delta z$ we set $p_{-1}(u) = 0$ and solve

\[
\begin{align*}
\frac{p_2(u)}{p_2(u)2\Delta z} + \frac{p_1(u)}{p_1(u)\Delta z} + p_0(u) &= 1 - p_h(u), \\
\frac{p_2(u)(2\Delta z)^2}{p_2(u)(2\Delta z)^2} + p_1(u)(\Delta z)^2 &= v(u), \quad (49) \]
\]
with solution

\[ p_2 (u) = \frac{1}{2} (v' (u) - e' (u)), \quad (50) \]

\[ p_1 (u) = 2e' (u) - v' (u), \quad (51) \]

\[ p_0 (u) = 1 - p_h (u) - p_2 (u) - p_1 (u), \quad (52) \]

\[ p_{-1} (u) = 0. \quad (53) \]

In the second case, if it is not possible to find valid probabilities that match two moments we instead revert to binomial branching, setting \( p_2 (u) = 0 \) and solving

\[ \frac{p_1 (u)}{p_1 (u) \Delta z} + p_0 (u) = 1 - p_h (u), \quad (54) \]

as before.

We note that probabilities found for the American Dirichlet lattice are similar to those computed for the Dirichlet approach described in section 3.1. However the American Dirichlet probabilities exactly match the moments of the defective distribution and hitting probabilities.

5 Numerical Results

We use the American Dirichlet lattice to vanilla American put values. We consider two underlyings and two sets of options. The first underlying has high volatility with \( \sigma = 0.4 \); the second underlying has medium volatility with \( \sigma = 0.2 \). In both cases we set \( S_0 = 100 \). The first set of options matures in half a year with strikes of 98, 100, and 102, and we use \( r = 0.06 \); the second set of options matures in one year with strikes of 95, 100, and 105, and we set \( r = 0.05 \). We shall write, for instance, \((o3, u2)\) to denote the case of the third option valued with the the second underlying.

Benchmark values were obtained in two ways: the first using a PSOR Crank-Nicolson finite difference routine with 500 time steps and 5,000 space steps, with the space boundaries set at 2 and 5,000; the second using a heptanomial lattice with 10,000 time steps, truncated at 8 standard deviations from the mean. Table 1 gives the benchmark values. The top number is the lattice value, the bottom number the Crank-Nicolson value. These values are accurate to at most four decimal places.

The appendix gives tables and figures showing convergence for both the plain and American Dirichlet lattices. The tables give option values as the number of time steps increases from 100 to 5,000. The figures plot \( \ln (c_N) \) and \( \ln (p_N) \) (the log of the plain lattice value) against \( \ln (N) \). Slopes of these plots can be interpreted as rates of convergence.

\[ ^7 \text{Values found on the heptanomial lattice tend to be greater than the values found by Crank-Nicolson, but took about one tenth the time to compute. Figures in the appendix show the heptanomial lattice values.} \]
**Benchmark values**

<table>
<thead>
<tr>
<th>Underlyings</th>
<th>Underlyings</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top value: lattice</td>
<td>Bottom value: PDE</td>
</tr>
</tbody>
</table>

**Option 1:**
- \( r = 0.05 \)
- \( T = 0.5 \)
- \( X = 98 \)
- \( \sigma = 0.4 \)
- Value: 9.12288
- Value: 3.75928

**Option 2:**
- \( r = 0.05 \)
- \( T = 0.5 \)
- \( X = 100 \)
- \( \sigma = 0.2 \)
- Value: 10.14141
- Value: 4.65564

**Option 3:**
- \( r = 0.05 \)
- \( T = 0.5 \)
- \( X = 102 \)
- \( \sigma = 0.2 \)
- Value: 11.21794
- Value: 5.67524

**Option 4:**
- \( r = 0.06 \)
- \( T = 1 \)
- \( X = 95 \)
- \( \sigma = 0.4 \)
- Value: 10.81207
- Value: 3.77635

**Option 5:**
- \( r = 0.06 \)
- \( T = 1 \)
- \( X = 100 \)
- \( \sigma = 0.2 \)
- Value: 13.29563
- Value: 5.79887

**Option 6:**
- \( r = 0.06 \)
- \( T = 1 \)
- \( X = 105 \)
- \( \sigma = 0.2 \)
- Value: 16.04444
- Value: 8.41660

**Table 1:** Benchmark values for vanilla American options

Times are not reported. The American Dirichlet lattice runs more slowly than the plain lattice, but not significantly so. This is firstly because branching probabilities on the American Dirichlet lattice can be pre-computed and so do not contribute to individual run times, and secondly because Dirichlet branching need only be used close to the exercise boundary. Away from the boundary the option is either surely exercised (with \( u_j \) set to \( S_{j,i} \)) or never exercised (with \( u_j \) set to 0). We found that, for reasonable values of \( N \), Dirichlet branching had a significant influence only within three space steps of the exercise boundary (a not unexpected result).\(^8\)

For both the plain and American Dirichlet lattices option values converge non-uniformly toward the benchmark values, sometimes with rapid oscillations ((o1, u2), (o4, u2)) and sometimes with slow ((o1, u1), (o3, u2)).

In cases where oscillation is slow, and convergence is initially uniform for small values of \( N \), one may investigate rates of convergence for the methods. If \( c_N \) is converging uniformly towards \( c_\infty \) at a rate \( M \) then \( c_N - c_\infty = O(N^{-M}) \), so that \( \frac{d \ln(c_N - c_\infty)}{d \ln(N)} \sim -M \). This corresponds to the slope of the \( \ln - \ln \) plots given in the figures. We see from the figures, for instance for cases (o2, u1), (o3, u2) and (o4, u1), that the American Dirichlet lattice is converging faster than the plain lattice, although usually with greater error.

The American Dirichlet value is always greater than the plain value. In cases where the plain value is biased low, tending to converge upwards towards the benchmark value, the advantage of the American Dirichlet lattice is clear ((o5, u2), (o6, u2)). In other cases both methods appear to have similar convergence properties, with the American Dirichlet value biased upwards by as much as the

\(^8\) Generally, with \( N \) taking sizes reported in the tables, the plain and American Dirichlet lattices exercised at the same levels in the lattice.
plain values are biased downwards \((o_2, u_2), (o_4, u_1)\). In other cases the plain lattice is converging faster \((o_1, u_1), (o_4, u_2)\).

We make this more formal. Table 2 displays some error analysis results. It computes the summed absolute error for time steps \(N = 1,000\) to \(N = 5,000\), for values given in the tables; 
\[
\text{error} = \sum_{N=1000}^{N=5000} |c_N - b|,
\]
where \(c_N\) is the tabulated value and \(b\) is the benchmark value. In the table, numbers in bold are significantly smaller than their paired counterparts.

In five cases \((o_2, u_1), (o_5, u_1), (o_6, u_1), (o_5, u_2), (o_6, u_2)\) the American Dirichlet lattice has significantly smaller error that the plain lattice. In four cases the there is little to distinguish the two methods. In three cases \((o_1, u_1), (o_3, u_1), (o_4, u_2)\) the plain method seems to be converging with less error. In three-quarters of the cases the American Dirichlet lattice is doing no worse than the plain lattice.

Although by no means conclusive, this analysis indicates that the American Dirichlet lattice may have better convergence properties than the plain lattice.

### 6 Conclusions

We have presented a new lattice method to value American options, the American Dirichlet lattice. It is based on the valuation of a nominated barrier option. The method is slightly slower than a plain lattice, but not significantly so. It has greater initial bias than the plain lattice, but may have superior convergence properties once the number of time steps is sufficient for the bias to have been removed.

We applied the American Dirichlet lattice to value vanilla American puts, but it is more generally applicable. In other cases its convergence advantage
over the plain lattice method may be more clear cut.
References


### Appendix: Tables and Figures

<table>
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<th>N_steps</th>
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Table 3: Dirichlet and plain values for option 1
Figure 1: Option 1, underlying 1: log-log convergence

Figure 2: Option 1, underlying 2: log-log convergence
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Table 4: Dirichlet and plain values for option 2
Figure 3: Option 2, underlying 1: log-log convergence

Figure 4: Option 2, underlying 2: log-log convergence
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Table 5: Dirichlet and plain values for option 3
Figure 5: Option 3, underlying 1: log-log convergence

Figure 6: Option 3, underlying 2: log-log convergence
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Table 6: Dirichlet and plain values for option 4
Figure 7: Option 4, underlying 1: log-log convergence

Figure 8: Option 4, underlying 2: log-log convergence
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Table 7: Dirichlet and plain values for option 5
Figure 9: Option 5, underlying 1: log-log convergence

Figure 10: Option 5, underlying 2: log-log convergence
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Table 8: Dirichlet and plain values for option 6
Figure 11: Option 6, underlying 1: log-log convergence

Figure 12: Option 6, underlying 2: log-log convergence
8 Appendix: An optimization procedure for $u$

To proceed set $u = S_{j+1,i+t} \exp \left( \frac{\sigma \varepsilon \sqrt{\kappa \Delta t}}{2} \right)$ for some $0 \leq \varepsilon < 1$. Then for $k \in B_t$ the distribution function of the conditional minimum can be rewritten as

$$F_{j,j+1}^{i,i+k}(u) = \exp \left( -\frac{2}{\sigma^2 \Delta t} \ln \left( \frac{u}{S_{j,i}} \right) \ln \left( \frac{u}{S_{j,j+1,i+k}} \right) \right)$$

$$= \left( \frac{u}{S_{j,j}} \right)^{-\frac{2n}{\sigma^2 \Delta t} (1-k+\varepsilon)}$$

and the ratio of distribution functions in equation (20) as

$$\frac{F_{j,j+1}^{i,i+q}(u)}{F_{j,j+1}^{i,i+k}(u)} = \left( \frac{u}{S_{j,i}} \right)^{-\frac{2n}{\sigma^2 \Delta t} (k-q)}$$

Set $v = \left( \frac{u}{S_{j,i}} \right)^{-\frac{2n}{\sigma^2 \Delta t}}$ then $F_{j,j+1}^{i,i+k}(u) = v^{1-k+\varepsilon}$ and $\frac{F_{j,j+1}^{i,i+k}(u)}{F_{j,j+1}^{i,i+k}(u)} = v^{k-q}$.

Also

$$\frac{\partial f_{j,j+1}^{i,i+k}}{\partial u} = -\frac{2}{u \sigma^2 \Delta t} \left( \left( r - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma (2l - k + 2\varepsilon) \sqrt{\kappa \Delta t} \right)$$

$$= \frac{A}{u} + \frac{B}{u} (k - 2\varepsilon)$$

where

$$A = -\frac{2}{\sigma^2 \Delta t} \left( \left( r - \frac{1}{2} \sigma^2 \right) \Delta t + 2\sigma l \sqrt{\kappa \Delta t} \right)$$

$$B = \frac{2}{\sigma \Delta t} \sqrt{\kappa \Delta t}.$$

Restricting ourselves to trinomial branching, so $l = -1$, $\kappa = 3$ and $k = 0, 1$, and given $\frac{\partial H(u)}{\partial u} = -1$ equation (28) becomes

$$p_1 F_{j,j+1}^{i,i+1}(u) \frac{\partial f_{j,j+1}^{i,i+1}}{\partial u} (H(u) - \hat{c}_{j+1,i+1}) + p_0 F_{j,j+1}^{i,i+0}(u) \frac{\partial f_{j,j+1}^{i,i+0}}{\partial u} (H(u) - \hat{c}_{j+1,i+0}) = p_1 F_{j,j+1}^{i,i+1}(u) + p_0 F_{j,j+1}^{i,i+0}(u) + p_{-1}.$$  \hspace{1cm} (60)

Hence, substituting in for $v$, $F_{j,j+1}^{i,i+k}(u)$, $\frac{F_{j,j+1}^{i,i+q}}{F_{j,j+1}^{i,i+k}}(u)$, and $\frac{\partial f_{j,j+1}^{i,i+k}}{\partial u}$ we get

$$u = \frac{A + B (1 - 2\varepsilon)}{1 + \frac{p_1}{p_0} v^{-\lambda} + \frac{p_1}{p_0} v^{-2} - \varepsilon} + \frac{A - 2B \varepsilon}{1 + \frac{p_1}{p_0} v^{-1} + \frac{p_1}{p_0} v^{-1} \varepsilon}$$

$$\hspace{1cm} (61)$$

But $p_1 = p_{-1}$, $u = S_{j,i} v^{-\lambda}$ where $\lambda = \frac{\sigma \sqrt{\kappa \Delta t}}{2\varepsilon}$ and $H(u) = X - u$ (note that $\lambda = 1/B$) so
\[ S_{j,i}v^{-\lambda} \left( 1 + \frac{p_0}{p_1}v + v^{2-\varepsilon} \right) = (A + B (1 - 2\varepsilon)) \left( X - S_{j,i}v^{-\lambda} - \tilde{e}_{j+1,i+1} \right) \]
\[ + \frac{p_0}{p_1}v (A - 2B\varepsilon) \left( X - S_{j,i}v^{-\lambda} - \tilde{e}_{j+1,i+0} \right) \] (62)

Moreover
\[ \varepsilon = \frac{1}{\sigma \sqrt{\kappa \Delta t}} \ln \left( \frac{S_{j,i}}{S_{j+1,i+1}} \right) - \frac{1}{2\kappa} \ln v \]
\[ = C - \frac{1}{2\kappa} \ln v \]

where \( C = \frac{1}{\sigma \sqrt{\kappa \Delta t}} \ln \left( \frac{S_{j,i}}{S_{j+1,i+1}} \right) \). Putting this result into equation (62) we get a nonlinear equation in \( v = \left( \frac{u}{S_{j,i}} \right)^{-\frac{1}{\sigma \sqrt{\kappa \Delta t}}} \) that can be solved, for instance, using Newton’s method. This obtains possible values for \( u \) that satisfy the first order condition. To be an optimal exercise policy \( u \) must also satisfy the second order condition and be within the range of allowed values of \( u \). Any such \( u \) becomes a candidate for the optimal exercise level.
9 Appendix: Hitting times and defective densities

The distribution of hitting times of a geometric Brownian motion

Given its value $S_t$ at time $t$, the probability of the process hitting a barrier at level $u < S_t$ before time $T$ is

$$\Pr[\tau_u \leq T] = N(-h'_2) + \left(\frac{u}{S_t}\right) \frac{2}{\pi^2(r-\frac{1}{2}\sigma^2)} N(h_2)$$

(63)

where

$$h'_2 = \frac{1}{\sigma \sqrt{T-t}} \ln \left(\frac{S_t}{ue^{-r(T-t)}}\right) - \frac{1}{2} \sigma \sqrt{T-t},$$

(64)

and

$$h_2 = \frac{1}{\sigma \sqrt{T-t}} \ln \left(\frac{u}{S_te^{-r(T-t)}}\right) - \frac{1}{2} \sigma \sqrt{T-t}.$$  

(65)

Hence the value $R(u)$ at time $t$ of a payoff of $(X-u)_+$ received at time $T$ if the barrier is hit is

$$R(u) = e^{-r(T-t)} (X-u)_+ \left(N(-h'_2) + \left(\frac{u}{S_t}\right) \frac{2}{\pi^2(r-\frac{1}{2}\sigma^2)} N(h_2)\right).$$

(66)

We use this formula to compute payoffs for the deferred nominated barrier option. For the nominated barrier option, not deferred, there is also an exact formula. The value $\hat{R}(u)$ at time $t$ of a payoff of $(X-u)_+$ received at the time the barrier is hit is (Suo and Wang, p20)

$$\hat{R}(u) = (X-u)_+ \int_t^T e^{-rs} dPr[T_s \leq T]$$

(67)

$$= (X-u)_+ \left[\left(\frac{u}{S_t}\right)^{\frac{\mu+\nu}{\sigma^2}} N(e'_2) + \left(\frac{u}{S_t}\right)^{\frac{\mu-\nu}{\sigma^2}} N(e_2)\right]$$

(68)

where $N$ is the standard normal distribution function and

$$e_2 = \frac{1}{\sigma \sqrt{T-t}} \ln \left(\frac{u}{S_t}\right) + \frac{\nu}{\sigma} \sqrt{T-t}$$

(69)

$$e'_2 = e_2 - 2\frac{\nu}{\sigma} \sqrt{T-t}$$

(70)

with $\mu = r - \frac{1}{2} \sigma^2$ and $\nu = \sqrt{(r-\frac{1}{2} \sigma^2)^2 + 2r}. $

The defective density of a geometric Brownian motion $\Pr[S_T \leq S, \tau_u > T \mid S_t]$ is the defective density at time $T > t$ for a geometric Brownian motion $S =$
(S_t)_{t \geq 0}$ and barrier level $u < S_t$. For $s \geq t$ set $W_s = \ln \left( \frac{S_t}{S_s} \right)$ so that $W_s = (r - \frac{1}{2} \sigma^2) s + \sigma z_s$ for a Wiener process $z_s$. We have

$$
\Pr [S_T \leq S, \tau_u > T | S_t] = \Pr [W_T \leq w, \tau_u^W > \hat{u} | W_t]
$$

(71)

where $w = \ln \left( \frac{S}{S_t} \right)$, $\hat{u} = \ln \left( \frac{u}{S_t} \right)$ and $\tau_u^W$ is the hitting of $W_s$ to $\hat{u}$. The defective density $f (w, T | w_t, t; \hat{u})$ of a Brownian motion with drift is known. Set $\mu = r - \frac{1}{2} \sigma^2$. Then

$$
f (w, T | w_t, t; \hat{u}) = d \Pr [W_T \leq w, \tau_u^W > \hat{u} | W_t]
$$

(72)

$$
= \frac{1}{\sigma \sqrt{T-t}} \left\{ n (e_1) - \exp \left( - \frac{2 (w_t - \hat{u}) \mu}{\sigma^2} \right) n (e_2) \right\}
$$

(73)

where $n$ is the standard normal density function and

$$
e_1 = \frac{w - w_t - \mu (T-t)}{\sigma \sqrt{T-t}},
$$

(74)

$$
e_2 = \frac{w + w_t - 2 \hat{u} - \mu (T-t)}{\sigma \sqrt{T-t}}.
$$

(75)