

The κ -word problem over DRH

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Abstract

Profinite techniques are explored in order to prove decidability of a word problem over a family of pseudovarieties of semigroups, which is parameterized by pseudovarieties of groups.

Let κ be the signature that naturally generalizes the usual signature on groups: it consists of the multiplication, and of the $(\omega - 1)$ -power. Given a pseudovariety of groups H , we denote by DRH the pseudovariety all finite semigroups whose regular \mathcal{R} -classes lie in H . We prove that the word problem for κ -terms is decidable over DRH provided it is decidable over H (in general, the word problem for κ -terms is said to be decidable over a pseudovariety V if it is decidable whether two κ -terms define the same element in every semigroup of V). Further, we present a canonical form for elements in the free κ -semigroup over DRH, based on the knowledge of a canonical form for elements in the free κ -semigroup over H . This extends work of Almeida and Zeitoun on the pseudovariety of all finite \mathcal{R} -trivial semigroups.

Keywords: word problem, kappa-word, \mathcal{R} -class, pseudovariety, free profinite semigroup, canonical form.

1. Introduction

A *pseudovariety* is a class of finite semigroups that is closed under taking binary products, subsemigroups, and homomorphic images. Since Eilenberg's correspondence was formulated in 1976 [1, Chapter VII, Theorem 3.4s] such classes play an important role in the study of rational languages. He showed that pseudovarieties are in a bijective correspondence with varieties of rational languages. In turn, the study of the latter is strongly motivated by its application in Computer Science, namely, via Automata Theory [2, 3].

Amongst the decision problems usually considered for classes of algebraic structures, the *identity problem* (which is the *word problem* for relatively free structures and which we call the *word problem* for the class of algebraic structures in question) and the *membership problem* are of great relevance (for a survey, see [4]).

While the former consists in deciding whether two terms (over a fixed signature) have the same interpretation in every algebra from the class, the latter asks whether a given algebra belongs to the given class. In the case where the class considered is a pseudovariety, the two problems are closely related as witnessed by the work of Albert, Baldinger, and Rhodes [5]: they used undecidability of a certain word problem in order to prove that decidability of the membership problem for pseudovarieties is not preserved under taking joins of pseudovarieties.

Besides being an interesting problem in itself, the word problem for pseudovarieties also appears as one of the ingredients of a stronger property of pseudovarieties named *tameness* [6]. As mentioned above, decidability of the membership problem is not preserved by taking joins of pseudovarieties. It turns out that this is not the unique relevant operator with this property. Similar results are known, for instance, with respect to the (two-sided) semidirect product, and the Mal'cev product [5, 7, 8]. When Almeida was looking for properties on pseudovarieties that are preserved by taking semidirect products, he came up with the notion of *hyperdecidability* [9], which seemed natural at the light of his and Weil's Pseudoidentity

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Basis Theorem [10, Theorem 5.3].² However, hyperdecidability is a decision problem that is very hard to solve in general. That sets the tone to consider the property of being *tame*, which although stronger than hyperdecidability, is in general easier to obtain.

When proving tameness, the main decision component consists in solving a word problem (the other crucial component of tameness is called *reducibility*; see [12] for an overview of related concepts). The word problem involved may be described as follows: we fix a pseudovariety \mathbf{V} and suppose that σ is a recursively enumerable signature with a computable interpretation in each semigroup of \mathbf{V} , which commutes with homomorphisms between elements of \mathbf{V} . Then, solving the σ -word problem over \mathbf{V} consists in deciding whether two given σ -terms have the same interpretation in every semigroup of \mathbf{V} . Note that this is equivalent to solving the word problem over the free σ -semigroup over \mathbf{V} [13]. In this paper we shall consider the word problem with respect to the *canonical signature* κ , which consists of the *multiplication* and of the $(\omega - 1)$ -power. This is the most natural generalization of the usual signature on groups, since the $(\omega - 1)$ -power is interpreted as inversion on finite groups.

On the other hand, one of the pseudovarieties shown to be of great relevance is \mathbf{R} , which consists of all finite semigroups whose regular \mathcal{R} -classes are trivial. It appears naturally in different contexts (see, for instance, [14, 15, 16, 17]), and has been the focus of many works (other examples are present in [18, 19, 20, 21]). In turn, a natural generalization of \mathbf{R} is found in the pseudovarieties of the form \mathbf{DRH} for a pseudovariety of groups \mathbf{H} . This class contains all finite semigroups whose regular \mathcal{R} -classes are groups from \mathbf{H} . Observe that, when \mathbf{H} is the trivial pseudovariety, the pseudovariety \mathbf{DRH} is precisely \mathbf{R} .

Also, the pseudovarieties \mathbf{DRH} may be seen as a specialization of the pseudovariety \mathbf{DS} , of all finite semigroups whose regular \mathcal{D} -classes are subsemigroups. The interest in the latter has been pointed out by Schützenberger in [22], where he characterizes the varieties of rational languages corresponding to some subpseudovarieties of \mathbf{DS} under Eilenberg’s correspondence, among which \mathbf{DRH} . Later, Almeida and Weil [18] described the structure of free pro- \mathbf{DRH} semigroups. Several other authors have shown interest in better understanding \mathbf{DS} [23, 24, 25]. Furthermore, the results found in the literature suggest that the investigation of subpseudovarieties of \mathbf{DS} may lead to a better understanding of \mathbf{DS} itself [26, 27, 17], and thus, \mathbf{DRH} is an interesting instance.

We mention some related work. In 2007, Almeida and Zeitoun [28] solved the κ -word problem over the pseudovariety \mathbf{R} . Their methods have been adapted by Moura [29] to the pseudovariety \mathbf{DA} , consisting of all finite semigroups whose regular \mathcal{D} -classes are aperiodic subsemigroups. In this paper, we solve the same problem for some of the pseudovarieties of the form \mathbf{DRH} . The only condition we impose on \mathbf{H} is quite reasonable: we require that it have a decidable κ -word problem. Further, it is expected that, combining Moura’s work with our own, the same approach may be extended to $\mathbf{DO} \cap \overline{\mathbf{H}}$, that is, to the pseudovariety of all finite semigroups whose regular \mathcal{D} -classes are orthodox semigroups and whose subgroups lie in \mathbf{H} . The pseudovariety $\mathbf{DO} \cap \overline{\mathbf{H}}$ may be considered as the simplest non aperiodic version of \mathbf{DA} . The key ingredient in these approaches is the study of certain factors of pseudowords over the considered pseudovariety. Such factors are obtained by taking successive refinements of factorizations of a given pseudoword. When it concerns a pseudovariety \mathbf{DRH} , each factor obtained in this way is characterized both by its projection onto \mathbf{R} and a component over \mathbf{H} . This somehow explains why we were not able to use the results on \mathbf{R} , but we had to extend them instead.

This paper is organized as follows. Section 2, of preliminaries, is divided into four subsections: in the first we set up the general notation; we recall some aspects related with theory of profinite semigroups in the second; we describe the κ -word problem in the third; and we reserve the fourth to the statement of some general facts on the structure of free pro- \mathbf{DRH} semigroups. In Section 3 we introduce \mathbf{DRH} -automata, which are a generalization of \mathbf{R} -automata defined in [28], and are useful to represent the \mathcal{R} -classes of free pro- \mathbf{DRH} semigroups. We devote Section 4 to the presentation of a canonical form for κ -words over \mathbf{DRH} assuming the knowledge of a canonical form for κ -words over \mathbf{H} . Section 5 presents a number of technical results that are instrumental in Section 6, in which we describe an algorithm to solve the κ -word problem over \mathbf{DRH} . Finally, in Section 7 we apply our results to the particular case of the pseudovariety \mathbf{DRG} .

²It was later realized that the referred theorem has a gap in its proof. Although it is not known so far the full generality of the result, it remains valid in some relevant cases (see also [11, Section 3.7]).

2. Preliminaries

We assume that the reader is familiar with pseudovarieties, (pro)finite semigroups, and the basics of topology. For further reading we refer to [30, 13, 31]. Some knowledge of automata theory may be useful, although no use of deep results is made. For this topic, we refer to [2]. A study of pseudovarieties of the form DRH may be found in [18].

2.1. Notation

Given a semigroup S , we let S^I represent the monoid obtained by adjoining an identity to S (even if S is already a monoid). If s_1, \dots, s_n are elements in S , then $\prod_{i=1}^n s_i$ denotes the product $s_1 \cdots s_n$. An infinite sequence $(s_i)_{i \geq 1} \subseteq S$ defines the *infinite product* $(\prod_{i=1}^n s_i)_{n \geq 1}$.

The free semigroup (respectively, monoid) on a (possibly infinite) set B is denoted B^+ (respectively, B^*). Elements in B^* are called *words*. The *empty word* of B^* is the identity element ε . The *length* of a word $u \in B^*$ is $|u| = 0$ if $u = \varepsilon$, and $|u| = n$ if $u = b_1 \cdots b_n$, for certain $b_1, \dots, b_n \in B$. The free group on B is denoted FG_B , and we denote by B^{-1} the set $\{b^{-1} : b \in B\}$ disjoint from B , where b^{-1} represents the inverse of b in FG_B .

We say that a set of symbols is an *alphabet*. Generic finite alphabets are denoted A , while $\Sigma = \{0, 1\}$ is a fixed two-element alphabet.

Let $\mathcal{A} = (V, \rightarrow, \mathbf{v}^0, F)$ be a *deterministic automaton* over an alphabet A (where V is the set of *states*, \rightarrow is the *transition function*, and $\{\mathbf{v}^0\}$ and F are the sets of *initial* and *terminal states*, respectively). We write transitions in \mathcal{A} as $\mathbf{v} \xrightarrow{a} \mathbf{v}.a$, for $\mathbf{v} \in V$ and $a \in A^*$. Given a state $\mathbf{v} \in V$, we denote by $\mathcal{A}_{\mathbf{v}}$ the sub-automaton of \mathcal{A} *rooted at* \mathbf{v} , that is, the (deterministic) automaton $(\mathbf{v}.A^*, \rightarrow|_{\mathbf{v}.A^*}, \mathbf{v}, F \cap (\mathbf{v}.A^*))$.

The symbols \mathcal{R} , \mathcal{H} , and \mathcal{D} denote some of *Green's relations*. We reserve the letter \mathbf{H} to denote an arbitrary pseudovariety of groups, and DRH stands for the pseudovariety of all finite semigroups whose regular \mathcal{R} -classes belong to \mathbf{H} . Other pseudovarieties playing a role in this work are \mathbf{S} , the pseudovariety of all finite semigroups; \mathbf{G} , the pseudovariety of all finite groups; and \mathbf{R} , the pseudovariety of all finite semigroups with trivial \mathcal{R} -classes.

2.2. Profinite semigroups

Let \mathbf{V} be a pseudovariety of semigroups. We denote the *free A -generated pro- \mathbf{V} semigroup* by $\overline{\Omega}_A \mathbf{V}$. Elements of $\overline{\Omega}_A \mathbf{V}$ are called *pseudowords over \mathbf{V}* (or simply *pseudowords*, when $\mathbf{V} = \mathbf{S}$). Let $\iota : A \rightarrow \overline{\Omega}_A \mathbf{V}$ be the generating mapping of $\overline{\Omega}_A \mathbf{V}$. We point out that, unless \mathbf{V} is the trivial pseudovariety, ι is injective. For that reason, we often identify the alphabet A with its image under ι . With this assumption, we obtain that the free semigroup A^+ is a subsemigroup of $\overline{\Omega}_A \mathbf{V}$ and thus, it is coherent to say that $I \in (\overline{\Omega}_A \mathbf{V})^I$ is the *empty word/pseudoword*. On the other hand, if $B \subseteq A$, then we have an injective continuous homomorphism $\overline{\Omega}_B \mathbf{V} \rightarrow \overline{\Omega}_A \mathbf{V}$, induced by the inclusion map $B \rightarrow A$. So, we consider $\overline{\Omega}_B \mathbf{V}$ as a subsemigroup of $\overline{\Omega}_A \mathbf{V}$. In turn, if \mathbf{W} is a subpseudovariety of \mathbf{V} , then we denote by $\rho_{\mathbf{V}, \mathbf{W}}$ the natural projection from $\overline{\Omega}_A \mathbf{V}$ onto $\overline{\Omega}_A \mathbf{W}$. We shall write $\rho_{\mathbf{W}}$ when \mathbf{V} is clear from the context. Whenever the pseudovariety \mathbf{Sl} of all finite semilattices is contained in \mathbf{V} , we denote the projection $\rho_{\mathbf{Sl}} = \rho_{\mathbf{V}, \mathbf{Sl}}$ by c and call it the *content function*.

Finally, a *pseudoidentity over \mathbf{V}* (or simply *pseudoidentity*, when $\mathbf{V} = \mathbf{S}$) is a formal equality $u = v$, with $u, v \in \overline{\Omega}_A \mathbf{V}$. We say that a pseudoidentity $u = v$ *holds* in a pseudovariety $\mathbf{W} \subseteq \mathbf{V}$ if the interpretations of u and v coincide in every semigroup of \mathbf{W} . If that is the case, then we may say that u and v are *equal modulo \mathbf{W}* , and we write $u =_{\mathbf{W}} v$.

2.3. The κ -word problem

The *canonical implicit signature*, denoted κ , consists of two implicit operations: the *multiplication* $(\cdot \cdot)$, and the $(\omega - 1)$ -*power* $(\cdot)^{\omega-1}$. Each of these operations has a natural interpretation over a given profinite semigroup S : the multiplication sends each pair (s_1, s_2) to its product $s_1 s_2$, and the $(\omega - 1)$ -power sends each element s to the limit $\lim_{n \geq 1} s^{n!-1}$. We define κ -*terms* over an alphabet A inductively as follows:

- the empty word ε and each letter $a \in A$ are κ -terms;

- if u and v are κ -terms, then $(u \cdot v)$ and $(u^{\omega-1})$ are also κ -terms.

Of course, each nonempty κ -term may naturally be seen as representing an element of the free κ -semigroup $\Omega_A^\kappa \mathbf{S}$, and conversely, for each element of $\Omega_A^\kappa \mathbf{S}$ there is a (usually non-unique) κ -term representing it. We call κ -words the elements of $(\Omega_A^\kappa \mathbf{S})^I$.

Let ℓ be an integer. We may generalize the $(\omega - 1)$ -power by letting $x^{\omega+\ell} = \lim_{n \geq 1} x^{n+1+\ell}$. Then, for every $q \geq 1$, the expressions $(x^{\omega-1})^q$ and $x^{\omega-1}x^q$ represent κ -words (by u^q we mean q times the product of u), and the equalities $(x^{\omega-1})^q = x^{\omega-q}$ and $x^{\omega-1}x^q = x^{\omega+q-1}$ hold in $\Omega_A^\kappa \mathbf{S}$. It is usual to consider the extended implicit signature $\bar{\kappa}$ that contains the multiplication and all $(\omega + q)$ -powers (for an integer q). We define both $\bar{\kappa}$ -term and $\bar{\kappa}$ -word in the same fashion as we defined κ -term and κ -word, respectively. Clearly, κ -words are $\bar{\kappa}$ -words and conversely, but a $\bar{\kappa}$ -term may not be a κ -term.

Saying that the κ -word problem over a pseudovariety \mathbf{V} is decidable amounts to saying that there exists an algorithm determining whether the interpretation of two given κ -terms coincides in every semigroup of \mathbf{V} , that is, whether they define the same element of $\Omega_A^\kappa \mathbf{V}$. Although our goal is to solve the κ -word problem over DRH (under certain reasonable conditions on \mathbf{H}), it shall be useful to consider $\bar{\kappa}$ -terms instead of κ -terms in the intermediate steps.

The implicit signature $\bar{\kappa}$ enjoys a nice property that we state here for later reference.

Lemma 2.1 ([28, Lemma 2.2]). *Let u be a $\bar{\kappa}$ -term and let $u = u_\ell a u_r$ be a factorization of u such that $c(u) = c(u_\ell) \uplus \{a\}$. Then, u_ℓ and u_r are $\bar{\kappa}$ -terms.*

2.4. Structure of free pro-DRH semigroups

We start with a uniqueness result on factorization of pseudowords.

Proposition 2.2 ([28, Proposition 2.1]). *Let $x, y, z, t \in \bar{\Omega}_A \mathbf{S}$ and $a, b \in A$ be such that $xa y = z b t$. Suppose that $a \notin c(x)$ and $b \notin c(z)$. If either $c(x) = c(z)$ or $c(xa) = c(zb)$, then $x = z$, $a = b$, and $y = t$.*

This motivates the definition of *left basic factorization* of a pseudoword $u \in \bar{\Omega}_A \mathbf{S}$: it is the unique triple $\text{lbf}(u) = (u_\ell, a, u_r)$ of $(\bar{\Omega}_A \mathbf{S})^I \times A \times (\bar{\Omega}_A \mathbf{S})^I$ such that $u = u_\ell a u_r$, $a \notin c(u_\ell)$, and $c(u) = c(u_\ell a)$. The left basic factorization is also well defined over each pseudovariety DRH.

Proposition 2.3 ([18, Proposition 2.3.1]). *Every element $u \in \bar{\Omega}_A \text{DRH}$ admits a unique factorization of the form $u = u_\ell a u_r$ such that $a \notin c(u_\ell)$ and $c(u_\ell a) = c(u)$.*

Then, whenever $u \in \bar{\Omega}_A \text{DRH}$, we also say that the triple $\text{lbf}(u) = (u_\ell, a, u_r)$ described in Proposition 2.3 is the left basic factorization of u .

We may iterate the left basic factorization of a pseudoword u (or of a pseudoword u over DRH) as follows. Set $u'_0 = u$. For $k \geq 0$, if $u'_k \neq I$, then we let $(u_{k+1}, a_{k+1}, u'_{k+1})$ be the left basic factorization of u'_k . Since the contents $(c(u_k a_k))_{k \geq 1}$ form a decreasing sequence for inclusion, there exists an index k such that either $u'_k = I$ (in which case we set $\lceil u \rceil = k$) or, for all $m \geq k$, $c(u_k a_k) = c(u_m a_m)$ (setting then $\lceil u \rceil = +\infty$). The *cumulative content* of u is $\bar{c}(u) = \emptyset$ in the former case, and it is $\bar{c}(u) = c(u_k a_k)$ in the latter. In particular, Proposition 2.3 yields that the cumulative content of a pseudoword is completely determined by its projection onto $\bar{\Omega}_A \mathbf{R}$. We denote the factor $u_k a_k$ by $\text{lbf}_k(u)$, whenever it is defined and we write $\text{lbf}_\infty(u) = (u_1 a_1, \dots, u_k a_k, I, I, \dots)$ if $u'_k = I$, and $\text{lbf}_\infty(u) = (u_k a_k)_{k \geq 1}$ otherwise. We further define the *irregular* and *regular* parts of u , respectively denoted $\text{irr}(u)$ and $\text{reg}(u)$: if $\bar{c}(u) = \emptyset$, then $\text{irr}(u) = u$ and $\text{reg}(u) = I$; if $\bar{c}(u) = c(u'_k)$ and k is minimal for this equality, then $\text{irr}(u) = \text{lbf}_1(u) \cdots \text{lbf}_k(u)$ and $\text{reg}(u) = u'_k$. This terminology is explained by the following result.

Proposition 2.4 ([18, Corollary 6.1.5]). *Let $u \in \bar{\Omega}_A \text{DRH}$. Then, u is regular if and only if $c(u) = \bar{c}(u)$ (and, hence, $\text{reg}(u) = u$).*

Suppose that $\bar{c}(u) \neq \emptyset$. Since $\bar{\Omega}_A \mathbf{S}$ is a compact monoid, the infinite product $(\text{lbf}_1(u) \cdots \text{lbf}_k(u))_{k \geq 1}$ has an accumulation point, and it is not hard to see that any two of its accumulation points are \mathcal{R} -equivalent. Furthermore, if all the factors $\text{lbf}_k(u)$ have the same content, then the \mathcal{R} -class in which the accumulation

points lie is regular [18, Proposition 2.1.4]. On the other hand, the regular \mathcal{R} -classes of $\overline{\Omega}_A\text{DRH}$ are groups. Hence, in this case, we may define the *idempotent designated* by the infinite product $(\text{lbf}_1(u) \cdots \text{lbf}_k(u))_{k \geq 1}$ to be the identity of the group to which its accumulation points belong.

Together with Lemma 2.1, the next result is behind the properties of $\overline{\Omega}_A\text{DRH}$ that we use most often in the sequel. It consists of a particular case of [18, Proposition 5.1.2].

Proposition 2.5. *Let \mathbf{H} be a pseudovariety of groups. If e is an idempotent of $\overline{\Omega}_A\text{DRH}$ and if H_e is its \mathcal{H} -class, then letting $\psi_e(a) = eae$ for each $a \in c(e)$ defines a unique homeomorphism $\psi_e : \overline{\Omega}_{c(e)}\mathbf{H} \rightarrow H_e$ whose inverse is the restriction of $\rho_{\text{DRH},\mathbf{H}}$ to H_e .*

The following consequence is not hard to derive.

Corollary 2.6. *Let u be a pseudoword and $v, w \in (\overline{\Omega}_A\mathbf{S})^I$ be such that $c(v) \cup c(w) \subseteq \overline{c}(u)$ and $v =_{\mathbf{H}} w$. Then, the pseudovariety DRH satisfies $uv = uw$.*

We proceed with the statement of two known facts about DRH . We include the proofs for the sake of completeness.

Lemma 2.7. *Let u, v be pseudowords. Then, $\rho_{\text{DRH}}(u)$ and $\rho_{\text{DRH}}(v)$ lie in the same \mathcal{R} -class if and only if the pseudovariety DRH satisfies $\text{lbf}_\infty(u) = \text{lbf}_\infty(v)$.*

Proof. Suppose that $u \mathcal{R} v$ modulo DRH and let u_0 and v_0 be possibly empty pseudowords such that DRH satisfies $uu_0 = v$ and $u = vv_0$. This implies that DRH also satisfies $u = uu_0v_0$, and thus, by uniqueness of left basic factorization in $\overline{\Omega}_A\text{DRH}$, the equalities

$$\text{lbf}_\infty(u) = \text{lbf}_\infty(uu_0v_0) = \text{lbf}_\infty(vv_0) = \text{lbf}_\infty(v)$$

hold modulo DRH .

Conversely, let us assume that $\text{lbf}_\infty(u) =_{\text{DRH}} \text{lbf}_\infty(v)$. Then, we may choose accumulation points of $(\text{lbf}_1(u) \cdots \text{lbf}_k(u))_{k \geq 1}$ and of $(\text{lbf}_1(v) \cdots \text{lbf}_k(v))_{k \geq 1}$, say u' and v' , respectively, having the same value in DRH . Since the accumulation points of these sequences are \mathcal{R} -above u and v , respectively, there exist possibly empty pseudowords u_0 and v_0 such that $u = u'u_0$ and $v = v'_0v_0$. Clearly, we have $\text{lbf}_\infty(v) = \text{lbf}_\infty(v')$ and so, the inclusion $c(v_0) \subseteq \overline{c}(v')$ holds. Therefore, the following equalities are valid in DRH

$$u = u'u_0 = v'u_0 \stackrel{\text{Corollary 2.6}}{=} v'(v_0v_0^{\omega-1})u_0 = v(v_0^{\omega-1}u_0).$$

Hence, u is \mathcal{R} -below v modulo DRH . By symmetry, we also get that DRH satisfies $v \leq_{\mathcal{R}} u$. \square

Lemma 2.8. *Let $u, v \in \overline{\Omega}_A\mathbf{S}$ and $u_0, v_0 \in (\overline{\Omega}_A\mathbf{S})^I$ be such that $c(u_0) \subseteq \overline{c}(u)$ and $c(v_0) \subseteq \overline{c}(v)$. Then, the pseudovariety DRH satisfies $uu_0 = vv_0$ if and only if it satisfies $u \mathcal{R} v$ and if, in addition, the pseudovariety \mathbf{H} satisfies $uu_0 = vv_0$. In particular, by taking $u_0 = I = v_0$, we get that $u =_{\text{DRH}} v$ if and only if $u \mathcal{R} v$ modulo DRH and $u =_{\mathbf{H}} v$.*

Proof. Suppose that $uu_0 = vv_0$ modulo DRH . Since $c(u_0) \subseteq \overline{c}(u)$ and $c(v_0) \subseteq \overline{c}(v)$, it follows that $\text{lbf}_\infty(u) \mathcal{R} \text{lbf}_\infty(v)$. By Lemma 2.7, u and v are \mathcal{R} -equivalent modulo DRH .

Conversely, suppose that u and v are in the same \mathcal{R} -class modulo DRH and that \mathbf{H} satisfies $uu_0 = vv_0$. From the fact that $u \mathcal{R} v$ modulo DRH it follows the existence of a possibly empty pseudoword v'_0 such that DRH satisfies $u = vv'_0 \mathcal{R} v$, and so $c(v'_0) \subseteq \overline{c}(v)$. On the other hand, since the pseudoidentities $\{u = vv'_0, uu_0 = vv_0\}$ are valid in \mathbf{H} , it follows that \mathbf{H} satisfies $v'_0u_0 = v_0$. Therefore, Corollary 2.20 may be used to conclude that DRH satisfies $uu_0 = v(v'_0u_0) = vv_0$ as desired. \square

3. DRH -automata

The goal of this section is to characterize equality of pseudowords over DRH using certain kinds of automata—the so-called DRH -automata—and equalities over \mathbf{H} . We define an equivalence relation \sim on

the class of DRH-automata in such a way that the equivalence classes of DRH-automata are in one-to-one correspondence with the \mathcal{R} -classes of pseudowords over DRH (see Theorem 3.8).

We start by introducing the notion of a DRH-automaton.

An A -labeled DRH-automaton is a tuple $\mathcal{A} = \langle V, \rightarrow, v^0, F, \lambda_{\mathbf{H}}, \lambda \rangle$, where $\langle V, \rightarrow, v^0, F \rangle$ is a nonempty deterministic trim automaton over Σ , and $\lambda_{\mathbf{H}} : V \rightarrow (\overline{\Omega}_A \mathbf{H})^I$ and $\lambda : V \rightarrow A \uplus \{\varepsilon\}$ are functions. We further require that \mathcal{A} satisfies the following conditions (A.1)–(A.6).

- (A.1) the set of final states is $F = \lambda^{-1}(\varepsilon)$ and $\lambda_{\mathbf{H}}(F) = \{I\}$;
- (A.2) there is no outgoing transition from F ;
- (A.3) for every $v \in V \setminus F$, both $v.0$ and $v.1$ are defined;
- (A.4) for every $v \in V \setminus F$, the equality $\lambda(v.\Sigma^*) = \lambda(v.0\Sigma^*) \uplus \{\lambda(v)\}$ holds.

Recall that, by Lemma 2.7, an infinite tuple (v_1, v_2, v_3, \dots) of $(\overline{\Omega}_A \text{DRH})^I$ satisfying $c(v_1) \supseteq c(v_2) \supseteq \dots$, with each v_i either of the form $u_i a_i$ (where u_i is a pseudoword over DRH and a_i a letter) with $a_i \notin c(u_i)$ or the empty word I determines exactly one \mathcal{R} -class of $(\overline{\Omega}_A \text{DRH})^I$. The intuitive idea behind conditions (A.1)–(A.4) is that the label λ of the vertices $v^0, v^0.1, v^0.11, \dots$ represents the letters a_1, a_2, a_3, \dots appearing in such a tuple. Then, at each non-final vertex $v^0.1^i$, we find the information about u_i by following the edge labeled by 0, and we find the information about the tuple $(v_{i+1}, v_{i+2}, \dots)$ by following the edge labeled by 1. This explains requirement (A.4). Since with this we can only access information concerning the \mathcal{R} -classes, an automaton \mathcal{A} satisfying conditions (A.1)–(A.4) shall determine a unique pseudoword over \mathbf{R} . In fact, if conditions (A.1)–(A.4) hold for \mathcal{A} , then the *reduct* $\mathcal{A}_{\mathbf{R}} = \langle V, \rightarrow, v^0, F, \lambda \rangle$ is an A -labeled \mathbf{R} -automaton (see [28, Definition 3.11]), which is the structure used to solve the κ -word problem over \mathbf{R} . Since the cumulative content of a pseudoword over DRH depends only on its projection onto $\overline{\Omega}_A \mathbf{R}$, and hence, also its regularity, we may use the known results for the word problem in \mathbf{R} (namely, [28, Theorem 3.21]) as guidance for defining the *length* $\|\mathcal{A}\|$, the *regularity index* $r.\text{ind}(\mathcal{A})$ and the *cumulative content* $\vec{c}(\mathcal{A})$ of a DRH-automaton \mathcal{A} from the knowledge of its reduct $\mathcal{A}_{\mathbf{R}}$. In particular, the parameter $\|\mathcal{A}\|$ corresponds to the smallest index k such that $v_{k+1} = I$, while the parameter $r.\text{ind}(\mathcal{A})$ is the index in which the content of v_i stabilizes in case the above tuple is representing a pseudoword whose cumulative content is nonempty. We set:

$$\begin{aligned} \|\mathcal{A}\| &= \sup\{k \geq 0 : v^0.1^k \text{ is defined}\}; \\ r.\text{ind}(\mathcal{A}) &= \begin{cases} \infty, & \text{if } \|\mathcal{A}\| < \infty; \\ \min\{m \geq 0 : \forall k \geq m \quad \lambda(v^0.1^k \Sigma^*) = \lambda(v^0.1^m \Sigma^*)\}, & \text{otherwise;} \end{cases} \\ \vec{c}(\mathcal{A}) &= \begin{cases} \emptyset, & \text{if } \|\mathcal{A}\| < \infty; \\ \lambda(v^0.1^{r.\text{ind}(\mathcal{A})} \Sigma^*), & \text{otherwise.} \end{cases} \end{aligned}$$

We are now able to state the further required properties for \mathcal{A} :

- (A.5) if $v \in V \setminus F$, then $\|\mathcal{A}_{v.0}\| < \infty$ if and only if $\lambda_{\mathbf{H}}(v) = I$;
- (A.6) if $v \in V \setminus F$ and $\|\mathcal{A}_{v.0}\| = \infty$, then $\lambda_{\mathbf{H}}(v) \in \overline{\Omega}_{\vec{c}(\mathcal{A}_{v.0})} \mathbf{H}$.

We say that \mathcal{A} is a DRH-*tree* if it is a DRH-automaton such that for every $v \in V$ there exists a unique $\alpha \in \Sigma^*$ such that $v^0.\alpha = v$.

Example 3.1. In Figure 1 we represent an example of a DRH-automaton, call it \mathcal{A} . The first label in each state corresponds to its image under $\lambda_{\mathbf{H}}$ and the second to its image under λ . Let v^0 be the initial state. We have $\|\mathcal{A}\| = \infty$, and so, the regularity index of \mathcal{A} is finite, and its cumulative content is nonempty. Observing that $\lambda(v^0.\Sigma^*) = \{a, b, c\}$ and $\lambda(v^0.1^k \Sigma^*) = \{a, b\}$ for every $k \geq 1$, we conclude that $r.\text{ind}(\mathcal{A}) = 1$ and $\vec{c}(\mathcal{A}) = \{a, b\}$.

Definition 3.2. We say that two DRH-automata $\mathcal{A}_i = \langle V_i, \rightarrow_i, v_i^0, F_i, \lambda_{i,\mathbf{H}}, \lambda_i \rangle$, $i = 1, 2$, are isomorphic if there exists a bijection $f : V_1 \rightarrow V_2$ such that

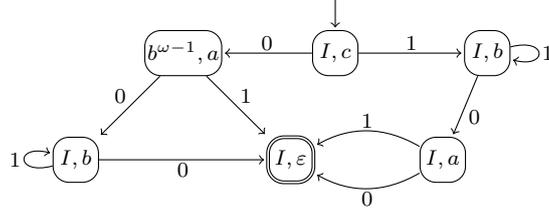


Figure 1: Example of an $\{a, b, c\}$ -labeled DRH-automaton.

- $f(v_1^0) = v_2^0$;
- for every $v \in V_1$ and $\alpha \in \Sigma$, $f(v) \cdot \alpha = f(v \cdot \alpha)$;
- for every $v \in V_1$, the equalities $\lambda_{1,H}(v) = \lambda_{2,H}(f(v))$ and $\lambda_1(v) = \lambda_2(f(v))$ hold.

Isomorphic DRH-automata are essentially the same, up to the name of the states. Therefore, we consider DRH-automata only up to isomorphism.

We denote the set of all A -labeled DRH-automata by \mathbb{A}_A .

Next, we shall assign a pseudoword over DRH to each DRH-automaton. For that, the important data about the DRH-automaton concerns the labels we can read along each path starting at the initial state. This leads to the next definition.

Definition 3.3. Let $\mathcal{A}_j = \langle V_j, \rightarrow_j, v_j^0, F_j, \lambda_{j,H}, \lambda_j \rangle$, $j = 1, 2$, be two DRH-automata. We say that \mathcal{A}_1 and \mathcal{A}_2 are equivalent if

$$\forall \alpha \in \Sigma^*, \lambda_1(v_1^0 \cdot \alpha) = \lambda_2(v_2^0 \cdot \alpha) \text{ and } \lambda_{1,H}(v_1^0 \cdot \alpha) = \lambda_{2,H}(v_2^0 \cdot \alpha). \quad (1)$$

We agree that (1) means that either both equalities hold or both $v_1^0 \cdot \alpha$ and $v_2^0 \cdot \alpha$ are undefined. We write $\mathcal{A}_1 \sim \mathcal{A}_2$ when \mathcal{A}_1 and \mathcal{A}_2 are equivalent.

Observe that equivalent DRH-trees are necessarily isomorphic. The following lemma is useful when defining a bijective correspondence between the equivalence classes of \mathbb{A}_A and the \mathcal{R} -classes of $\overline{\Omega}_A \text{DRH}$. Although its proof is analogous to the proof of [28, Lemma 3.16], we include it for the sake of completeness.

Lemma 3.4. Every DRH-automaton has a unique equivalent DRH-tree.

Proof. Take a DRH-automaton $\mathcal{A} = \langle V, \rightarrow, v^0, F, \lambda_H, \lambda \rangle$ and let $\mathcal{T} = \langle V_{\mathcal{T}}, \rightarrow_{\mathcal{T}}, v_{\mathcal{T}}^0, F_{\mathcal{T}}, \lambda_{\mathcal{T},H}, \lambda_{\mathcal{T}} \rangle$ be the DRH-tree defined as follows. We set $V_{\mathcal{T}} = \{\alpha \in \Sigma^* : v^0 \cdot \alpha \text{ is defined}\}$ and put $v_{\mathcal{T}}^0 = \varepsilon$. The labels of each state $\alpha \in V_{\mathcal{T}}$ are given by $\lambda_{\mathcal{T},H}(\alpha) = \lambda_{\mathcal{T},H}(v^0 \cdot \alpha)$ and by $\lambda_{\mathcal{T}}(\alpha) = \lambda(v^0 \cdot \alpha)$. We also take $F_{\mathcal{T}} = \lambda_{\mathcal{T}}^{-1}(\varepsilon)$. Finally, the transitions in \mathcal{T} are given by $\alpha \cdot 0 = \alpha 0$ and by $\alpha \cdot 1 = \alpha 1$, whenever $\lambda_{\mathcal{T}}(\alpha) \neq \varepsilon$. It is a routine matter to check that \mathcal{T} is a DRH-tree equivalent to \mathcal{A} . \square

Given a DRH-automaton \mathcal{A} , we denote by $\vec{\mathcal{A}} = \langle \vec{V}, \rightarrow, \vec{v}^0, \vec{F}, \vec{\lambda}_H, \vec{\lambda} \rangle$ the unique DRH-tree which is equivalent to \mathcal{A} . Denoting both transition functions of \mathcal{A} and of $\vec{\mathcal{A}}$ by \rightarrow is an abuse of notation justified by the construction made in the proof of Lemma 3.4. Given a DRH-tree \mathcal{T} with root v^0 and $0 \leq i \leq \|\mathcal{T}\| - 1$, we denote by $\mathcal{T}_{[i]}$ the DRH-subtree rooted at $v^0 \cdot 1^i$. In particular, $\vec{\mathcal{A}}_{[i]}$ denotes the DRH-subtree of $\vec{\mathcal{A}}$ rooted at $\vec{v}^0 \cdot 1^i$. Figure 2 provides the intuition for this definition.

Before defining the pseudoword over DRH given by a certain DRH-tree, we introduce the following notation.

Notation 3.5. Let $u \in \overline{\Omega}_A \text{DRH}$ and $v \in \overline{\Omega}_A H$ be such that $c(v) \subseteq \vec{c}(u)$. By Corollary 2.6, the set $u\rho_{\text{DRH},H}^{-1}(v)$ is a singleton. It is convenient to denote by w the unique element of $u\rho_{\text{DRH},H}^{-1}(v)$. In this case, the notation $\rho_H(w)$ refers to the element $\rho_H(w) = \rho_H(u) v$ of $\overline{\Omega}_A H$.

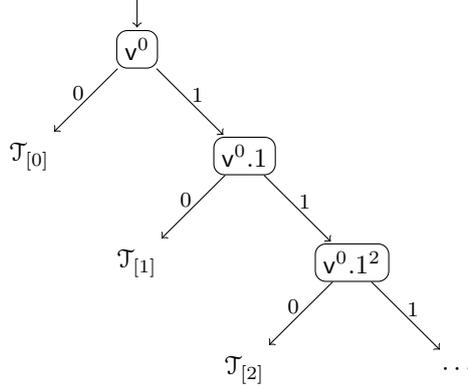


Figure 2: Representation of the DRH-subtrees of the form $\mathcal{T}_{[i]}$ of a tree \mathcal{T} .

Definition 3.6. Let $\mathcal{T} = \langle V, \rightarrow, v^0, F, \lambda_{\mathbb{H}}, \lambda \rangle$ be an A -labeled DRH-tree. The value of $\pi(\mathcal{T})$ is the element of $(\overline{\Omega}_A \text{DRH})^I$ with content $\lambda(v^0.\Sigma^*)$ that is recursively defined as follows:

- if \mathcal{T} is the trivial DRH-tree, then $\pi(\mathcal{T}) = I$;
- otherwise, we consider two different cases according to whether or not $\|\mathcal{T}\| < \infty$.
 - If $\|\mathcal{T}\| < \infty$, then we set

$$\pi(\mathcal{T}) = \prod_{i=0}^{\|\mathcal{T}\|-1} \pi(\mathcal{T}_{[i]}) \lambda_{\mathbb{H}}(v^0.1^i) \lambda(v^0.1^i).$$

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We remark that Properties (A.4)–(A.6) imply that $c(\pi(\mathcal{T})) = \lambda(v^0.\Sigma^*)$.

- If $\|\mathcal{T}\| = \infty$, then we first define the idempotent associated to \mathcal{T} , denoted $\text{id}(\mathcal{T})$. Observe that, again by Properties (A.4)–(A.6), we have $c(\pi(\mathcal{T}_{[k]}) \lambda_{\mathbb{H}}(v^0.1^k) \lambda(v^0.1^k)) = \lambda(v^0.1^k.\Sigma^*)$. Therefore, by definition of $\text{r.ind}(-)$, for $k \geq \text{r.ind}(\mathcal{T})$, all the elements $\pi(\mathcal{T}_{[k]}) \lambda_{\mathbb{H}}(v^0.1^k) \lambda(v^0.1^k)$ have the same content. We let $\text{id}(\mathcal{T})$ be the idempotent designated by the infinite product

$$(\pi(\mathcal{T}_{[\text{r.ind}(\mathcal{T})]} \lambda_{\mathbb{H}}(v^0.1^{\text{r.ind}(\mathcal{T})}) \lambda(v^0.1^{\text{r.ind}(\mathcal{T})}) \dots \pi(\mathcal{T}_{[k]} \lambda_{\mathbb{H}}(v^0.1^k) \lambda(v^0.1^k))_{k \geq \text{r.ind}(\mathcal{T})}. \quad (2)$$

Then, we take

$$\pi(\mathcal{T}) = \left(\prod_{i=0}^{\text{r.ind}(\mathcal{T})-1} \pi(\mathcal{T}_{[i]}) \lambda_{\mathbb{H}}(v^0.1^i) \lambda(v^0.1^i) \right) \cdot \text{id}(\mathcal{T}).$$

Observe that the value $\pi(\mathcal{T})$ is well-defined since, by Properties (A.1) and (A.4), every infinite path in \mathcal{T} contains only a finite number of edges labeled by 0 (recall that, by definition, $\mathcal{T}_{[i]} = \mathcal{T}_{v^0.1^i}$).

We also define the value of the irregular part of \mathcal{T} :

$$\pi_{\text{irr}}(\mathcal{T}) = \prod_{i=0}^{\min\{\|\mathcal{T}\|, \text{r.ind}(\mathcal{T})\}-1} \pi(\mathcal{T}_{[i]}) \lambda_{\mathbb{H}}(v^0.1^i) \lambda(v^0.1^i).$$

If $\|\mathcal{T}\| < \infty$, then we set $\text{id}(\mathcal{T}) = I$. Using this notation, we have the equality

$$\pi(\mathcal{T}) = \pi_{\text{irr}}(\mathcal{T}) \cdot \text{id}(\mathcal{T}).$$

Finally, the value of an A -labeled DRH-automaton \mathcal{A} , denoted $\pi(\mathcal{A})$, is the value $\pi(\vec{\mathcal{A}})$ of the unique A -labeled DRH-tree equivalent to \mathcal{A} . Similarly, the elements $\pi_{\text{irr}}(\mathcal{A})$ and $\text{id}(\mathcal{A})$ are defined to be $\pi_{\text{irr}}(\vec{\mathcal{A}})$ and $\text{id}(\vec{\mathcal{A}})$, respectively.

An example, the reader may check that the value of the DRH-automaton represented in Figure 1 is the pseudoword $b^{\omega-1}ac(ab)^{\omega}$ over DRH. The next result is a simple observation that we state for later reference, and that helps to understand the meaning of Definition 3.6.

Lemma 3.7. *Given a DRH-automaton $\mathcal{A} = \langle V, \rightarrow, v^0, F, \lambda_{\text{H}}, \lambda \rangle$, the following equalities hold:*

$$\begin{aligned} \text{lbf}_{i+1}(\pi(\mathcal{A})) &= \pi(\mathcal{A}_{[i]})\lambda_{\text{H}}(v^0.1^i)\lambda(v^0.1^i), \text{ for } 0 \leq i \leq \|\mathcal{A}\| - 1; \\ \text{lbf}_{i+1}(\pi(\mathcal{A})) &= I, \text{ for } i \geq \|\mathcal{A}\|; \\ \text{irr}(\pi(\mathcal{A})) &= \pi_{\text{irr}}(\mathcal{A}); \\ \vec{c}(\mathcal{A}) &= \vec{c}(\pi(\mathcal{A})). \end{aligned}$$

In particular, for a certain $u \in \overline{\Omega}_A\text{DRH}$, the elements $\pi(\mathcal{A})$ and u are \mathcal{R} -equivalent if and only if $\pi_{\text{irr}}(\mathcal{A}) = \text{irr}(u)$ and $\text{id}(\mathcal{A}) \mathcal{R} \text{reg}(u)$. \square

Since the value of a DRH-automaton \mathcal{A} depends only on the unique DRH-tree $\vec{\mathcal{A}}$ lying in the \sim -class of \mathcal{A} , there is a well defined map $\bar{\pi} : \mathbb{A}_A/\sim \rightarrow (\overline{\Omega}_A\text{DRH})^I/\mathcal{R}$ which sends a class \mathcal{A}/\sim to the \mathcal{R} -class of the value of $\vec{\mathcal{A}}$. This map is, in effect, a bijection.

Theorem 3.8. *The map $\bar{\pi}$ is bijective.*

Proof. To prove that $\bar{\pi}$ is injective we shall consider two DRH-trees \mathcal{T}_1 and \mathcal{T}_2 whose values belong to the same \mathcal{R} -class of $\overline{\Omega}_A\text{DRH}$ and show that they are necessarily equal. We argue inductively on the content of $\pi(\mathcal{T}_j)$ (which is the same for $j = 1, 2$ by hypothesis) by combining two facts: (a) the \mathcal{R} -class to which $\pi(\mathcal{T}_j)$ belongs is completely characterized by the sequence $\text{lbf}_{\infty}(\pi(\mathcal{T}_j))$ (see Lemma 2.7); and (b) the sequence $\text{lbf}_{\infty}(\pi(\mathcal{T}_j))$ is determined by the values of the DRH-subtrees of \mathcal{T}_j of the form $\mathcal{T}_{j,[i]}$ together with the labels of the states $v_j^0.1^i$ (for $i \geq 0$), where v_j^0 is the initial state of \mathcal{T}_j (see Lemma 3.7). Note that, the content of the value of each subtree $\mathcal{T}_{j,[i]}$ is strictly contained in the content of the value of \mathcal{T}_j , by Property (A.4).

We write $\mathcal{T}_j = \langle V_j, \rightarrow_j, v_j^0, F_j, \lambda_{j,\text{H}}, \lambda_j \rangle$, for $j = 1, 2$. If $\pi(\mathcal{T}_j)$ has empty content, then \mathcal{T}_1 and \mathcal{T}_2 are both the trivial DRH-tree. Let us assume that $|c(\pi(\mathcal{T}_j))| > 0$. By the observations above, the equality $\|\mathcal{T}_1\| = \|\mathcal{T}_2\|$ holds. Therefore, in order to prove that $\mathcal{T}_1 = \mathcal{T}_2$ it is enough to prove that for every $0 \leq i \leq \|\mathcal{T}_j\| - 1$, both the subtrees $\mathcal{T}_{1,[i]}$ and $\mathcal{T}_{2,[i]}$, and the labels of the states $v_1^0.1^i$ and $v_2^0.1^i$ are equal. Fix an index $0 \leq i \leq \|\mathcal{T}_j\| - 1$. By Lemma 3.7, we have

$$\begin{aligned} \pi(\mathcal{T}_{1,[i]})\lambda_{1,\text{H}}(v_1^0.1^i) &= \pi(\mathcal{T}_{2,[i]})\lambda_{2,\text{H}}(v_2^0.1^i), \\ \lambda_1(v_1^0.1^i) &= \lambda_2(v_2^0.1^i), \end{aligned} \tag{3}$$

and therefore, using Properties (A.5) and (A.6), it follows that $\pi(\mathcal{T}_{1,[i]})$ and $\pi(\mathcal{T}_{2,[i]})$ are \mathcal{R} -equivalent, which by induction hypothesis implies that $\mathcal{T}_{1,[i]} = \mathcal{T}_{2,[i]}$. It remains to show the equality $\lambda_{1,\text{H}}(v_1^0.1^i) = \lambda_{2,\text{H}}(v_2^0.1^i)$. In the case where $\vec{c}(\mathcal{T}_{j,[i]}) = \emptyset$, the equality follows immediately from (A.5). Otherwise, we may compute

$$\begin{aligned} \pi_{\text{irr}}(\mathcal{T}_{1,[i]})\text{id}(\mathcal{T}_{1,[i]})\lambda_{1,\text{H}}(v_1^0.1^i) &= \pi(\mathcal{T}_{1,[i]})\lambda_{1,\text{H}}(v_1^0.1^i) \stackrel{(3)}{=} \pi(\mathcal{T}_{2,[i]})\lambda_{2,\text{H}}(v_2^0.1^i) \\ &= \pi_{\text{irr}}(\mathcal{T}_{2,[i]})\text{id}(\mathcal{T}_{2,[i]})\lambda_{2,\text{H}}(v_2^0.1^i), \end{aligned}$$

which in turn implies that $\text{id}(\mathcal{T}_{1,[i]})\lambda_{1,\text{H}}(v_1^0.1^i) = \text{id}(\mathcal{T}_{2,[i]})\lambda_{2,\text{H}}(v_2^0.1^i)$. Since $\rho_{\text{H}}(\text{id}(\mathcal{T}_{1,[i]}))$ and $\rho_{\text{H}}(\text{id}(\mathcal{T}_{2,[i]}))$ are both the identity of $\overline{\Omega}_A\text{H}$ we obtain the equality $\lambda_{1,\text{H}}(v_1^0.1^i) = \lambda_{2,\text{H}}(v_2^0.1^i)$ as intended.

The proof of surjectivity follows the same general idea. Take some arbitrary $w \in (\overline{\Omega}_A\text{DRH})^I$. We have to show the existence of a DRH-tree \mathcal{T} whose value is \mathcal{R} -equivalent to w . In order to do so, we proceed by induction on the content of w , by considering the iteration of the left basic factorization of w to the

right, say $w = w_0 a_0 \cdots w_k a_k w'_k$. Then, the tree whose value lies in the same \mathcal{R} -class of the factor w_i shall correspond to the subtree $\mathcal{T}_{[i]}$ of \mathcal{T} . The factor $w_i a_i$ determines the labels of the state $v^0.1^i$, being v^0 the initial state of \mathcal{T} . Let us see the details.

If $c(w) = \emptyset$, then we have $[w]_{\mathcal{R}} = \{I\}$, which is the \mathcal{R} -class of the value of the trivial DRH-tree. Otherwise, we let $w = w_0 a_0 \cdots w_k a_k w'_k$ be the k -th iteration of the left basic factorization of w (whenever it is defined). For each $0 \leq i \leq \lceil w \rceil - 1$, we have $c(w_i) \subsetneq c(w)$ and so, by induction hypothesis, there exists a DRH-tree $\mathcal{T}_i = \langle V_i, \rightarrow_i, v_i^0, F_i, \lambda_{i,H}, \lambda_i \rangle$ such that $\pi(\mathcal{T}_i) \mathcal{R} w_i$. In particular, the equality $\pi_{\text{irr}}(\mathcal{T}_i) = \text{irr}(w_i)$ holds and consequently, H satisfies

$$\pi(\mathcal{T}_i) \cdot \text{reg}(w_i) = \pi_{\text{irr}}(\mathcal{T}_i) \cdot \text{id}(\mathcal{T}_i) \cdot \text{reg}(w_i) = \text{irr}(w_i) \cdot 1 \cdot \text{reg}(w_i) = w_i. \quad (4)$$

280 On the other hand, since $c(\text{reg}(w_i)) = \bar{c}(\text{id}(\mathcal{T}_i))$, we deduce that $\text{id}(\mathcal{T}_i) \cdot \text{reg}(w_i)$ is \mathcal{R} -equivalent to $\text{id}(\mathcal{T}_i)$. Consequently, the pseudowords w_i and $\pi(\mathcal{T}_i) \cdot \text{reg}(w_i)$ are \mathcal{R} -equivalent as well. This relation together with (4) imply, by Lemma 2.8, that the equality $\pi(\mathcal{T}_i) \cdot \text{reg}(w_i) = w_i$ holds.

Now, we construct a DRH-tree $\mathcal{T} = \langle V, \rightarrow, v^0, F, \lambda_H, \lambda \rangle$ as follows:

- $V = \begin{cases} \{v \in V_i : i \geq 0\} \uplus \{v_i\}_{i \geq 0}, & \text{if } \lceil w \rceil = \infty; \\ \{v \in V_i : i = 0, \dots, \lceil w \rceil - 1\} \uplus \{v_i\}_{i=0}^{\lceil w \rceil - 1} \uplus \{v_\varepsilon\}, & \text{if } \lceil w \rceil < \infty; \end{cases}$
- 285 • $v^0 = v_0$;
- $F = \begin{cases} \{v \in F_i : i \geq 0\}, & \text{if } \lceil w \rceil = \infty; \\ \{v \in F_i : i = 0, \dots, \lceil w \rceil - 1\} \uplus \{v_\varepsilon\}, & \text{if } \lceil w \rceil < \infty; \end{cases}$
- $\lambda_H(v_i) = \rho_H(\text{reg}(w_i))$ and $\lambda(v_i) = a_i$ for $i = 0, \dots, \lceil w \rceil - 1$;
- $\lambda(v_\varepsilon) = \varepsilon$, if $\lceil w \rceil$ is finite;
- $v_i.0 = v_i^0$ and $v_i.1 = \begin{cases} v_{i+1}, & \text{if } i < \lceil w \rceil - 1; \\ v_\varepsilon, & \text{if } i = \lceil w \rceil - 1; \end{cases}$
- 290 • transitions and labelings on V_i are given by those of \mathcal{T}_i .

Then it is easy to check that \mathcal{T} is a DRH-tree and that $\bar{\pi}(\mathcal{T}/\sim) = [w]_{\mathcal{R}}$. □

The construction we just made to prove surjectivity of $\bar{\pi}$ may be abstracted as follows. Suppose that we are given two DRH-automata $\mathcal{A}_i = \langle V_i, \rightarrow_i, v_i^0, F_i, \lambda_{i,H}, \lambda_i \rangle$, $i = 0, 1$, a letter $a \in A$ such that $\lambda_1(V_1) \subseteq \lambda_0(V_0) \uplus \{a\}$ and a pseudoword u such that $c(u) \subseteq \bar{c}(\mathcal{A}_0)$. Then, we denote by $(\mathcal{A}_0, u \mid a, \mathcal{A}_1)$ the DRH-automaton $\mathcal{A} = \langle V, \rightarrow, v^0, F, \lambda_H, \lambda \rangle$, where

- $V = V_0 \uplus V_1 \uplus \{v^0\}$;
- $v^0.0 = v_0^0$ and $v^0.1 = v_1^0$;
- $F = F_0 \uplus F_1$;
- $\lambda_H(v^0) = \rho_H(u)$ and $\lambda(v^0) = a$;
- 300 • all the other transitions and labels are given by those of \mathcal{A}_0 and \mathcal{A}_1 .

Given an element w of $(\bar{\Omega}_A S)^I$, we denote by $\mathcal{T}(w)$ the DRH-tree representing the \sim -class $\bar{\pi}^{-1}([\rho_{\text{DRH}}(w)]_{\mathcal{R}})$. With a little abuse of notation, when $w \in (\bar{\Omega}_A \text{DRH})^I$, we use $\mathcal{T}(w)$ to denote the unique DRH-tree in the \sim -class $\bar{\pi}^{-1}([w]_{\mathcal{R}})$. Later, we shall see that, for every κ -word w , there exists a finite DRH-automaton \mathcal{A} in the \sim -class of $\mathcal{T}(w)$ (Corollary 4.7).

Lemma 3.9. *Let w be a pseudoword and write $\text{lbf}(w) = (w_\ell, a, w_r)$. Then, we have the equality*

$$\mathcal{T}(w) = (\mathcal{T}(w_\ell), \text{reg}(w_\ell) \mid a, \mathcal{T}(w_r)).$$

305 *Proof.* The claim follows immediately from the fact that if $\text{lbf}_\infty(w) = (v_1, v_2, v_3 \dots)$, then $v_1 = w_\ell a$ and $\text{lbf}_\infty(w_r) = (v_2, v_3, \dots)$; together with the construction of a DRH-tree representing the \mathcal{R} -class of a given pseudoword made in the proof of Theorem 3.8. In order to clarify the ideas, we depict such construction in Figure 3. As before, the first label in each state corresponds to its image under λ_H and the second to its

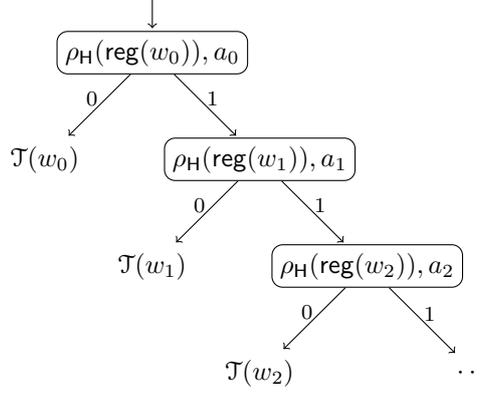


Figure 3: The DRH-tree $\mathcal{J}(w)$ when w is such that $\text{lbf}_\infty(w) = (w_0 a_0, w_1 a_1, w_2 a_2, \dots)$.

image under λ . Note that the tree contains an infinite path starting at the root with all the edges labeled by 1 if and only if w has nonempty cumulative content. \square

The value of a path $v_0 \xrightarrow{\alpha_0} v_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} v_{n+1}$ in a DRH-automaton \mathcal{A} is given by

$$\prod_{i=0}^n (\alpha_i, \lambda_{H, \alpha_i}(v_i), \lambda(v_i)) \in (\Sigma \times (\overline{\Omega}_A H)^I \times A)^+,$$

where $\lambda_{H, \alpha_i}(v_i) = \lambda_H(v_i)$ if $\alpha_i = 0$, and $\lambda_{H, \alpha_i}(v_i) = I$ otherwise. Given a state v of \mathcal{A} , the language associated to v is the set $\mathcal{L}(v)$ of all values of successful paths in \mathcal{A}_v . The language associated to \mathcal{A} , denoted $\mathcal{L}(\mathcal{A})$, is the language associated to its root. Finally, the language associated to the pseudoword w is $\mathcal{L}(w) = \mathcal{L}(\mathcal{J}(w))$.

315 **Lemma 3.10.** *Let $\mathcal{A}_1, \mathcal{A}_2$ be DRH-automata. Then, the languages $\mathcal{L}(\mathcal{A}_1)$ and $\mathcal{L}(\mathcal{A}_2)$ coincide if and only if the DRH-trees $\vec{\mathcal{A}}_1$ and $\vec{\mathcal{A}}_2$ are the same.*

Proof. Recall that, by Lemma 3.4, if $\vec{\mathcal{A}}_1 = \vec{\mathcal{A}}_2$, then \mathcal{A}_1 and \mathcal{A}_2 are equivalent DRH-automata. Hence, Definition 3.3 makes clear the reverse implication. Conversely, let $\mathcal{A}_j = \langle V_j, \rightarrow_j, v_j^0, F_j, \lambda_{j,H}, \lambda_j \rangle$ ($j = 1, 2$) be two DRH-automata such that $\mathcal{L}(\mathcal{A}_1) = \mathcal{L}(\mathcal{A}_2)$. We first observe that, for $j = 1, 2$ and $\alpha \in \Sigma^*$, the state $v_j^0 \cdot \alpha$ is defined if and only if there exists an element in $\mathcal{L}(\mathcal{A}_j)$ of the form $(\alpha\beta, -, -)$, for a certain $\beta \in \Sigma^*$ (we are using the fact that DRH-automata are trim). Hence, the state $v_1^0 \cdot \alpha$ is defined if and only if so is the state $v_2^0 \cdot \alpha$. Choose $\alpha = \alpha_0 \alpha_1 \dots \alpha_n \in \Sigma^*$, with each $\alpha_i \in \Sigma$ and such that $v_1^0 \cdot \alpha$ is defined. If $v_1^0 \cdot \alpha \in F_1$, then we have a successful path $v_1^0 \xrightarrow{\alpha_0} v_1^1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} v_1^{n+1}$, and so, the element $\prod_{i=0}^n (\alpha_i, \lambda_{1,H, \alpha_i}(v_1^i), \lambda_1(v_1^i))$ belongs to $\mathcal{L}(\mathcal{A}_1)$ and hence, to $\mathcal{L}(\mathcal{A}_2)$. But that implies that, in \mathcal{A}_2 , there is a successful path $v_2^0 \xrightarrow{\alpha_0} v_2^1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} v_2^{n+1}$, which in turn yields that both $v_1^0 \cdot \alpha$ and $v_2^0 \cdot \alpha$ are terminal states. In particular the equalities in (1) hold. On the other hand, if $v_1^0 \cdot \alpha$ is not a terminal state, then condition (A.3) implies that $v_1^0 \cdot \alpha 0$ is defined. Since any DRH-automaton is trim, there exists $\beta = \alpha_{n+2} \dots \alpha_m \in \Sigma^*$ such that

$$v_1^0 \xrightarrow{\alpha_0} v_1^1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} v_1^{n+1} \xrightarrow{0} v_1^{n+2} \xrightarrow{\alpha_{n+2}} \dots \xrightarrow{\alpha_m} v_1^{m+1} \quad (5)$$

is a successful path in \mathcal{A}_1 . Again, since $\mathcal{L}(\mathcal{A}_1) = \mathcal{L}(\mathcal{A}_2)$, this determines a successful path in \mathcal{A}_2 given by $v_2^0 \xrightarrow{\alpha_0} v_2^1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_n} v_2^{n+1} \xrightarrow{0} v_2^{n+2} \xrightarrow{\alpha_{n+2}} \dots \xrightarrow{\alpha_m} v_2^{m+1}$, with the same value as the path (5). In particular, the $(n+2)$ -nd letter (of the alphabet $\Sigma \times (\overline{\Omega}_A H)^I \times A$) of that value is

$$(0, \lambda_{1,H,0}(v_1^{n+1}), \lambda_1(v_1^{n+1})) = (0, \lambda_{2,H,0}(v_2^{n+1}), \lambda_2(v_2^{n+1})).$$

But that means precisely that the desired equalities in (1) hold. Therefore, \mathcal{A}_1 and \mathcal{A}_2 are equivalent and so, $\vec{\mathcal{A}}_1 = \vec{\mathcal{A}}_2$. \square

The next result provides a characterization of equalities of pseudowords over DRH.

Proposition 3.11. *Let $u, v \in \overline{\Omega}_A \mathcal{S}$. Then the equality $\rho_{\text{DRH}}(u) = \rho_{\text{DRH}}(v)$ holds if and only if $\mathcal{L}(u) = \mathcal{L}(v)$ and \mathbf{H} satisfies $u = v$.*

Proof. Let u and v be two equal pseudowords modulo DRH. In particular, the \mathcal{R} -classes $[\rho_{\text{DRH}}(u)]_{\mathcal{R}}$ and $[\rho_{\text{DRH}}(v)]_{\mathcal{R}}$ coincide and so, the DRH-trees $\mathcal{T}(u)$ and $\mathcal{T}(v)$ are the same, by Theorem 3.8. Therefore, we have $\mathcal{L}(u) = \mathcal{L}(\mathcal{T}(u)) = \mathcal{L}(\mathcal{T}(v)) = \mathcal{L}(v)$. As \mathbf{H} is a subpseudovariety of DRH, we also have $u =_{\mathbf{H}} v$. Conversely, suppose that $\mathcal{L}(u) = \mathcal{L}(v)$ and $u =_{\mathbf{H}} v$. By Lemma 3.10, it follows that $\mathcal{T}(u) = \mathcal{T}(v)$. Thus, by Theorem 3.8, the pseudovariety DRH satisfies $u \mathcal{R} v$. As, in addition, the pseudowords u and v are equal modulo \mathbf{H} , we conclude by Lemma 2.8 that DRH satisfies $u = v$. \square

4. A canonical form for κ -words over DRH

As observed in Subsection 2.3, every element of $\Omega_A^\kappa \mathcal{S}$, may be represented by some $\bar{\kappa}$ -term. In turn, given a pseudovariety \mathbf{V} , there is a natural projection $\rho_{\mathbf{V}} : \overline{\Omega}_A \mathcal{S} \rightarrow \overline{\Omega}_A \mathbf{V}$ whose restriction to κ -words maps to $\Omega_A^\kappa \mathbf{V}$. We say that a *canonical form* for the elements of $\Omega_A^\kappa \mathbf{V}$ is a map $\text{cf}_{\mathbf{V}}$ that assigns to each element w of $\Omega_A^\kappa \mathbf{V}$ a $\bar{\kappa}$ -term $\text{cf}_{\mathbf{V}}(w)$ such that the κ -word it represents projects to w under $\rho_{\mathbf{V}}$. In the case the map $\text{cf}_{\mathbf{V}}$ is computable, we obtain decidability of the κ -word problem over \mathbf{V} .

In this section, we exhibit a canonical form for κ -words over DRH starting from a canonical form for κ -words over \mathbf{H} . The key ingredient is the fact that each \mathcal{R} -class of $\Omega_A^\kappa \text{DRH}$ admits a representation by a finite DRH-automaton (see Corollary 4.7). In turn, to each finite DRH-automaton we may uniquely assign a $\bar{\kappa}$ -term (see Definition 4.1). By standardizing the choice of a finite DRH-automaton representing the \mathcal{R} -class of an element $w \in \Omega_A^\kappa \text{DRH}$, we obtain the desired canonical form. In particular, it is a consequence of the results of Section 6 that if $\text{cf}_{\mathbf{H}}$ is computable, then so is cf_{DRH} .

Throughout this section, we fix a pseudovariety of groups \mathbf{H} and a canonical form $\text{cf}_{\mathbf{H}}$ for the elements of $\Omega_A^\kappa \mathbf{H}$.

Definition 4.1. *Given a finite DRH-automaton $\mathcal{A} = \langle V, \rightarrow, v^0, F, \lambda_{\mathbf{H}}, \lambda \rangle$ such that $\lambda_{\mathbf{H}}(V) \subseteq (\Omega_A^\kappa \mathbf{H})^I$, let us define the expression $\pi_{\text{cf}}(\mathcal{A})$ inductively on the number $|V|$ of states as follows.*

- If $|V| = 1$, then \mathcal{A} is the trivial DRH-automaton and we take $\pi_{\text{cf}}(\mathcal{A}) = I$.
- If $|V| > 1$ and $\|\mathcal{A}\| < \infty$, then we put

$$\pi_{\text{cf}}(\mathcal{A}) = \prod_{i=0}^{\|\mathcal{A}\|-1} \pi_{\text{cf}}(\mathcal{A}_{v^0 \cdot 1^i}) \text{cf}_{\mathbf{H}}(\lambda_{\mathbf{H}}(v^0 \cdot 1^i)) \lambda(v^0 \cdot 1^i).$$

- Finally, we suppose that $|V| > 1$ and $\|\mathcal{A}\| = \infty$. Since \mathcal{A} is a finite automaton, we necessarily have a cycle of the form $v^0 \cdot 1^\ell \xrightarrow{1} v^0 \cdot 1^{\ell+1} \xrightarrow{1} \dots \xrightarrow{1} v^0 \cdot 1^{\ell+n} \xrightarrow{1} v^0 \cdot 1^\ell$, where ℓ is a certain integer greater than or equal to $\text{r.ind}(\mathcal{A})$. Choose ℓ to be the least possible. Then, we make $\pi_{\text{cf}}(\mathcal{A})$ be given by

$$\prod_{i=0}^{\text{r.ind}(\mathcal{A})-1} \pi_{\text{cf}}(\mathcal{A}_{v^0 \cdot 1^i}) \text{cf}_{\mathbf{H}}(\lambda_{\mathbf{H}}(v^0 \cdot 1^i)) \lambda(v^0 \cdot 1^i) \cdot \left(\prod_{i=\text{r.ind}(\mathcal{A})}^{\ell-1} \pi_{\text{cf}}(\mathcal{A}_{v^0 \cdot 1^i}) \text{cf}_{\mathbf{H}}(\lambda_{\mathbf{H}}(v^0 \cdot 1^i)) \lambda(v^0 \cdot 1^i) \cdot \left(\prod_{i=0}^n \pi_{\text{cf}}(\mathcal{A}_{v^0 \cdot 1^{\ell+i}}) \text{cf}_{\mathbf{H}}(\lambda_{\mathbf{H}}(v^0 \cdot 1^{\ell+i})) \lambda(v^0 \cdot 1^{\ell+i}) \right)^\omega \right)^\omega.$$

We point out that, by definition, the value of the κ -word over DRH naturally induced by $\pi_{\text{cf}}(\mathcal{A})$ is precisely $\pi(\mathcal{A})$. On the other hand, it is easy to check that, for every $w \in \overline{\Omega}_A \text{DRH}$, if $w \mathcal{R} \pi(\mathcal{A})$, then the identity $w = \pi(\mathcal{A})\text{reg}(w)$ holds. Thus, in view of Theorem 3.8, we wish to standardize a choice of a finite DRH-automaton, say $\mathcal{A}(w)$, equivalent to $\mathcal{T}(w)$, for each $w \in \Omega_A^\kappa \text{DRH}$. After that, we may let the canonical form of w be given by $\pi_{\text{cf}}(\mathcal{A}(w))\text{cf}_H(\text{reg}(w))$.

Fix a DRH-automaton $\mathcal{A} = \langle V, \rightarrow, \mathbf{v}^0, F, \lambda_H, \lambda \rangle$. We say that two states $\mathbf{v}_1, \mathbf{v}_2 \in V$ are *equivalent* if $\pi(\mathcal{A}_{\mathbf{v}_1})$ and $\pi(\mathcal{A}_{\mathbf{v}_2})$ lie in the same \mathcal{R} -class. Clearly, this defines an equivalence relation on V , say \sim (it should be clear from the context when we are referring to this equivalence relation or to the equivalence relation on \mathbb{A}_A introduced in Definition 3.3). We write $[\mathbf{v}]$ for the equivalence class of $\mathbf{v} \in V$.

Lemma 4.2. *Let $\mathcal{A} = \langle V, \rightarrow, \mathbf{v}^0, F, \lambda_H, \lambda \rangle$ be a DRH-automaton and consider the equivalent class on V defined above. Then, for every $\mathbf{v}_1, \mathbf{v}_2 \in V \setminus F$, we have*

$$[\mathbf{v}_1] = [\mathbf{v}_2] \iff \begin{cases} [\mathbf{v}_1.0] = [\mathbf{v}_2.0] \text{ and } [\mathbf{v}_1.1] = [\mathbf{v}_2.1]; \\ \lambda_H(\mathbf{v}_1) = \lambda_H(\mathbf{v}_2) \text{ and } \lambda(\mathbf{v}_1) = \lambda(\mathbf{v}_2). \end{cases}$$

Proof. Let $\mathbf{v}_1, \mathbf{v}_2 \in V \setminus F$ be non-terminal states of \mathcal{A} . We first observe that since each DRH-automaton has a unique equivalent DRH-tree and represents a unique \mathcal{R} -class of $\overline{\Omega}_A \text{DRH}$ (see, respectively, Lemma 3.4 and Theorem 3.8), having $[\mathbf{v}_1] = [\mathbf{v}_2]$ amounts to having $\vec{\mathcal{A}}_{\mathbf{v}_1} = \vec{\mathcal{A}}_{\mathbf{v}_2}$. In turn, the latter condition means that the labels λ and λ_H of the states $\mathbf{v}_1, \mathbf{v}_2$ are the same, and $\vec{\mathcal{A}}_{\mathbf{v}_1.\beta} = \vec{\mathcal{A}}_{\mathbf{v}_2.\beta}$ for $\beta \in \{0, 1\}$. Finally, invoking again Lemma 3.4 and Theorem 3.8, the equalities $\vec{\mathcal{A}}_{\mathbf{v}_1.\beta} = \vec{\mathcal{A}}_{\mathbf{v}_2.\beta}$, for $\beta \in \{0, 1\}$, are equivalent to the equalities $[\mathbf{v}_1.0] = [\mathbf{v}_2.0]$ and $[\mathbf{v}_1.1] = [\mathbf{v}_2.1]$. This concludes the proof. \square

We define the *wrapping* of a DRH-automaton $\mathcal{A} = \langle V, \rightarrow, \mathbf{v}^0, F, \lambda_H, \lambda \rangle$ to be the DRH-automaton $[\mathcal{A}] = \langle V/\sim, \rightarrow, [\mathbf{v}^0], F/\sim, \bar{\lambda}_H, \bar{\lambda} \rangle$, where

- $[\mathbf{v}].0 = [\mathbf{v}.0]$ and $[\mathbf{v}].1 = [\mathbf{v}.1]$, for $\mathbf{v} \in V \setminus F$;
- $\bar{\lambda}_H([\mathbf{v}]) = \lambda_H(\mathbf{v})$ and $\bar{\lambda}([\mathbf{v}]) = \lambda(\mathbf{v})$, for $\mathbf{v} \in V$.

By Lemma 4.2, this automaton is well defined. Furthermore, its definition ensures that $\mathcal{A} \sim [\mathcal{A}]$. The *wrapped DRH-automaton* of $w \in \overline{\Omega}_A \text{DRH}$ is $\mathcal{A}(w) = [\mathcal{T}(w)]$. Observe that, by Lemmas 2.1 and 3.9, the label λ_H of $\mathcal{T}(w)$ takes values in $\Omega_A^\kappa H$ when w is a κ -word. Our next goal is to prove that $\mathcal{A}(w)$ is finite, provided w is a κ -word.

Let us associate to a pseudoword $w \in (\overline{\Omega}_A \text{DRH})^I$ a certain set of its factors. For $\alpha \in \Sigma^*$, we define $f_\alpha(w)$ inductively on $|\alpha|$:

$$\begin{aligned} f_\varepsilon(w) &= w; \\ (f_{\alpha 0}(w), a, f_{\alpha 1}(w)) &= \text{lbf}(f_\alpha(w)), \text{ for a certain } a \in A, \text{ whenever } f_\alpha(w) \neq I. \end{aligned}$$

Then, the set of DRH-factors of w is given by

$$\mathcal{F}(w) = \{f_\alpha(w) : \alpha \in \Sigma^* \text{ and } f_\alpha(w) \text{ is defined}\}.$$

The relevance of the definition of the set $\mathcal{F}(w)$ is explained by the following result.

Lemma 4.3. *Let $w \in \overline{\Omega}_A \text{DRH}$ and $\mathcal{T}(w) = \langle V, \rightarrow, \mathbf{v}^0, F, \lambda_H, \lambda \rangle$. Then, for every $\alpha \in \Sigma^*$ such that $f_\alpha(w)$ is defined, the relation $f_\alpha(w) \mathcal{R} \pi(\mathcal{T}(w)_{\mathbf{v}^0.\alpha})$ holds.*

Proof. We prove the statement by induction on $|\alpha|$. When $\alpha = \varepsilon$, the result follows from Theorem 3.8. Let $\alpha \in \Sigma^*$ and invoke the induction hypothesis to assume that $f_\alpha(w)$ and $\pi(\mathcal{T}(w)_{\mathbf{v}^0.\alpha})$ are \mathcal{R} -equivalent. Writing $\text{lbf}(\pi(\mathcal{T}(w)_{\mathbf{v}^0.\alpha})) = (w_\ell, a, w_r)$, Lemma 3.7 yields the relations $w_\ell \mathcal{R} \pi(\mathcal{T}(w)_{\mathbf{v}^0.\alpha 0})$ and $w_r \mathcal{R} \pi(\mathcal{T}(w)_{\mathbf{v}^0.\alpha 1})$. On the other hand, since $\text{lbf}(f_\alpha(w)) = (f_{\alpha 0}(w), b, f_{\alpha 1}(w))$, using Lemma 2.7 we deduce that $f_{\alpha 0}(w) = w_\ell$, $a = b$, and $f_{\alpha 1}(w) \mathcal{R} w_r$, leading to the desired conclusion. \square

375 Hence, in order to prove that $\mathcal{A}(w)$ is finite for every κ -word w , it suffices to prove that so is $\mathcal{F}(w)/\mathcal{R}$. The next two lemmas are useful to achieve that target.

Lemma 4.4. *Let w be a regular κ -word over DRH. Then, there exist κ -words x, y and z over DRH (with x and z possibly empty) such that $w = xy^{\omega-1}z$, $c(y) = c(w)$, $\bar{c}(x) \subsetneq c(w)$, and y is not regular.*

Proof. By definition of κ -word, we may write $w = w_1 \cdots w_n$, where each w_i is either a letter in A or an
380 $(\omega - 1)$ -power of another κ -word. Since any letter of the cumulative content of w occurs in $\text{lbf}_\infty(w)$ infinitely many times, there must be an $(\omega - 1)$ -power under which they all appear. Hence, since w is regular (and so, $c(w) = \bar{c}(w)$), there exists an index $i \in \{1, \dots, n\}$ such that $w_i = v^{\omega-1}$ and $c(v) = c(w)$. Let j be the minimum such i . We have $w = u_0 v_0^{\omega-1} z_0$, where $u_0 = w_1 \cdots w_{j-1}$, $v_0^{\omega-1} = w_j$, and $z_0 = w_{j+1} \cdots w_n$. Also, minimality of j yields that $\bar{c}(u_0) \subsetneq \bar{c}(w) = c(w)$. So, if v_0 is not regular, then we just take $x = u_0$, $y = v_0$,
385 and $z = z_0$. Suppose that v_0 is regular. Using the same reasoning, we may write $v_0 = u_1 v_1^{\omega-1} z_1$, with $\bar{c}(u_1) \subsetneq c(w)$ and $c(v_1) = c(v_0) = c(w)$. Again, if v_1 is not regular, then we may choose $x = u_0 u_1$, $y = v_1$ and $z = z_1 v_0^{\omega-2} z_0$. Otherwise, we repeat the process with v_1 . Since w is a κ -word, there is only a finite number of occurrences of $(\omega - 1)$ -powers, so that this iteration cannot run forever. Therefore, we eventually get κ -words x, y and z satisfying the desired properties. \square

390 **Lemma 4.5.** *Let $w \in \Omega_A^\kappa \text{DRH}$ be regular. For each $m \geq 1$, let w'_m be the unique κ -word over DRH satisfying the equality $w = \text{lbf}_1(w) \cdots \text{lbf}_m(w) w'_m$. Then, both sets $\{\text{lbf}_m(w) : m \geq 1\}$ and $\{[w'_m]_{\mathcal{R}} : m \geq 1\}$ are finite.*

Proof. Write $\text{lbf}_m(w) = w_m a_m$, for every $m \geq 1$, and $w = xy^{\omega-1}z$, with x, y and z satisfying the properties stated in Lemma 4.4. We define a sequence of pairs of possibly empty κ -words $\{(u_i, v_i)\}_{i \geq 0}$ and a strictly increasing sequence of non-negative integers $\{k_i\}_{i \geq 0}$ inductively as follows. We start with $(u_0, v_0) = (I, x)$ and we let k_0 be the maximum index such that $\text{lbf}_1(w) \cdots \text{lbf}_{k_0}(w)$ is a prefix of x . If x has no prefix of this form, then we set $k_0 = 0$. We also write $v_0 = v'_0 v''_0$, with $v'_0 = \text{lbf}_1(w) \cdots \text{lbf}_{k_0}(w)$ (by Proposition 2.3, given v'_0 there is only one possible value for v''_0). For each $i \geq 0$, we let u_{i+1} be such that $w_{k_{i+1}} = v''_i u_{i+1}$ and v_{i+1} is such that $y = u_{i+1} a_{k_{i+1}} v_{i+1}$. Observe that, by uniqueness of first-occurrences factorizations, there is only one pair (u_{i+1}, v_{i+1}) satisfying these conditions. The integer k_{i+1} is the maximum such that $\text{lbf}_{k_{i+2}}(w) \cdots \text{lbf}_{k_{i+1}}(w)$ is a prefix of v_{i+1} (or $k_{i+1} = k_i + 1$ if there is no such prefix) and we factorize $v_{i+1} = v'_{i+1} v''_{i+1}$, with $v'_{i+1} = \text{lbf}_{k_{i+2}}(w) \cdots \text{lbf}_{k_{i+1}}(w)$. By construction, for all $i \geq 0$, the pseudoidentity $w'_{k_{i+1}} = v_{i+1} y^{\omega-(i+2)} z$ holds. In particular, for every $m \geq 1$, there exist $i \geq 0$ and $\ell \in \{2, \dots, k_{i+1} - k_i\}$ such that

$$w'_m = \text{lbf}_{k_i+\ell}(w) \text{lbf}_{k_i+\ell+1}(w) \cdots \text{lbf}_{k_{i+1}}(w) v''_{i+1} y^{\omega-(i+2)} z. \quad (6)$$

On the other hand, for all $i \geq 0$, the factorization $y = u_{i+1} a_{k_{i+1}} v_{i+1}$ is such that $a_{k_{i+1}} \notin c(u_{i+1})$ (recall that $a_{k_{i+1}} \notin c(w_{k_{i+1}})$ and u_{i+1} is a factor of $w_{k_{i+1}}$). By uniqueness of first-occurrences factorization over DRH, it follows that the set $\{(u_i, v_i)\}_{i \geq 0}$ is finite. Consequently, the set

$$\{\text{lbf}_{k_i+\ell}(w) \text{lbf}_{k_i+\ell+1}(w) \cdots \text{lbf}_{k_{i+1}}(w) v''_{i+1} : i \geq 0, \ell \in \{2, \dots, k_{i+1} - k_i\}\}$$

is also finite. In particular, there is only a finite number of κ -words $\text{lbf}_m(w)$. Finally, taking into account that $c(z) \subseteq c(y)$ and (6) we may conclude that there are only finitely many \mathcal{R} -classes of the form $[w'_m]_{\mathcal{R}}$ ($m \geq 1$). \square

395 Now, we are able to prove that $\mathcal{F}(w)/\mathcal{R}$ is finite for every κ -word w over DRH.

Proposition 4.6. *Let w be a possibly empty κ -word over DRH. Then, the quotient $\mathcal{F}(w)/\mathcal{R}$ is finite.*

Proof. We prove the result by induction on $|c(w)|$. If $|c(w)| = 0$, then it is trivial. Suppose that $|c(w)| \geq 1$. We distinguish two possible scenarios.

Case 1. The κ -word w is not regular, that is, $\bar{c}(w) \subsetneq c(w)$.

400 Then, there exists $k \geq 1$ such that $w = w_1 a_1 \cdots w_m a_m w'_m$, with $\text{lbf}_k(w) = w_k a_k$, for $k = 1, \dots, m$ and $c(w'_m) \subsetneq c(w)$. By definition of $f_\alpha(w)$, we have the identities $f_{1^{k-1}0}(w) = w_k$ (for $k = 1, \dots, m$) and

$f_{1^m}(w) = w'_m$. Hence, we may deduce that $\mathcal{F}(w)$ is the union of the sets $\mathcal{F}(w_k)$ (for $k = 1, \dots, m$) together with $\mathcal{F}(w'_m)$. Using the induction hypothesis on each one of the intervening sets, we conclude that $\mathcal{F}(w)/\mathcal{R}$ is finite.

405 **Case 2.** The κ -word w is regular.

Again, write $\text{lbf}_k(w) = w_k a_k$ and $w = \text{lbf}_1(w) \cdots \text{lbf}_k(w) w'_k$, for $k \geq 1$. Since $f_{1^{k-1}0}(w) = w_k$ and $f_{1^k}(w) = w'_k$, for every $k \geq 1$, by Lemma 4.5, we know that the sets $\{f_{1^{k-1}0}(w) : k \geq 1\}$ and $\{[f_{1^k}(w)]_{\mathcal{R}} : k \geq 1\}$ are both finite. Applying the induction hypothesis to each factor w_k , we derive that $\{[f_{1^{k-1}0\alpha}(w)]_{\mathcal{R}} : \alpha \in \Sigma^*, k \geq 1\}$ is also a finite set. Therefore, since any element of $\mathcal{F}(w)/\mathcal{R}$ is of one of the forms $[f_{1^{k-1}0\alpha}(w)]_{\mathcal{R}}$ and $[f_{1^k}(w)]_{\mathcal{R}}$, we conclude that $\mathcal{F}(w)/\mathcal{R}$ is finite as well. \square

As an immediate consequence (recall Lemma 4.3), we obtain:

Corollary 4.7. *Let w be a possibly empty κ -word. Then, the wrapped DRH-automaton $\mathcal{A}(w)$ is finite.* \square

415 Unlike the aperiodic case \mathbf{R} , the converse of Corollary 4.7 does not hold in general. For instance, taking $\mathbf{H} = \mathbf{G}$, it is not hard to see that $\mathcal{A}(a^{p^\omega} b)$ (with p a prime number) is finite, although $a^{p^\omega} b$ is not a κ -word over DRG (see Figure 4). A converse is achieved when we further require that the labels $\lambda_{\mathbf{H}}$ are valued by κ -words over \mathbf{H} and that $\rho_{\mathbf{H}}(\text{reg}(w))$ is itself a κ -word.

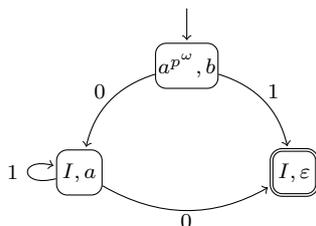


Figure 4: The DRH-automaton $\mathcal{A}(a^{p^\omega} b)$.

For a given $w \in (\Omega_{\mathbf{A}}^{\kappa} \text{DRH})^I$, the expression

$$\text{cf}(w) = \pi_{\text{cf}}(\mathcal{A}(w)) \text{cf}_{\mathbf{H}}(\rho_{\mathbf{H}}(\text{reg}(w)))$$

is said the *canonical form of w* . We write $\text{cf}(u) \equiv \text{cf}(v)$ (with $u, v \in (\Omega_{\mathbf{A}}^{\kappa} \text{DRH})^I$) when both sides coincide. We have just proved the claimed existence of a canonical form for elements of $\Omega_{\mathbf{A}}^{\kappa} \text{DRH}$.

420 **Theorem 4.8.** *Let \mathbf{H} be a pseudovariety of groups such that the elements of $\Omega_{\mathbf{A}}^{\kappa} \mathbf{H}$ have a canonical form $\text{cf}_{\mathbf{H}}$. Then, for all κ -words u and v over DRH, the equality $u = v$ holds if and only if $\text{cf}(u) \equiv \text{cf}(v)$.* \square

5. $\bar{\kappa}$ -terms seen as well-parenthesized words

In Section 3, we characterized \mathcal{R} -classes over DRH by means of certain equivalence classes of automata. In order to solve the κ -word problem over DRH, the next goal is to find an algorithm to construct such automata, which we do in Section 6. This section serves the purpose of preparing this construction.

5.1. General definitions

425 Let B be a possibly infinite alphabet and consider the associated alphabet $B_{[]} = B \uplus \{[{}^q,]^q : q \in \mathbb{Z}\}$. We say that a word in $B_{[]}^*$ is *well-parenthesized over B* if it does not contain $[{}^q,]^q$ as a factor and if it can be reduced to the empty word ε by applying the rewriting rules $[{}^q,]^q \rightarrow \varepsilon$ and $a \rightarrow \varepsilon$, for $q \in \mathbb{Z}$ and $a \in B$. We denote the set of all well-parenthesized words over B by $\text{Dyck}(B)$. The *content* of a well-parenthesized word x is the set of letters in B that occur in x and it is denoted $c(x)$.

To each $\bar{\kappa}$ -term over A we may associate a well-parenthesized word over A inductively as follows:

$$\begin{aligned} \text{word}(I) &= \varepsilon; \\ \text{word}(a) &= a, \quad \text{if } a \in A; \\ \text{word}(u \cdot v) &= \text{word}(u)\text{word}(v), \quad \text{if } u \text{ and } v \text{ are } \bar{\kappa}\text{-terms}; \\ \text{word}(u^{\omega+q}) &= [{}^q\text{word}(u)]^q, \quad \text{if } u \text{ is a } \bar{\kappa}\text{-term.} \end{aligned}$$

Conversely, we associate a κ -word to each well-parenthesized word over A as follows:

$$\begin{aligned} \text{om}(\varepsilon) &= I; \\ \text{om}(a) &= a, \quad \text{if } a \in A; \\ \text{om}(xy) &= \text{om}(x) \cdot \text{om}(y), \quad \text{if } x, y \in \text{Dyck}(A); \\ \text{om}([{}^q x]^q) &= \text{om}(x)^{\omega+q}, \quad \text{if } x \in \text{Dyck}(A). \end{aligned}$$

Note that, due to the associative property in both $\text{Dyck}(A)$ and $\Omega_A^\kappa \mathcal{S}$, $\text{om}(_)$ is well-defined. With the aim of distinguishing the occurrences of each letter in A in a well-parenthesized word x over A , we assign to each $x \in \text{Dyck}(A)$ a well-parenthesized word $x_{\mathbb{N}}$ over $A \times \mathbb{N}$ containing all the information about the position of the letters. With that in mind we define recursively the following family of functions $\{p_k : \text{Dyck}(A) \rightarrow \text{Dyck}(A \times \mathbb{N})\}_{k \geq 0}$:

$$\begin{aligned} p_k(a) &= (a, k+1), \quad \text{if } a \in A; \\ p_k([{}^q _]^q) &= [{}^q \text{ and } p_k(_]^q], \quad \text{if } q \in \mathbb{Z}; \\ p_k(ay) &= p_k(a)p_{k+1}(y), \quad \text{if } a \in A_{[\]} \text{ and } y \in A_{[\]}^*. \end{aligned}$$

We define $x_{\mathbb{N}} = p_0(x)$. For instance, if $x = a[{}^q b[{}^r c]^r]^q$, then $x_{\mathbb{N}}$ is the word $(a, 1)[{}^q (b, 2)[{}^r (c, 3)]^r]^q$. It is often convenient to denote the pair (a, i) by a_i . Let $x \in \text{Dyck}(A \times \mathbb{N})$. Then, we may associate to x two well-parenthesized words $\pi_A(x)$ and $\pi_{\mathbb{N}}(x)$ corresponding to the projection of x onto $A_{[\]}^*$ and onto $\mathbb{N}_{[\]}^*$, respectively. We denote $c_A(x) = c(\pi_A(x))$ and $c_{\mathbb{N}}(x) = c(\pi_{\mathbb{N}}(x))$. Given a $\bar{\kappa}$ -term w , we denote by \bar{w} the well-parenthesized word $(\$, 0)\text{word}(w\#)_{\mathbb{N}}$ over the alphabet $(A \uplus \{\$, \#\}) \times \mathbb{N}$. For instance, if $w = a(bc^{\omega+r})^{\omega+q}$, then $\text{word}(w)$ is $x = a[{}^q b[{}^r c]^r]^q$ as given above, and we have $\bar{x} = (\$, 0)(a, 1)[{}^q (b, 2)[{}^r (c, 3)]^r]^q(\#, 4)$, or alternatively, $\bar{x} = \$_0[{}^q b_2[{}^r c_3]^r]^q \#_4$. The map $\eta : \text{Dyck}(A \times \mathbb{N}) \rightarrow \Omega_A^\kappa \mathcal{S}$ assigns to each well-parenthesized word $x \in \text{Dyck}(A \times \mathbb{N})$ the κ -word $\eta(x) = \text{om}(\pi_A(x))$.

Let x be a well-parenthesized word over $A \times \mathbb{N}$. We define its *tail* $\mathbf{t}_i(x)$ from position $i \in \mathbb{N}$ inductively as follows

$$\begin{aligned} \mathbf{t}_i(\varepsilon) &= \varepsilon; \\ \mathbf{t}_i(yz) &= \mathbf{t}_i(z), \quad \text{if } y, z \in \text{Dyck}(A \times \mathbb{N}) \text{ and } i \notin c_{\mathbb{N}}(y); \\ \mathbf{t}_i(a_i y) &= y, \quad \text{if } y \in \text{Dyck}(A \times \mathbb{N}); \\ \mathbf{t}_i([{}^q y]^q z) &= \mathbf{t}_i(y)[{}^{q-1} y]^{q-1} z, \quad \text{if } y, z \in \text{Dyck}(A \times \mathbb{N}) \text{ and } i \in c_{\mathbb{N}}(y). \end{aligned}$$

The *prefix* of $x \in \text{Dyck}(A \times \mathbb{N})$ until $a \in A$ is defined by

$$\begin{aligned} \mathbf{p}_a(\varepsilon) &= \varepsilon; \\ \mathbf{p}_a(yz) &= y\mathbf{p}_a(z), \quad \text{if } y, z \in \text{Dyck}(A \times \mathbb{N}) \text{ and } a \notin c_A(y); \\ \mathbf{p}_a(a_i y) &= \varepsilon, \quad \text{if } y \in \text{Dyck}(A \times \mathbb{N}); \\ \mathbf{p}_a([{}^q y]^q z) &= \mathbf{p}_a(y), \quad \text{if } y, z \in \text{Dyck}(A \times \mathbb{N}) \text{ and } a \in c_A(y). \end{aligned}$$

The *factor* of a well-parenthesized word $x \in \text{Dyck}(A \times \mathbb{N})$ from $i \in \mathbb{N}$ until $a \in A$ is given by

$$x(i, a) = \mathbf{p}_a(\mathbf{t}_i(x)).$$

If instead, we are given a $\bar{\kappa}$ -term w , then we write $w(i, a)$ to mean the κ -word $\eta(\bar{w}(i, a))$. If a is a letter occurring in $\pi_A(x)$, for a well-parenthesized word x over $A \times \mathbb{N}$, then it is possible to write $x = ya_iz$ with y and z possibly empty but not necessarily well-parenthesized words over $A \times \mathbb{N}$ such that $a \notin c_A(y)$. In this case we say that a_i is a *marker of x* . If a_i is the last first occurrence of a letter, that is, if the inclusion $c_A(z) \subseteq c_A(ya_i)$ holds, then we say that a_i is the *principal marker of x* .

5.2. Properties of tails and prefixes of well-parenthesized words

The next results state some properties concerning tails and prefixes of well-parenthesized words. Some of the proofs are omitted since they are rather technical and entirely similar to the proofs of the analogous results in [28]. When that is the case, we refer the reader to the corresponding result.

Lemma 5.1 (cf. [28, Lemma 5.3]). *Let $x \in \text{Dyck}(A \times \mathbb{N})$ and let $a, b \in A$. If $b \in c_A(\mathfrak{p}_a(x))$, then*

$$\mathfrak{p}_b(\mathfrak{p}_a(x)) = \mathfrak{p}_b(x).$$

Lemma 5.2 (cf. [28, Lemma 5.4]). *Let $x \in \text{Dyck}(A \times \mathbb{N})$ be such that a belongs to $c_A(x)$. If $k \in c_{\mathbb{N}}(\mathfrak{p}_a(x))$, then $a \in c_A(\mathfrak{t}_k(x))$.*

Lemma 5.3 (cf. [28, Lemma 5.5]). *Let $x \in \text{Dyck}(A \times \mathbb{N})$ and let $k \in c_{\mathbb{N}}(\mathfrak{p}_a(x))$. Then, we have*

$$\mathfrak{t}_k(\mathfrak{p}_a(x)) = \mathfrak{p}_a(\mathfrak{t}_k(x)).$$

In the next result we explicitly describe the outcome of computing a tail of a well-parenthesized word with nested parentheses. The element $\mu_n(\vec{x}, \vec{y}, \vec{q})$ defined in the statement describes the general form of a word with n pairs of nested parentheses.

Lemma 5.4. *Let $\vec{x} = (x_j)_{j \geq 0}$ and $\vec{y} = (y_j)_{j \geq 0}$ be two sequences of possibly empty well-parenthesized words over $A \times \mathbb{N}$ such that $x_0y_0 \neq \varepsilon$, and for every $i, j \geq 0$, the index i occurs in $\pi_{\mathbb{N}}(x_0y_0x_1y_1 \cdots x_jy_j)$ at most once. Let $\vec{q} = (q_j)_{j \geq 0}$ be a sequence of integers. For each $n \geq 0$, we define the well-parenthesized words $\mu_n(\vec{x}, \vec{y}, \vec{q})$ and $\xi_n(\vec{x}, \vec{y}, \vec{q})$ as follows:*

$$\begin{aligned} \mu_0(\vec{x}, \vec{y}, \vec{q}) &= x_0y_0 \\ \mu_{n+1}(\vec{x}, \vec{y}, \vec{q}) &= x_{n+1}[\mu_n(\vec{x}, \vec{y}, \vec{q})]^{q_n}y_{n+1}, \text{ if } n \geq 0 \\ \xi_n(\vec{x}, \vec{y}, \vec{q}) &= [\mu_n(\vec{x}, \vec{y}, \vec{q})]^{q_n}y_{n+1}, \text{ if } n \geq 0. \end{aligned}$$

When the triple $(\vec{x}, \vec{y}, \vec{q})$ is clear from the context we simply write μ_n and ξ_n .

Let i be a natural number and suppose that $i \in c_{\mathbb{N}}(x_\ell y_\ell)$ for a certain $\ell \geq 0$. Then, for every $n \geq \ell$, the following equality holds:

$$\mathfrak{t}_i(\mu_n) = \mathfrak{t}_i(\mu_\ell) \cdot \xi_\ell \cdot \xi_{\ell+1} \cdots \xi_{n-1}. \quad (7)$$

Proof. We argue by induction on n . If $n = \ell$, then the result holds clearly, since the factor $\xi_\ell \cdot \xi_{\ell+1} \cdots \xi_{n-1}$ vanishes in (7). Suppose that $n > \ell$ and that the result holds for any smaller n . We may compute

$$\begin{aligned} \mathfrak{t}_i(\mu_n) &= \mathfrak{t}_i(x_n[\mu_{n-1}]^{q_{n-1}}y_n) \\ &= \mathfrak{t}_i(\mu_{n-1}) \cdot [\mu_{n-1}]^{q_{n-1}-1}y_n \quad \text{since } i \notin c_{\mathbb{N}}(x_n) \text{ and } i \in c_{\mathbb{N}}(\mu_{n-1}) \\ &= \mathfrak{t}_i(\mu_{n-1}) \cdot \xi_{n-1} \\ &= \mathfrak{t}_i(\mu_\ell) \cdot \xi_\ell \cdots \xi_{n-2} \cdot \xi_{n-1} \quad \text{by induction hypothesis} \end{aligned}$$

obtaining the desired equality (7). □

By successively applying Lemma 5.4, we obtain the following:

Corollary 5.5. *Using the same notation and assuming the same hypothesis as in the previous lemma, suppose that $k \in c_{\mathbb{N}}(y_0)$. Then,*

(a) if $i \in c_{\mathbb{N}}(x_\ell)$ for a certain $\ell \geq 0$, then the equality

$$\mathbf{t}_k(\mathbf{t}_i(\mu_n)) = \mathbf{t}_k(y_0) \cdot \xi_0 \cdot \xi_1 \cdots \xi_{n-2} \cdot \xi_{n-1}$$

holds for every $n \geq \ell$;

(b) if $i \in c_{\mathbb{N}}(y_\ell)$ for a certain $\ell \geq 1$, then the equality

$$\mathbf{t}_k(\mathbf{t}_i(\mu_n)) = \mathbf{t}_k(y_0) \cdot \xi_0 \cdot \xi_1 \cdots \xi_{\ell-1} \cdot [\mu_\ell]^{q_\ell-2} y_{\ell+1} \cdot \xi_{\ell+1} \cdots \xi_{n-1}$$

holds for every $n \geq \ell$. □

460 The reader may wish to compare the next result with [28, Lemma 5.8]. While in (a) we obtain a similar result, in (b) we do not obtain in general an equality, but only an \mathcal{R} -equivalence. In the case where \mathbf{H} is the trivial pseudovariety, and so $\text{DRH} = \mathbf{R}$, we get an equality on (b) thus recovering [28, Lemma 5.8].

Lemma 5.6. *Let w be a $\bar{\kappa}$ -term, $i \geq 0$, and $a \in c(w) \uplus \{\#\}$. Assume that b_k is the principal marker of $\bar{w}(i, a)$. Then, the following properties hold:*

465 (a) $\mathbf{p}_b(\bar{w}(i, a)) = \bar{w}(i, b)$;

(b) DRH satisfies $\eta(\mathbf{t}_k(\bar{w}(i, a))) \mathcal{R} w(k, a)$.

Moreover, if the projection of $w(i, a)$ onto $\bar{\Omega}_A \text{DRH}$ is not regular, then the relation in (b) becomes an equality in $\bar{\Omega}_A \mathbf{S}$.

470 *Proof.* By definition, we have $\bar{w}(i, a) = \mathbf{p}_a(\mathbf{t}_i(\bar{w}))$. Since $b \in c_A(\bar{w}(i, a))$, it follows from Lemma 5.1 that $\mathbf{p}_b(\bar{w}(i, a)) = \mathbf{p}_b(\mathbf{p}_a(\mathbf{t}_i(\bar{w}))) = \mathbf{p}_b(\mathbf{t}_i(\bar{w})) = \bar{w}(i, b)$.

Let us prove the second assertion. By definition of \bar{w} , we know that b_k appears exactly once in \bar{w} and the same happens with the index i . Let $\bar{w} = x \cdot b_k \cdot y$. We distinguish the cases where x and y are both possibly empty well-parenthesized words and where neither of x nor y is a well-parenthesized word. In the first case, since $b_k \in c(\bar{w}(i, a)) \subseteq c(\mathbf{t}_i(\bar{w}))$, the index i must belong to $c_{\mathbb{N}}(x)$. So, we get $\mathbf{t}_k(\bar{w}(i, a)) = \mathbf{t}_k(\mathbf{p}_a(\mathbf{t}_i(\bar{w}))) = \mathbf{t}_k(\mathbf{p}_a(\mathbf{t}_i(x)b_k y))$. Should a occur in $\mathbf{t}_i(x)b_k$, then b_k would not appear in $\bar{w}(i, a)$. So, it follows that

$$\mathbf{t}_k(\mathbf{p}_a(\mathbf{t}_i(x)b_k y)) = \mathbf{t}_k(\mathbf{t}_i(x)b_k \mathbf{p}_a(y)) = \mathbf{p}_a(y). \quad (8)$$

On the other hand, we have the equalities $\bar{w}(k, a) = \mathbf{p}_a(\mathbf{t}_k(\bar{w})) = \mathbf{p}_a(y) \stackrel{(8)}{=} \mathbf{t}_k(\bar{w}(i, a))$, and so the desired relation (b) follows.

Now, we suppose that

$$\begin{aligned} x &= x_n [x_{n-1}^{q_{n-1}} \cdots [x_1^{q_0} x_0, \\ b_k y &= y_0]^{q_0} y_1]^{q_1} \cdots y_{n-1}]^{q_{n-1}} y_n, \end{aligned}$$

where all the x_j 's and y_j 's are possibly empty well-parenthesized words, for $j = 0, \dots, n$. We note that, since $k \in c_{\mathbb{N}}(\bar{w}(i, a)) = c_{\mathbb{N}}(\mathbf{p}_a(\mathbf{t}_i(\bar{w})))$, Lemma 5.3 yields the equalities

$$\mathbf{t}_k(\bar{w}(i, a)) = \mathbf{t}_k(\mathbf{p}_a(\mathbf{t}_i(\bar{w}))) = \mathbf{p}_a(\mathbf{t}_k(\mathbf{t}_i(\bar{w}))). \quad (9)$$

With that in mind, we start by computing the elements $\mathbf{t}_k(\bar{w})$ and $\mathbf{t}_k(\mathbf{t}_i(\bar{w}))$. Let

$$\begin{aligned} \vec{x} &= (x_0, x_1, \dots, x_n, \varepsilon, \varepsilon, \dots); \\ \vec{y} &= (y_0, y_1, \dots, y_n, \varepsilon, \varepsilon, \dots); \\ \vec{q} &= (q_0, q_1, \dots, q_{n-1}, 0, 0, \dots). \end{aligned}$$

and let $\ell \in \{0, 1, \dots, n\}$ be such that $i \in c_{\mathbb{N}}(x_{\ell}y_{\ell})$. Observe that, since $\bar{w} = \mu_n = \mu_n(\vec{x}, \vec{y}, \vec{q})$ is a well-parenthesized word, the sequences \vec{x} and \vec{y} satisfy the hypothesis of Lemma 5.4. Noticing that k belongs to $c_{\mathbb{N}}(y_0)$, we may apply Lemma 5.4 and obtain

$$\mathbf{t}_k(\bar{w}) = \mathbf{t}_k(\mu_0) \cdot \xi_0 \cdot \xi_1 \cdots \xi_{n-1} = \mathbf{t}_k(y_0) \cdot \xi_0 \cdot \xi_1 \cdots \xi_{n-1} \quad (10)$$

Now, we have two possible situations.

- (i) $i \in c_{\mathbb{N}}(x_{\ell})$, for a certain $\ell \in \{0, \dots, n\}$;
- 475 (ii) $i \in c_{\mathbb{N}}(y_{\ell})$, for a certain $\ell \in \{n, \dots, 1\}$.

If we are in Case (i), then we may use Corollary 5.5(a) and get

$$\mathbf{t}_k(\mathbf{t}_i(\bar{w})) = \mathbf{t}_k(y_0) \cdot \xi_0 \cdot \xi_1 \cdots \xi_{n-2} \cdot \xi_{n-1}.$$

Hence, we have an equality between $\mathbf{t}_k(\bar{w}(i, a)) = \mathbf{p}_a(\mathbf{t}_k(\mathbf{t}_i(\bar{w})))$ and $\bar{w}(k, a) = \mathbf{p}_a(\mathbf{t}_k(\bar{w}))$, thereby proving (b).

On the other hand, when the situation occurring is (ii), Corollary 5.5(b) yields

$$\mathbf{t}_k(\mathbf{t}_i(\bar{w})) = \mathbf{t}_k(y_0) \cdot \xi_0 \cdot \xi_1 \cdots \xi_{\ell-1} \cdot [\mu_{\ell}]^{q_{\ell}-2} y_{\ell+1} \cdot \xi_{\ell+1} \cdots \xi_{n-1}.$$

If the first occurrence of a in $\mathbf{t}_k(\mathbf{t}_i(\bar{w}))$ is in $\mathbf{t}_k(y_0) \cdot \xi_0 \cdot \xi_1 \cdots \xi_{\ell-1}$ or in μ_{ℓ} , then the first occurrence of a in $\mathbf{t}_k(\bar{w})$ is also in one of these factors and we easily conclude that

$$\mathbf{p}_a(\mathbf{t}_k(\mathbf{t}_i(\bar{w}))) = \mathbf{p}_a(\mathbf{t}_k(y_0) \cdot \xi_0 \cdot \xi_1 \cdots \xi_{\ell-1} \cdot \mu_{\ell}) = \mathbf{p}_a(\mathbf{t}_k(\bar{w})),$$

thereby proving again an equality in (b).

Otherwise, the first occurrence of a in $\mathbf{t}_k(\mathbf{t}_i(\bar{w}))$ is in $y_{\ell+1} \cdot \xi_{\ell+1} \cdots \xi_{n-1}$. Analyzing the equality (10), we deduce that a occurs for the first time in $\mathbf{t}_k(\bar{w})$ also in the factor $y_{\ell+1} \cdot \xi_{\ell+1} \cdots \xi_{n-1}$. Then, we may compute

$$\mathbf{p}_a(\mathbf{t}_k(\mathbf{t}_i(\bar{w}))) = \mathbf{t}_k(y_0) \cdot \xi_0 \cdot \xi_1 \cdots \xi_{\ell-1} \cdot [\mu_{\ell}]^{q_{\ell}-2} \cdot \mathbf{p}_a(y_{\ell+1} \cdot \xi_{\ell+1} \cdots \xi_{n-1}) \quad (11)$$

$$\mathbf{p}_a(\mathbf{t}_k(\bar{w})) = \mathbf{t}_k(y_0) \cdot \xi_0 \cdot \xi_1 \cdots \xi_{\ell-1} \cdot [\mu_{\ell}]^{q_{\ell}-1} \cdot \mathbf{p}_a(y_{\ell+1} \cdot \xi_{\ell+1} \cdots \xi_{n-1}). \quad (12)$$

Moreover, using again Lemma 5.4, we obtain

$$\begin{aligned} \bar{w}(i, a) &= \mathbf{p}_a(\mathbf{t}_i(\bar{w})) = \mathbf{p}_a(\mathbf{t}_i(\mu_n)) = \mathbf{p}_a(\mathbf{t}_i(y_{\ell}) \cdot \xi_{\ell} \cdot \xi_{\ell+1} \cdots \xi_{n-1}) = \mathbf{t}_i(y_{\ell}) [\mu_{\ell}]^{q_{\ell}-1} \cdot \mathbf{p}_a(y_{\ell+1} \xi_{\ell+1} \cdots \xi_{n-1}) \\ &= \mathbf{t}_i(y_{\ell}) [x_{\ell}]^{q_{\ell}-1} x_{\ell} [x_{\ell-1}]^{q_{\ell-2}} \cdots [x_0]^{q_0} y_0 \cdots y_{\ell-1} [\mu_{\ell}]^{q_{\ell}-1} y_{\ell} \cdot \mathbf{p}_a(y_{\ell+1} \xi_{\ell+1} \cdots \xi_{n-1}) \end{aligned} \quad (13)$$

Since b_k is the principal marker of $\bar{w}(i, a)$, we know that the following inclusion holds:

$$c_A(y_0 y_1 \cdots y_{\ell} \cdot \mathbf{p}_a(y_{\ell+1} \cdot \xi_{\ell+1} \cdots \xi_{n-1})) \subseteq c_A(\mathbf{t}_i(y_{\ell}) x_{\ell} \cdots x_0 b_k).$$

Also, by definition of μ_{ℓ} , we have an inclusion $c_A(\mathbf{t}_i(y_{\ell}) x_{\ell} \cdots x_0 b_k) \subseteq c_A(\mu_{\ell})$. Consequently, we obtain $c_A(\mathbf{p}_a(y_{\ell+1} \cdot \xi_{\ell+1} \cdots \xi_{n-1})) \subseteq c_A(\mu_{\ell})$. Observing that

$$\vec{c}(\eta([\mu_{\ell}]^{q_{\ell}-2})) = c(\eta(\mathbf{p}_a(y_{\ell+1} \cdot \xi_{\ell+1} \cdots \xi_{n-1}))), \quad (14)$$

we end up with the desired relations, which are valid in DRH:

$$\begin{aligned} \eta(\mathbf{t}_k(\bar{w}(i, a))) &\stackrel{(9),(11)}{=} \eta(\mathbf{t}_k(y_0) \cdot \xi_0 \cdot \xi_1 \cdots \xi_{\ell-1}) \cdot \eta([\mu_{\ell}]^{q_{\ell}-2}) \cdot \eta(\mathbf{p}_a(y_{\ell+1} \cdot \xi_{\ell+1} \cdots \xi_{n-1})) \\ &\stackrel{(14)}{\mathcal{R}} \eta(\mathbf{t}_k(y_0) \cdot \xi_0 \cdot \xi_1 \cdots \xi_{\ell-1}) \cdot \eta(\mu_{\ell})^{\omega+q_{\ell}-1} \cdot \eta(\mathbf{p}_a(y_{\ell+1} \cdot \xi_{\ell+1} \cdots \xi_{n-1})) \\ &\stackrel{(12)}{=} \eta(\bar{w}(k, a)) = w(k, a). \end{aligned}$$

480 We finally observe that we actually proved an equality in $\bar{\Omega}_A \mathbf{S}$ rather than a relation modulo DRH, except in the last situation. But that scenario only occurs when $w(i, a)$ is regular modulo DRH. Indeed, since $b_k \in c(y_0)$ is the principal marker of $w(i, a)$, from the equality (13), we may deduce that $\vec{c}(w(i, a)) = c(w(i, a))$, which by Proposition 2.4 implies that $\rho_{\text{DRH}}(w(i, a))$ is regular. \square

The well-parenthesized words we are interested in are those of the form $\bar{w}(i, a)$ for some $\bar{\kappa}$ -term w . Such words have the property that whenever a_i, b_i are letters of \bar{w} , a and b are necessarily the same letter. That is easily seen as follows: clearly that is the case for $\bar{w}(0, \#)$ by definition, and computing tails and prefixes preserves that property. Based on that, for a well-parenthesized word x over $A \times \mathbb{N}$, we consider the following property:

$$\forall a, b \in A, \quad \forall i \in \mathbb{N}, \quad a_i, b_i \in c(x) \implies a = b \quad (H(x))$$

The proof of the next result may be easily adapted from the proof of [28, Lemma 5.9]. The main difference is that, since in the aperiodic case \mathbf{R} the identity $z^\omega = z^{\omega+1}$ is valid for every pseudoword z we do not need to distinguish the brackets $[^a _]^q$ for different values of q . That is not the case for pseudovarieties of the form DRH in general, and so that should be taken into account when computing elements of the form $\mathbf{t}_i(x)$.

Lemma 5.7. *Let $x \in \text{Dyck}(A \times \mathbb{N}) \setminus \{\varepsilon\}$ satisfy $(H(x))$ and suppose that a_i is a marker of x . Then the equality $\eta(x) = \eta(\mathbf{p}_a(x) \cdot a_i \cdot \mathbf{t}_i(x))$ holds.*

Corollary 5.8. *Let w be a $\bar{\kappa}$ -term. Let $i \in \mathbb{N}$ and $a \in A \uplus \{\#\}$, and let b_k be the principal marker of $\bar{w}(i, a)$. Suppose that $\text{lbf}(w(i, a)) = (w_\ell, m, w_r)$. Then, $m = b$ and DRH satisfies $w_\ell = w(i, b)$, and $w_r \mathcal{R} w(k, a)$. Moreover, if $\rho_{\text{DRH}}(w(i, a))$ is not regular, then $\text{lbf}(w(i, a)) = (w(i, b), b, w(k, a))$.*

Proof. As b_k is the principal marker of $\bar{w}(i, a)$, we can write $\bar{w}(i, a) = xb_ky$, where $c_A(y) \subseteq c_A(xb_k)$ and $b \notin c_A(x)$. Since $(H(\bar{w}(i, a)))$ holds, Lemma 5.7 yields

$$\eta(\bar{w}(i, a)) = \eta(\mathbf{p}_b(\bar{w}(i, a)) \cdot b_k \cdot \mathbf{t}_k(\bar{w}(i, a))) = \eta(\mathbf{p}_b(\bar{w}(i, a))) \cdot b \cdot \eta(\mathbf{t}_k(\bar{w}(i, a))).$$

Furthermore, since $b \notin c_A(x)$, we also have $c_A(\mathbf{p}_b(\bar{w}(i, a))) = c_A(x)$ and consequently, the left basic factorization of $w(i, a)$ is precisely

$$(\eta(\mathbf{p}_b(\bar{w}(i, a))), b, \eta(\mathbf{t}_k(\bar{w}(i, a)))).$$

In particular, we have $m = b$ and, by Lemma 5.6, the pseudovariety DRH satisfies $w_\ell = w(i, b)$ and $w_r \mathcal{R} w(k, a)$, with an equality in \mathbf{S} in the latter relation when $w(i, a)$ is not regular modulo DRH. \square

6. DRH-graphs and their computation

We begin this section with the definition of a DRH-graph. Through these structures, we are able to decide whether two κ -words are \mathcal{R} -equivalent over DRH. If we further assume that the word problem is decidable in $\Omega_A^\kappa \mathbf{H}$, then the word problem is decidable in $\Omega_A^\kappa \text{DRH}$ as well.

Definition 6.1. *Let w be a $\bar{\kappa}$ -term. The DRH-graph of w is the finite DRH-automaton*

$$\mathcal{G}(w) = \langle V(w), \rightarrow, \mathbf{v}(0, \#), \{\varepsilon\}, \lambda_{\mathbf{H}}, \lambda \rangle,$$

defined as follows. The set of states is

$$V(w) = \{\mathbf{v}(i, a) : 0 \leq i < |\bar{w}|, a \in c_A(\bar{w}) \text{ and } w(i, a) \neq I\} \uplus \{\varepsilon\}.$$

Let $\mathbf{v}(i, a) \in V(w) \setminus \{\varepsilon\}$ and b_k be the principal marker of $\bar{w}(i, a)$. The transitions of $\mathbf{v}(i, a)$ are $\mathbf{v}(i, a).0 = \mathbf{v}(i, b)$ and $\mathbf{v}(i, a).1 = \mathbf{v}(k, a)$. The labels are $\lambda_{\mathbf{H}}(\mathbf{v}(i, a)) = \rho_{\mathbf{H}}(\text{reg}(w(i, b)))$ and $\lambda(\mathbf{v}(i, a)) = b$. If a state $\mathbf{v}(i, a)$ is not reached from the root $\mathbf{v}(0, \#)$, then we discard it from $V(w)$.

The following result suggests that the construction of $\mathcal{G}(w)$ might be a starting point to solve the κ -word problem over DRH algorithmically.

Proposition 6.2. *For every $\bar{\kappa}$ -term w , $\mathcal{G}(w)$ is a DRH-automaton equivalent to $\mathcal{T}(w(0, \#))$.*

Proof. Let $\mathcal{T}(w(0, \#)) = \langle V, \rightarrow_{\mathcal{T}}, \mathbf{v}^0, F, \lambda_{\mathcal{T}, \mathbb{H}}, \lambda_{\mathcal{T}} \rangle$ and $\mathcal{G}(w) = \langle V(w), \rightarrow_{\mathcal{G}}, \mathbf{v}(0, \#), \{\varepsilon\}, \lambda_{\mathcal{G}, \mathbb{H}}, \lambda_{\mathcal{G}} \rangle$. We first claim that, for every $\alpha \in \Sigma^*$, we have

$$\mathbf{v}(0, \#).\alpha = \mathbf{v}(i, a) \implies \mathcal{T}(w(0, \#))_{\mathbf{v}^0.\alpha} = \mathcal{T}(w(i, a)). \quad (15)$$

To prove this, we argue by induction on $|\alpha|$. If $|\alpha| = 0$, then the result holds trivially. Let $\alpha \in \Sigma^*$ be such that $|\alpha| \geq 1$ and suppose that the result holds for every other shorter word α . We can write $\alpha = \beta\gamma$, with $\gamma \in \{0, 1\}$. Let $\mathbf{v}(0, \#).\beta = \mathbf{v}(i, a)$. By induction hypothesis, it follows that $\mathcal{T}(w(0, \#))_{\mathbf{v}^0.\beta} = \mathcal{T}(w(i, a))$. Let b_k be the principal marker of $\bar{w}(i, a)$. By definition of $\mathcal{G}(w)$, we have $\mathbf{v}(0, \#).\beta 0 = \mathbf{v}(i, b)$ and $\mathbf{v}(0, \#).\beta 1 = \mathbf{v}(k, a)$. On the other hand, Lemma 3.9 gives that if $\text{lbf}(w(i, a)) = (w_\ell, b, w_r)$, then $\mathcal{T}(w(i, a)) = (\mathcal{T}(w_\ell), \text{reg}(w_\ell) \mid b, \mathcal{T}(w_r))$, which in turn, by Corollary 5.8, is equivalent to

$$\mathcal{T}(w(i, a)) = (\mathcal{T}(w(i, b)), \text{reg}(w(i, b)) \mid b, \mathcal{T}(w(k, a))). \quad (16)$$

In particular, we conclude that $\mathcal{T}(w(0, \#))_{\mathbf{v}^0.\beta 0} = \mathcal{T}(w(i, b))$ and $\mathcal{T}(w)_{\mathbf{v}^0.\beta 1} = \mathcal{T}(w(k, a))$. It is now enough to notice that, for each pair $(i, a) \in [0, |\bar{w}|[\times c_A(\bar{w})$, the labels of the node $\mathbf{v}(i, a)$ of $\mathcal{G}(w)$ and the labels of the root of $\mathcal{T}(w(i, a))$ coincide. In fact, if b_k is the principal marker of $\bar{w}(i, a)$, then the construction of $\mathcal{G}(w)$ yields the equalities $\lambda_{\mathcal{G}}(\mathbf{v}(i, a)) = b$ and $\lambda_{\mathcal{G}, \mathbb{H}}(\mathbf{v}(i, a)) = \rho_{\mathbb{H}}(\text{reg}(w(i, b)))$, which, by (16), are precisely the labels of the root of $\mathcal{T}(w(i, a))$. \square

Consider a $\bar{\kappa}$ -term w . We may assume that w is given by a labeled tree. For instance, if

$$w = (((b^{\omega-2}) \cdot a) \cdot c) \cdot (((a \cdot b) \cdot (a^{\omega+1}))^{\omega-1}),$$

then the tree representing w is depicted in Figure 5. Since from such a tree representation we may compute

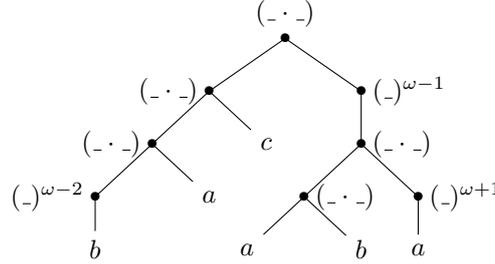


Figure 5: The tree representing $(((((b^{\omega-2}) \cdot a) \cdot c) \cdot (((a \cdot b) \cdot (a^{\omega+1}))^{\omega-1}))$.

\bar{w} in linear time, we assume that we are already given \bar{w} . We say that the *length* of a given $\bar{\kappa}$ -term w , denoted $|w|$, is the number of nodes of its tree representation. It is clear that $O(|w|) = O(|\bar{w}|)$. To actually compute the DRH-graph $\mathcal{G}(w)$ we essentially need to compute the principal marker of the words $\bar{w}(i, a)$ as well as the regular parts of $w(i, a)$. Almeida and Zeitoun [28] exhibited an algorithm to compute the first occurrences of each letter of a well-parenthesized word x . Given a word x , $\text{first}(x)$ consists of a list of the first occurrences of each letter in x . In particular, this computes the principal marker of x : it is the last entry of the outputted list. Moreover, if b_k is the principal marker of x , then the penultimate entry of the list is the principal marker of $\mathbf{p}_b(x)$, and so on. Hence, this is enough to almost compute $\mathcal{G}(w)$. More precisely, the knowledge of $\text{first}(\bar{w}(i, a))$, for every pair (i, a) , allows us to compute the reduct $\mathcal{G}_{\mathbb{R}}(w) = \langle V(w), \rightarrow, \mathbf{v}(0, \#), \{\varepsilon\}, \lambda \rangle$ in time $O(|w| |c(w)|)$.

Lemma 6.3 ([28, Lemma 5.15]). *Let w be a $\bar{\kappa}$ -term. Then, one may compute in time $O(|w| |c(w)|)$ a table giving, for each i such there exists $a_i \in c(\bar{w}) \cap A \times \mathbb{N}$, the word $\text{first}(\bar{w}(i, \#))$.*

It remains to find the labels of the states under $\lambda_{\mathbb{H}}$. For that purpose, we observe that the regular part of a pseudoword u depends deeply on the content of the factors of the form $\text{lbf}_k(u)$, which we may compute using Lemma 5.7; and of the cumulative content of u . Also, it follows from Lemma 3.7 and from

Proposition 6.2 that the cumulative content of any pseudoword of the form $w(i, a)$ is completely determined by the reduct $\mathcal{G}_R(w)$. Thus, we may start by computing the cumulative content of $w(i, a)$ and then compare it with the content of $\text{lbf}_k(w(i, a))$, for increasing values of k . When we achieve an equality, we know what is the regular part of $w(i, a)$. Algorithm 1 does that job. We assume that we already have the table described in Lemma 6.3, so that, computing $c(w(i, a))$ and the principal marker of $\bar{w}(i, a)$ takes $O(1)$ -time. Further, we may assume that we are given $\mathcal{G}_R(w)$, since we already explained how to get it from the table of Lemma 6.3 in $O(|w| |c(w)|)$ -time.

Algorithm 1

Input: A $\bar{\kappa}$ -term w and $(i, a) \in [0, |\bar{w}|[\times c_A(\bar{w})$ (with $\bar{w}(i, a) \neq \varepsilon$)

Output: $\text{reg}(w(i, a)) = I$, if $\bar{c}(w(i, a)) = \emptyset$ or k such that $\text{reg}(w(i, a)) = w(k, a)$, otherwise

```

1:  $L \leftarrow \{\}$ ,  $j \leftarrow i$ 
2: while  $j \notin L$  and  $\bar{w}(j, a) \neq \varepsilon$  do
3:    $L \leftarrow L \cup \{j\}$ 
4:    $j \leftarrow \pi_{\mathbb{N}}$ (principal marker of  $\bar{w}(j, a)$ )           ▷ So that, if  $v(j, a).1 \neq \varepsilon$ , then  $v(j, a) \leftarrow v(j, a).1$ 
5: end while
6: if  $\bar{w}(j, a) = \varepsilon$  then
7:   return  $I$ 
8: else
9:    $C \leftarrow c(w(j, a))$                                    ▷ The set  $C$  is the cumulative content of  $w(i, a)$ 
10:   $k \leftarrow i$ 
11:  while  $c_A(\bar{w}(k, a)) \neq C$  do
12:     $k \leftarrow \pi_{\mathbb{N}}$ (principal marker of  $\bar{w}(k, a)$ )
13:  end while
14:  return  $k$ 
15: end if

```

Lemma 6.4. *Algorithm 1 returns I if and only if $\bar{c}(w(i, a)) = \emptyset$. Otherwise, the value k outputted is such that $\text{reg}(w(i, a)) = w(k, a)$. Moreover, the algorithm runs in linear time, provided we have the knowledge of $\text{first}(w(i, a))$.*

Proof. By Property (A.3) of a DRH-automaton, and since there is only a finite number of possible states in $\mathcal{G}_R(w)_{v(i, a)}$, either there exists $k \geq 0$ such that $v(i, a).1^k = \varepsilon$, or there exists $\ell > k \geq 0$ such that $v(i, a).1^k = v(i, a).1^\ell$. Therefore, the **while** loop in line 2 does not run forever. If the occurring situation is the former, then $\bar{c}(\mathcal{G}(w)_{v(i, a)}) = \emptyset$. On the other hand, by Proposition 6.2, we have $\mathcal{G}(w)_{v(i, a)} \sim \mathcal{T}(w(i, a))$ which in turn, by Theorem 3.8, implies $\pi(\mathcal{G}(w)_{v(i, a)}) \mathcal{R} w(i, a)$ modulo DRH. Also, Lemma 3.7 yields $\bar{c}(w(i, a)) = \bar{c}(\mathcal{G}(w)_{v(i, a)}) = \emptyset$, and therefore, $\text{reg}(w(i, a)) = I$. This is the case where the symbol I is returned in line 7.

Now, suppose that $\ell > k \geq 0$ are such that $v(i, a).1^k = v(i, a).1^\ell$. Observe that, at the n -th running of the **while** loop in line 2, the value assigned to j in line 4 is the unique such that $v(i, a).1^n = v(j, a)$. Thus, in this case the **while** loop is exited after the ℓ -th iteration because j takes a value that is already in L . By Property (A.4), we have a chain of inclusions: $\lambda(\mathcal{G}(w)_{v(i, a).1^k}) \supseteq \lambda(\mathcal{G}(w)_{v(i, a).1^{k+1}}) \supseteq \dots \supseteq \lambda(\mathcal{G}(w)_{v(i, a).1^\ell})$. As $v(i, a).1^k = v(i, a).1^\ell$, these inclusions are actually equalities, implying that k is greater than or equal to $\text{r.ind}(\mathcal{G}(w)_{v(i, a)})$. Combining again Proposition 6.2, Theorem 3.8 and Lemma 3.7, we may deduce that $\bar{c}(w(i, a)) = \bar{c}(\mathcal{G}(w)_{v(i, a)}) = \lambda(\mathcal{G}(w)_{v(i, a).1^k})$, where the last member is precisely $c(w(j, a))$ provided that $v(i, a).1^k = v(j, a)$. Therefore, in line 9 we assign to C the cumulative content of $w(i, a)$. Until now, since we are assuming that we are given all the information about $\mathcal{G}_R(w)$, we only spend time $O(|w|)$, because that is the number of possible values of j that may appear in line 2.

Let us prove that, if we get to line 9, then the value k outputted in line 14 is such that $\text{reg}(w(i, a)) = w(k, a)$. We write

$$w(i, a) = \text{lbf}_1(w(i, a)) \cdots \text{lbf}_m(w(i, a))w'_m,$$

for every $m \geq 1$ (notice that $\text{lbf}_m(w(i, a))$ is defined for all $m \geq 1$ because we are assuming that $\bar{c}(w(i, a)) \neq \emptyset$). Then, the regular part of $w(i, a)$ is given by w'_ℓ , where $\ell = \min\{m : c(w'_m) = \bar{c}(w(i, a))\}$. In particular, the projection of w'_m onto $\bar{\Omega}_A\text{DRH}$ is not regular, for every $m < \ell$. Set $(c_0, k_0) = (a, i)$ and, for $m \geq 0$, let (c_{m+1}, k_{m+1}) be the principal marker of $\bar{w}(k_m, a)$. By Corollary 5.8, if $w(k_m, a)$ is not regular modulo DRH, then we have $\text{lbf}(w(k_m, a)) = (w(k_m, c_{m+1}), c_{m+1}, w(k_{m+1}, a))$. Therefore, the equality $w'_m = w(k_m, a)$ holds, for every $m \leq \ell$. Thus, the value k returned in line 14 is precisely k_ℓ , implying that $\text{reg}(w(i, a)) = w(k, a)$ as intended.

Since there are only $O(|\bar{w}|)$ possible values for k and we are assuming that we already know $\text{first}(w(i, \#))$ for all $i \in [0, |\bar{w}|[$, it follows that lines 8–15 run in time $O(|\bar{w}|)$.

Therefore, the overall time complexity of Algorithm 1 is $O(|w|)$. \square

So far, we possess all the needed information for computing $\mathcal{G}(w)$. Putting all the steps together, we obtain the following.

Theorem 6.5. *Given a $\bar{\kappa}$ -term w , it is possible to compute the DRH-graph of w in time $O(|w|^2 |c(w)|)$. \square*

The next question we should answer is how can we decide whether two DRH-graphs $\mathcal{G}(u)$ and $\mathcal{G}(v)$ represent the same \mathcal{R} -class of $\bar{\Omega}_A\text{DRH}$, that is, whether $\mathcal{G}(u) \sim \mathcal{G}(v)$. A possible strategy consists in visiting states in both DRH-graphs, comparing their labels (in a certain order). When we find a pair of mismatching labels, we stop, concluding that $\mathcal{G}(u)$ and $\mathcal{G}(v)$ are not equivalent. Otherwise, we conclude that they are equivalent after visiting all the states. More precisely, starting in the roots of $\mathcal{G}(u)$ and $\mathcal{G}(v)$, we mark the current states, say $v_u \in V(u)$ and $v_v \in V(v)$, as visited, and then repeat the process relatively to the pairs of DRH-automata $(\mathcal{G}(u)_{v_u.0}, \mathcal{G}(v)_{v_v.0})$ and $(\mathcal{G}(u)_{v_u.1}, \mathcal{G}(v)_{v_v.1})$. For a better understanding of the procedure, we sketch it in Algorithm 2.

Algorithm 2

Input: two DRH-graphs $\mathcal{G}_i = \langle V_i, \rightarrow_i, v_i^0, \lambda_{i,H}, \lambda_i \rangle$ ($i = 1, 2$)

Output: logical value of “ $\mathcal{G}_1 \sim \mathcal{G}_2$ ”

```

1: if  $v_1^0 = \varepsilon$  then
2:   return logical value of  $v_2^0 = \varepsilon$ 
3: else if  $v_1^0$  or  $v_2^0$  is unvisited then
4:   mark  $v_1^0$  and  $v_2^0$  as visited
5:   if  $\lambda_{1,H}(v_1^0) = \lambda_{2,H}(v_2^0)$  and  $\lambda_1(v_1^0) = \lambda_2(v_2^0)$  then
6:     return logical value of “ $(\mathcal{G}_1)_{v_1^0.0} \sim (\mathcal{G}_2)_{v_2^0.0}$  and  $(\mathcal{G}_1)_{v_1^0.1} \sim (\mathcal{G}_2)_{v_2^0.1}$ ”
7:   else
8:     return False
9:   end if
10: else
11:   return True
12: end if

```

Lemma 6.6. *Algorithm 2 returns the logical value of “ $\mathcal{G}_1 \sim \mathcal{G}_2$ ” for two input DRH-graphs \mathcal{G}_1 and \mathcal{G}_2 . Moreover, it runs in time $O(p \max\{|V_1|, |V_2|\})$, where p is a function of \mathcal{G}_1 and \mathcal{G}_2 (more precisely, of V_i and $\lambda_{i,H}$, for $i = 1, 2$) such that the word problem modulo H for any pair of labels $\lambda_{1,H}(v_1)$ and $\lambda_{2,H}(v_2)$ (with $v_1 \in V_1$ and $v_2 \in V_2$) may be solved in time $O(p)$.*

Proof. The correctness follows straightforwardly from the definition of the relation \sim . On the other hand, it runs in time $O(p \max\{|V_1|, |V_2|\})$, since each call of the algorithm takes time $O(p)$ (line 5) and each pair of states of the form $(v_1^0.\alpha, v_2^0.\alpha)$ is visited exactly once. \square

Given $\bar{\kappa}$ -terms u and v , we use $p(u, v)$ to denote a function depending on u and v such that the time for solving the word problem over H for any pair of factors of the form $u(i, a)$ and $v(j, b)$ is in $O(p(u, v))$. Note that such a function $p(u, v)$ is not unique, but the upcoming results are valid for any such function. Then,

summing up the time complexities of all the intermediate steps considered above, we have just proved the following.

Theorem 6.7. *Let \mathbf{H} be a pseudovariety of groups with decidable κ -word problem, and let u and v be $\bar{\kappa}$ -terms. Then, the equality of the pseudowords represented by u and v over DRH can be tested in time $O((p(u, v) + m)m |A|)$, where $m = \max\{|u|, |v|\}$. \square*

We remark that the real meaning of the time complexity exhibited in Theorem 6.7 is only well understood when we fix a pseudovariety of groups \mathbf{H} and estimate the parameter $p(u, v)$. In the extreme situation in which we take \mathbf{H} to be the trivial pseudovariety, our algorithm solves the κ -word problem in time $O(m^2 |A|)$. Of course, in this case, there exists an obvious algorithm that is more efficient, since we do not even need to run Algorithm 1 (this is the result of [28]). In general, the parameter $p(u, v)$ is expected to depend on the length of the intervening $\bar{\kappa}$ -terms, since we need at least to read u and v before deciding whether or not two certain factors represent the same κ -word over \mathbf{H} . Hence, m belongs to $O(p(u, v))$. Consequently, the overall time complexity stated in Theorem 6.7 becomes $O(p(u, v)m |A|)$. Since we are doing the same approach as in [28], this result is somehow the expected one. Roughly speaking, this may be interpreted as the time complexity of solving the κ -word problem in \mathbf{R} , together with a κ -word problem in \mathbf{H} for each state, that is, for each DRH-factor of the involved pseudowords (recall Lemmas 2.8 and 4.3).

Just as a complement, we mention that another possible approach would be to transform the DRH-graph $\mathcal{G}(w)$ in an automaton in the classical sense, say $\mathcal{G}'(w)$, recognizing the language $\mathcal{L}(w)$ (recall Proposition 3.11). That is easily done (time linear on the number of states), by moving the labels of a state to the arrows leaving it. More precisely, the automaton $\mathcal{G}'(w)$ shares the set of states with $\mathcal{G}(w)$ and each non terminal state $\mathbf{v}(i, a)$ has two transitions:

$$\begin{aligned} \mathbf{v}(i, a).(0, \lambda_{\mathbf{H}}(\mathbf{v}(i, a)), \lambda(\mathbf{v}(i, a))) &= \mathbf{v}(i, 1).0 \\ \mathbf{v}(i, a).(1, I, \lambda(\mathbf{v}(i, a))) &= \mathbf{v}(i, a).1. \end{aligned}$$

Then, we could use the results in the literature in order to minimize the automaton, obtaining a unique automaton representing each \mathcal{R} -class of the semigroup $(\bar{\Omega}_A \text{DRH})^I$. The unique issue in that approach is that the algorithms are usually prepared to deal with alphabets whose members may be compared in constant time. Hence, we should previously prepare the input automaton by renaming the subset of the alphabet $\Sigma \times (\bar{\Omega}_A \mathbf{H})^I \times A$, in which the labels of transitions are being considered. Let $p(u, v)$ and m have the same meaning as in Theorem 6.7. Since, a priori, we do not possess any information about the possible values for $\lambda_{\mathbf{H}}$, that would take $O(p(u, v)(m |A|)^2)$ -time (each time we rename an element in $(\bar{\Omega}_A \mathbf{H})^I$ we should first verify whether we already encountered another element with the same value over \mathbf{H}). Thereafter, we could use the linear time algorithm presented in [32], which works for this kind of automaton. Thus, a rough upper bound for the complexity spent using this method is $O(p(u, v)m^2 |A|^2)$, which although a bit worse, is still polynomial.

The following result gives us a family of pseudovarieties of the form DRH with decidable κ -word problem. It is a consequence of the fact that the free group is residually in \mathbf{G}_p .

Corollary 6.8. *Let p be a prime number. If $\mathbf{H} \supseteq \mathbf{G}_p$ is a pseudovariety of groups, then the pseudovariety DRH has decidable κ -word problem. \square*

7. An application: solving the word problem over DRG

Let us illustrate the previous results by considering the particular case of the pseudovariety DRG. By Theorem 6.7, the time complexity of our procedure for testing identities of $\bar{\kappa}$ -terms modulo DRG depends on a certain parameter $p(-, -)$. Recalling that $\Omega_A^{\kappa} \mathbf{G} = \text{FG}_A$, we have that the most natural first step to solve the κ -word problem over \mathbf{G} is to represent the $\bar{\kappa}$ -terms whose equality we wish to check by words over the alphabet $A \cup A^{-1}$. After that, solving the word problem in the free group is known to require only time linear on the size of the input. It turns out that $\bar{\kappa}$ -terms provide a much more compact representation of κ -words rather than κ -terms. For instance, for every $n > 0$, there is a $\bar{\kappa}$ -term of length 1, namely $a^{\omega^{-n}}$,

whose shortest representation as a κ -term has length $2n - 1$. Although we are not able to prove that there is not a more efficient algorithm than the one just suggested, it seems reasonable to require the input to be given by κ -terms. We do that assumption from hereon.

In order to discover the parameter $p(-, -)$, we should first analyze the (length of the) projection onto $\Omega_A^\kappa \mathbf{G} = \text{FG}_A$ of the elements of the form $w(i, a)$, where w is a κ -term.

Consider the alphabets $B_1 = (A \times \mathbb{N}) \uplus \{[-^1,]^{-1}\}$ and $B_2 = (A \times \mathbb{N}) \uplus \{[-^1, [-^2,]^{-1},]^{-2}\}$. Let x be a well-parenthesized word over B_2 . The *expansion* of x is the well-parenthesized word $\text{exp}(x)$ obtained by successively applying the rewriting rule $[-^2y]^{-2} \rightarrow [-^1y]^{-1}[-^1y]^{-1}$, whenever y is a well-parenthesized word. It is clear that $\text{om}(x)$ and $\text{om}(\text{exp}(x))$ represent the same κ -word and that x is a well-parenthesized word over B_1 . Further, we have the following.

Lemma 7.1. *Let x be a nonempty well-parenthesized word over B_1 and $i \in c_{\mathbb{N}}(x)$. Then, $\mathbf{t}_i(x)$ is a well-parenthesized word over B_2 and $|\text{exp}(\mathbf{t}_i(x))| \leq \frac{1}{2}(|x|^2 + 2|x| - 3)$. Moreover, this upper bound is tight for all odd values of $|x|$.*

Proof. The fact that $\mathbf{t}_i(x)$ is a well-parenthesized word over B_2 follows immediately from the definition of \mathbf{t}_i . To prove the inequality, we proceed by induction on $|x|$. If $x = a_i$, then $\mathbf{t}_i(x)$ is the empty word and so, the result holds. Let x be a well-parenthesized word over B_1 such that $|x| > 1$. The inequality holds clearly, unless x is of the form $x = [-^1y]^{-1}z$, with y and z well-parenthesized words over B_1 , y nonempty and $i \in c_{\mathbb{N}}(y)$. In that case, we have $\mathbf{t}_i(x) = \mathbf{t}_i(y)[-^2y]^{-2}z$. Using induction hypothesis on y , one may deduce that $|\text{exp}(\mathbf{t}_i(x))| \leq \frac{1}{2}(|x|^2 + 2|x| - 3)$. Finally, let $\vec{x} = (a_1, \varepsilon, \varepsilon, \dots)$, $\vec{y} = (\varepsilon, \varepsilon, \dots)$, $\vec{q} = (-1, -1, \dots)$, and $u_{2n+1} = \mu_n = \mu_n(\vec{x}, \vec{y}, \vec{q})$ (recall the notation used in Lemma 5.4). Then, u_{2n+1} is a well-parenthesized word over B_1 of length $2n + 1$. Moreover, using Lemma 5.4, we may compute

$$\begin{aligned} |\text{exp}(\mathbf{t}_1(u_{2n+1}))| &= |\text{exp}(\mathbf{t}_1(\mu_0) \cdot \xi_0 \cdot \xi_1 \cdots \xi_{n-1})| \\ &= |\text{exp}([-^2\mu_0]^{-2} \cdots [-^2\mu_{n-1}]^{-2})| \\ &= \sum_{k=0}^{n-1} 2(|\mu_k| + 2) \\ &= 2n^2 + 4n \quad \text{because } |\mu_k| = 2k + 1 \\ &= \frac{1}{2}(|u_{2n+1}|^2 + 2|u_{2n+1}| - 3) \end{aligned}$$

and the result follows. \square

Also, as a straightforward consequence of the definition of \mathbf{p}_a , the following holds.

Lemma 7.2. *Let x be a nonempty well-parenthesized word over B_1 and $a \in A$. Then, $\mathbf{p}_a(x)$ is also a well-parenthesized word over B_1 and $|\text{exp}(\mathbf{p}_a(x))| = |\mathbf{p}_a(x)| \leq |x|$. \square*

Given a well-parenthesized word x over B_2 , we define the *linearization over A* of x to be the word $\text{lin}(x)$ over the alphabet $A \uplus A^{-1}$ obtained by applying the rewriting rules $[-^1a_i]^{-1} \rightarrow a^{-1}$, $[-^1yz]^{-1} \rightarrow [-^1z]^{-1}[-^1y]^{-1}$ and $[-^2y]^{-2} \rightarrow [-^1y]^{-1}[-^1y]^{-1}$ to x (with $a_i \in c(x)$ and y, z well-parenthesized words). It is easy to see that $\text{lin}(x) = \text{lin}(\text{exp}(x))$ and that if x is a well-parenthesized word over B_1 , then $O(|\text{lin}(x)|) = O(|x|)$. Consequently, we have the next result.

Corollary 7.3. *Let w be an κ -term and $(i, a) \in [0, |\bar{w}|[\times c_A(\bar{w})$. Then, $|\text{lin}(\bar{w}(i, a))|$ belongs to $O(|w|^2)$. \square*

Now, we wish to compute $\text{lin}(x)$, for a given well-parenthesized word over B_2 . Recall the tree representation of κ -terms exemplified in Figure 5 (κ -terms are, in particular, $\bar{\kappa}$ -terms). We may recover, in linear time, such a tree representation for $\text{om}(x)$, for a well-parenthesized word x over B_1 . Furthermore, if we are given a well-parenthesized word over B_2 , we may compute, also in linear time, a tree representation for $\text{om}(\text{exp}(x))$. That amounts to, whenever we have a factor of the form $[-^2y]^{-2}$ in x , to include twice a subtree representing $[-^1y]^{-1}$.

650 On the other hand, since solving the word problem in FG_A (for words written over the alphabet $A \cup A^{-1}$) is a linear issue in the size of the input, by Corollary 7.3, we may take $p(u, v) = \max\{|u|^2, |v|^2\}$. Thus, we have proved the following.

Proposition 7.4. *The κ -word problem over DRG is decidable in $O(m^3 |A|)$ -time, where m is the maximum length of the inputs. \square*

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