

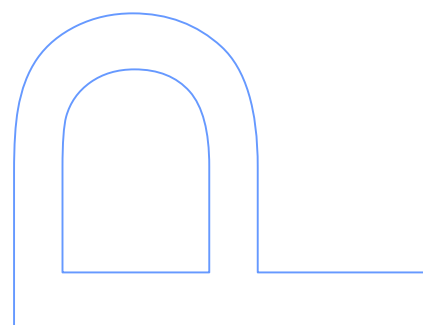
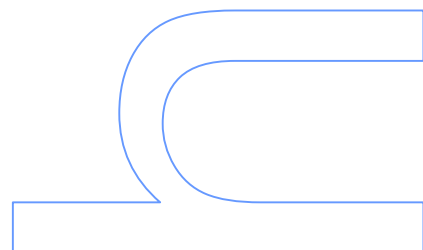
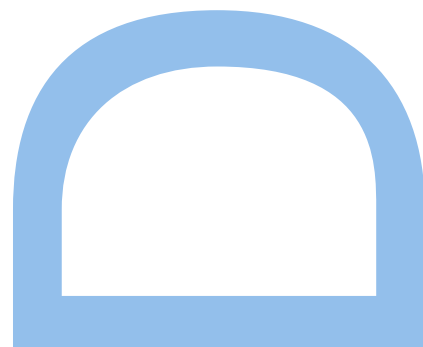
# Forward-backward stochastic differential equations and applications

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*Aos pais,*



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# Abstract

In this dissertation, we obtain the existence and uniqueness of a strong solution to fully-coupled forward-backward stochastic differential equations (FBSDEs) with jumps, on intervals of arbitrary length, and for a class of admissible FBSDE coefficients large enough to allow a range of useful applications. We then use this result in the formulation of a new model in finance allowing the selection of an optimal hedging portfolio for contingent claims, in a market with asynchronous jumps and in the presence of a large investor.

Existence and uniqueness results are well known for the case when the FBSDEs are driven by a Brownian motion. When the FBSDEs contain jumps there are very few such results. Moreover, they hold for classes of admissible coefficients that are too restrictive and not naturally suited to some applications. Our result is not purely probabilistic and relies on a certain generalization of a known link between FBSDEs driven by Brownian motion and second order partial differential equations (PDEs). In our situation, we have to deal with the more delicate case of non-local PDEs.

We also obtain other results of a more auxiliary nature, but nevertheless previously unknown. In particular, we give an alternative and probabilistic method to obtain the uniqueness of a solution to FBSDEs with jumps on a short-time interval. The main content of this dissertation is divided into 5 chapters.

In Chapter 1, we introduce the problem, describe the state of art on the theory of FBSDEs, and outline the plan of the thesis.

In Chapter 2, we gather some preliminary material on the theory of stochastic integration. In particular, we recall Itô's formula for Lévy-type integrals, which is a fundamental tool in the construction of explicit solutions to FBSDEs.

In Chapter 3, we review a family of martingales satisfying some useful properties that we later employ in our application. We also give an auxiliary result that we use in a more advanced stage.

In Chapter 4, we review a method ([24]) that can be used to solve explicitly FBSDEs driven by a Brownian motion, and give an application to the problem of finding a hedging portfolio for a contingent claim, in the presence of a large investor and in Brownian markets.

Finally, in Chapter 5, we offer the main contribution of this work, the existence and uniqueness of an adapted càdlàg solution to FBSDEs with jumps,

on an interval of time of arbitrary duration and for a large class of coefficients. We then show how solutions to certain non-local PDEs can be used to build an explicit solution to FBSDEs with jumps. We conclude by offering an asset pricing model to hedge contingent claims in the presence of a large investor, and markets with jumps.

The added contribution of this dissertation to the existent body of knowledge is thus two-fold: First, we give a new existence and uniqueness result for FBSDEs with jumps: our results hold for a sufficiently large class of admissible coefficients and on intervals of arbitrary duration. Second, a new and improved hedging model in finance is introduced, that makes use of FBSDEs to find optimal hedging portfolios for contingent claims, in markets where, unlike previously known models, not only the stocks may jump, but the jumps may occur at different times.

The contents of this dissertation gave raise to one published paper ([26]) and two submitted manuscripts ([27, 11]).

# Resumo

Nesta dissertação, é obtida a existência e unicidade de soluções fortes para sistemas de equações diferenciais estocásticas progressivas-regressivas (EDEPRs) com saltos, em intervalos de comprimento arbitrário, e para uma classe de coeficientes suficientemente rica para vários tipos de aplicações. Este resultado será depois usado na formulação de um novo modelo financeiro, permitindo encontrar uma carteira optimal para activos contingentes, num mercado com saltos assíncronos e na presença de um investidor de tamanho não-negligenciável.

Resultados de existência e unicidade são sobejamente conhecidos para o caso das EDEs serem dirigidas por movimentos Brownianos. No caso das EDEs conterem saltos, há um número muito reduzido deste tipo de resultados, e sob condições demasiado restritivas. O nosso resultado não é puramente probabilístico e depende duma generalização de uma ligação entre EDEPRs e equações diferenciais parciais (EDPs) de segunda ordem, que foi utilizada para o caso das equações dirigidas por movimentos Brownianos ([24]).

Obteremos outros resultados de natureza auxiliar, mas contudo, desconhecidos previamente. Em particular, introduzimos um método puramente probabilístico através do qual é possível obter um resultado de existência e unicidade para EDEPRs com saltos em intervalos de curta duração. A parte principal da dissertação é dividida em 5 capítulos.

No Capítulo 1, apresentamos o problema, fazemos uma revisão de literatura, e delineamos o plano da tese.

No Capítulo 2, reunimos algum material preliminar na teoria da integração estocástica. Em particular, recordamos a formula de Itô para integrais do tipo Lévy, que será fundamental na construção de soluções explícitas para EDSPR.

No Capítulo 3, introduzimos uma família de martingalas que satisfaz propriedades que serão depois utilizadas no nosso modelo.

No Capítulo 4, revemos os principais passos na construção de um método já existente para obter soluções explícitas para EDEPRs dirigidas por um movimento Browniano. Daremos depois uma aplicação sob a forma de um modelo para encontrar carteiras replicantes para activos contingentes em mercados Brownianos, na presença de um investidor de tamanho não negligenciável.

Finalmente, no Capítulo 5, apresentamos a nossa principal contribuição, nomeadamente a existência e unicidade de soluções fortes para EDEPRs com saltos, em intervalos de duração arbitrária, e para uma classe admissível de coeficientes suficientemente grande. Fechamos a dissertação com um modelo

inovador para replicar activos contingentes na presença de um investidor em mercados em que as acções subjacentes admitem saltos assíncronos.

A contribuição desta tese para a área de conhecimento assume uma natureza dupla: Primeiro, é obtido um novo resultado de existência e unicidade para EDEPRs com saltos. Mais, o resultado é válido para uma classe admissível de coeficientes suficientemente rica para um conjunto de aplicações, entre as quais se insere a nossa. Em segundo lugar, um modelo novo de "hedging" em finanças é apresentado, onde portfolios óptimos, replicantes de activos contingentes, são obtidos através de soluções para EDEPRs. Esse modelo, ao contrário dos seus antecessores, assume não só que o mercado possa ter saltos, mas que tais saltos possam ocorrer entre as diversas acções em alturas diferentes.

O conteúdo desta dissertação deu origem a um artigo publicado ([26]) e a dois artigos submetidos ([27, 11]).

# Acknowledgments

First of all, I thank my supervisor Professor Evelina Shamarova, without who this work would not be possible. I thank Prof. Evelina for teaching me to be a more effective researcher, learning from her keen eye to identify, on a mathematical proof, potential issues arising from selecting a certain path, that often lied way ahead in the demonstration. I am also grateful for her constant availability and willingness to help me, and all the fruitful discussions we had, even in non-academic topics.

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Finally, to you, Elena, thank you for making myself a better person everyday.

# Selected Symbols

## Basic elements

$(\Omega, \mathcal{F}, P)$ ,	a probability space, page 4
$B_t$ ,	a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ , page 4
$\ell_t$ ,	a Lévy process defined on a probability space $(\Omega, \mathcal{F}, P)$ , page 4
$N(t, \cdot)$ ,	a Poisson random measure on $\mathbb{R}_+ \times \mathcal{B}(\mathbb{R}^k)$ , page 5
$\tilde{N}(t, \cdot)$ ,	the compensated Poisson random measure, page 6
$\mathcal{F}_t$ ,	the natural filtration augmented with its null sets, page 29
$(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ ,	a filtered probability space, page 29

## Classes of stochastic processes

$\mathcal{P}(0, T; \mathbb{R}^k)$ ,	predictable, square-integrable, $\mathcal{F}_t$ -adapted processes, page 29
$\mathcal{F}(0, T; \mathbb{R}^k)$ ,	square-integrable, $\mathcal{F}_t$ -adapted processes, page 29
$\mathcal{M}^2(0, T; \mathbb{R}^k)$ ,	square-integrable martingales, page 30

## Other symbols

$\{H_t^{(i)}\}_{i=1}^\infty$ ,	the family of orthonormalized Teugels martingales, page 12
$L_t^i$ ,	the $i$ -th component of an $l$ -dimensional Lévy process $L_t$ , page 51
$\{H_t^{(ij)}\}_{j=1}^\infty$ ,	the family of orthonormalized Teugels martingales associated to $L_t^i$ , page 51

## Derivative notation

$\partial_x \phi$ or $\phi_x$ ,	the partial gradient of $\phi(t, x, u, p, w)$ with respect to the vector-valued variable $x$ , where $\phi : [0, T] \times \overline{F} \times \mathbb{R}^m \times E \times \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^l$ for some $l = 1, 2, \dots$ , page 44
$\partial_t \phi$ or $\phi_t$ ,	the partial derivative $\frac{\partial}{\partial t} \phi(t, x, u, p, w)$ , page 44

$\partial_u \phi$ ,	the partial gradient of $\phi(t, x, u, p, w)$ with respect to $u$ , page 44
$\partial_p \phi$ ,	the partial gradient of $\phi(t, x, u, p, w)$ with respect to $p$ , page 44
$\partial_w \phi$ ,	the partial Hadamard derivative of $\phi(t, x, u, p, w)$ with respect to $w$ , page 44
$\partial_{x_i} \phi$ or $\phi_{x_i}$ ,	the partial derivative $\frac{\partial}{\partial x_i} \phi(t, x, u, p, w)$ , page 44
$\partial_{u_i} \phi$ or $\phi_{u_i}$ ,	the partial derivative $\frac{\partial}{\partial u_i} \phi(t, x, u, p, w)$ , page 44
$\partial_{p_i} \phi$ or $\phi_{p_i}$ ,	the partial gradient of $\phi(t, x, u, p, w)$ with respect to the $i$ th line $p_i$ of the matrix $p$ , page 44
$\partial_{x_i x_j}^2 \phi$ or $\phi_{x_i x_j}$ ,	the second partial derivative $\frac{\partial^2}{\partial x_i \partial x_j} \phi(t, x, u, p, w)$ , page 44

# Chapter 1

## Introduction

### 1.1 Description of the problem

Let  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$  be a filtered probability space with the augmented filtration  $\mathcal{F}_t$  containing the  $P$ -null sets. Further, let  $B_t$  be a  $d$ -dimensional standard  $\mathcal{F}_t$ -Brownian motion,  $N(t, A)$  be an  $\mathcal{F}_t$ -adapted Poisson random measure on  $\mathbb{R}_+ \times \mathcal{B}(\mathbb{R}^l)$ , where  $\mathcal{B}(\mathbb{R}^l)$  is the  $\sigma$ -algebra of Borel sets on  $\mathbb{R}^l$ , and  $\tilde{N}(t, A) = N(t, A) - t\nu(A)$  be the associated compensated Poisson random measure on  $\mathbb{R}_+ \times \mathcal{B}(\mathbb{R}^l)$  with the intensity  $\nu(A)$  assumed to be a Lévy measure. In this work, we obtain the existence and uniqueness of an  $\mathcal{F}_t$ -adapted càdlàg strong solution to FBSDEs with jumps on arbitrary time intervals, of type

$$\begin{cases} X_t = x + \int_0^t f(s, X_s, Y_s, Z_s, \tilde{Z}_s(\cdot)) ds + \int_0^t \sigma(s, X_s, Y_s) dB_s \\ \quad + \int_0^t \int_{\mathbb{R}^l} \varphi(s, X_{s-}, Y_{s-}, u) \tilde{N}(ds, du), \\ Y_t = h(X_T) + \int_t^T g(s, X_s, Y_s, Z_s, \tilde{Z}_s(\cdot)) ds - \int_t^T Z_s dB_s \\ \quad - \int_t^T \int_{\mathbb{R}^l} \tilde{Z}_s(u) \tilde{N}(ds, du). \end{cases} \quad (1.1)$$

We then apply the results obtained to the field of asset pricing, where we solve the problem of finding an optimal portfolio for a contingent claim in the presence of a large investor in a market with asynchronous jumps.

### 1.2 State of the art

The main objective in studying systems of FBSDEs is to prove the existence and uniqueness of a solution to those systems, to determine the class of functions the FBSDE coefficients must belong to, and to study properties of such solutions. The investigation of FBSDEs has been the focus of several authors, and with recourse to different methods (see [1, 24, 10, 23, 20, 22]). Such investigation has been done for FBSDEs driven by Brownian motions, with coefficients under various regularity and growth conditions, and by means of one of the follow-

ing three methods: The method of continuation developed by Hu and Peng ([10]), the contraction mapping method on a short-time interval introduced by Delarue ([4]), and the four step scheme obtained by Ma et al.([25]). The first two methods are purely probabilistic. In particular, the method of continuation uses a certain monotonicity condition on the FBSDEs coefficients, which is too restrictive, while the contraction mapping method works only on a short time interval. The four step scheme holds on a time interval of arbitrary duration and for a large class of FBSDEs coefficients, and relies heavily on Ladyzhenskaya's theory on second order parabolic PDE ([21]). In fact, it is by means of gradient estimates for solutions of such PDEs that Delarue extends the existence of a solution given by the contraction mapping method on a short-time interval, to an arbitrary large interval.

In the context of systems of FBSDEs with jumps, Wu gave an extension of Peng's continuation method to FBSDEs of type (1.1) ([23]). This result holds, however, only for a class of admissible coefficients that is not naturally suited to a range of potential applications. In another work, Nualart and Schoutens introduced a family of one-dimensional martingales, having an orthogonality property, and admitting a useful predictable representation property ([16]). In a follow-up paper, the authors used this representation property to obtain the existence and uniqueness of a solution to a BSDE driven by this family of martingales ([17]).

In this work, we obtain the existence and uniqueness of a strong solution to FBSDEs containing Lévy-type integrals on intervals of arbitrary length, and for a large class of FBSDE coefficients. Namely, given a solution to a certain non-local PDE, we obtain an explicit solution to FBSDEs with jumps. During the development of our results, we encounter non-local parabolic second order PDEs, for which an existence and uniqueness result is a crucial result for the construction of an explicit solution to FBSDEs with jumps. The derivation of such result, using purely PDE techniques, is of rather technical nature, and beyond the area of this thesis which is stochastic analysis. We refer the reader to the original paper for the complete treatment of the subject ([11]).

### 1.3 Organization of the Thesis

After the introduction, we collect in Chapter 2 some preliminary elements on the theory of stochastic integration, that we use in the sequel.

In Chapter 3, we review the theory on orthonormalized Teugels martingales (see Nualart and Schoutens [16]). In the first section, we recall a certain representation property, allowing the representation of a random variable via stochastic integrals driven by elements of this family. This property is then extended to square-integrable martingales, and consequently allows the formulation of well-posed BSDEs driven by that family. In the second section, we give an additional result, namely a representation of the orthonormalized Teugels martingales as sums of Brownian motions and Poisson integrals that will be useful in the application to be proposed in the last chapter.

In Chapter 4, we review a well-known method (known as the four step scheme) to obtain the existence and uniqueness of solutions to FBSDEs driven by Brownian motion (see Ma et al. [24]). This method establishes a connection between those FBSDEs and second order parabolic PDEs. In particular, a solution to (1.1) is built with the help of Itô's formula, which applied to the forward component of the FBSDE yields a second-order PDE. The existence of a solution to this derived PDE is then a consequence of well-established theory on parabolic second order PDEs (see Ladyzenskaya et al. [21]). In the last section, we give an FBSDE hedging model for Brownian markets in the presence of a large investor. The model we present follows the treatment of previous work, but under somehow different assumptions (see the original model of Cvitanić and Ma [9]). In fact, some of the techniques used for the treatment of FBSDEs driven by Brownian motion can be adapted, and in some cases revised to the investigation of FBSDE with jumps.

Chapter 5 is dedicated to our main contribution, the existence and uniqueness of a strong solution to fully-coupled FBSDEs with jumps, holding on intervals of arbitrary duration, and for a sufficiently large class of coefficients. While in the first section we offer some preliminary remarks, we have split the second section into two further sub-sections: In the first part we briefly review an existence and uniqueness result based on an extension of Peng's continuation method (see Wu [23]). Here, we see how the so-called weak monotonicity conditions impose conditions over the admissible coefficients that are not compatible with the requirements of our application. In the second part, we give conditions under which we can extend Delarue's contraction mapping to FBSDEs with jumps. We stress that this method is only valid on a short-time interval. The third section is entirely dedicated to the derivation of our main result. With a view to an application to FBSDEs with jumps, we start by briefly outlining an existence and uniqueness result on non-local PDEs (see Shamarova and Pereira [11]). We then show how precisely we can make use of those results to build an explicit solution to FBSDEs with jumps.

It is important to remark that our method constitutes a step forward over both methods presented in the previous section: Not only we obtain a class of admissible coefficients much more natural for applications, but also solve the problem on intervals of arbitrary duration. In the last section, we formulate a model to find optimal replicating portfolios for contingent claims in markets with asynchronous jumps, and in the presence of a large investor. Those portfolios are obtained as a solution to a system of FBSDEs that we derive, and then solve with the help of our existence and uniqueness result. Our model is able not only to account for jumps in stocks, a definite improvement over Brownian models, but also assumes that jumps do not need to occur simultaneously. We believe that accounting for asynchronous jumps in stocks, i.e., jumps that occur at different times across different stocks, is an important step in selecting a model that can more accurately depict the reality observed in actual markets.

## Chapter 2

# Preliminaries

For convenience, we recall some elements and results from the theory of stochastic integration with respect to martingale valued measures, which we will use extensively throughout this dissertation. The main references are [2], [12] and [5].

### 2.1 Lévy processes. Lévy-Itô decomposition

**Definition 2.1.1.** A Lévy Process is a stochastic process  $(\ell_t, t \geq 0)$  defined on a probability space  $(\Omega, \mathcal{F}, P)$  such that a.s.  $\ell_0 = 0$  and the following conditions hold:

- (i) (independent increments) For any  $0 \leq s \leq t$ ,  $\ell_t - \ell_s$  is independent from  $\mathcal{F}_s$ .
- (ii) (stationary increments) For any  $0 \leq s \leq t$ ,  $\ell_{t-s} \stackrel{d}{=} \ell_t - \ell_s$ .
- (iii) (stochastic continuity) For any  $\varepsilon > 0$ ,  $0 \leq s \leq t$ ,  $\lim_{\tau \rightarrow 0} P(|\ell_{s+\tau} - \ell_s| \geq \varepsilon) = 0$ .

The following are two important families of Lévy processes:

- i) The *Brownian Motion* in  $\mathbb{R}^d$  is a Lévy process  $B = (B_t, t \geq 0)$ , such that  $B_t \stackrel{d}{=} N(0, tI)$ . The distribution  $N(0, tI)$  has a probability density function of the form  $f(x) = \frac{1}{\sqrt{(2\pi t)^k}} e^{-\frac{1}{2t}|x|^2}$ . It is possible to show that  $\phi_{B_t}(u) = e^{-\frac{1}{2t}u^2}$ .
- ii) The *Poisson process with intensity  $\lambda$*  in  $\mathbb{R}^k$  is a Lévy process  $N = (N_t, t \geq 0)$ , such that  $N_t$  has a discrete distribution taking values on the set of positive integers. For each  $t \geq 0$ , its probability density function is  $f_{N_t}(n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$ , and  $\phi_{N_t}(u) = e^{\lambda t(e^{iu} - 1)}$ .

Given two stochastic processes  $Y = (Y_t, t \geq 0)$  and  $X = (X_t, t \geq 0)$  defined on the same probability space,  $Y$  is called a *modification* of  $X$  if  $P(X_t = Y_t) = 1$ , for each  $t \geq 0$ . Note that the set at which the processes differ may change across time. The following result warrants the possibility of substituting a Lévy

process by a modification better suited for analysis. We will make use of this result in chapter 5.

**Theorem 2.1.2.** *Every Lévy process admits a right-continuous modification with left limits (càdlàg) that is itself a Lévy process.*

*Proof.* See §2.1.2 in [2] (p. 87).  $\square$

Let now  $\mathcal{M}_1(\mathbb{R}^k)$  be the set of all Borel probability measures on  $\mathbb{R}^k$ . A Borel measure  $\nu$  defined in  $\mathbb{R}$  is said to be a *Lévy measure* if  $\nu(\{0\}) = 0$  and  $\int_{\mathbb{R}} \min(|y|^2, 1) \nu(dy) < \infty$ . If we let  $A_i = (-i, -\frac{1}{i})^k \cup (\frac{1}{i}, i)^k$ , then  $A_i \in \mathcal{B}(\mathbb{R}^k \setminus \{0\})$ , and  $\nu(A_i) < \infty$ , for  $i = 1, 2, \dots$ . Furthermore  $(A_i, i \in \mathbb{N})$  is a cover of  $\mathbb{R}^k - \{0\}$ . Thus, every Lévy measure is  $\sigma$ -finite.

Define now for any  $t \geq 0$ ,  $A \in \mathcal{B}(\mathbb{R}^k \setminus \{0\})$ , and  $\omega \in \Omega$  the eventually infinite process

$$N(t, A)(\omega) = \sum_{0 \leq s \leq t} \chi_A(\Delta X_s(\omega)),$$

where  $\Delta X_t = X_t - X_{t-}$  denotes the *jump process* of a Lévy process  $X_t$ . In particular, for each  $\omega$ ,  $N(t, \cdot)(\omega) : \mathcal{B}(\mathbb{R}^k \setminus \{0\}) \rightarrow \mathbb{N} \cup \{0\}$ ,  $A \mapsto N(t, A)(\omega)$  is a counting measure. The quantity  $\mathbb{E}(N(1, A))$  is called the *intensity measure* associated with  $X$ . We say that  $A$  is *bounded below*, if  $0 \notin \overline{A}$ . This implies that there are only finite accumulations of small jumps.

**Proposition 2.1.3.** *If  $A$  is a Borel set bounded from below, then  $N(t, A) < \infty$  (a.s) for all  $t \geq 0$ . Furthermore,  $N(t, A)$  is a Poisson Process with intensity  $\mu(A) = \mathbb{E}(N(1, A))$ . Finally, if  $A$  and  $B$  are two disjoint borel sets, both bounded below, and  $s \neq t$ , then  $N(s, A)$  is independent from  $N(t, B)$ .*

*Proof.* See §2.3 in [2] (p. 101).  $\square$

We assume henceforth that  $A$  is a Borel set bounded from below. We are now ready to introduce the first main object in the theory of stochastic integration. Define thus the *Poisson integral* by

$$\int_A f(x) N(t, dx)(\omega) = \sum_{x \in A} f(x) N(t, \{x\})(\omega) = \sum_{0 \leq s \leq t} f(\Delta X_s(\omega)) \chi_A(\Delta X_s(\omega)),$$

whose characteristic function is  $e^{t \int_{\mathbb{R}^k} (e^{i(u, x)} - 1) \mu_{f, A}(dx)}$ , where  $\mu_{f, A}(B) = \mu(A \cap f^{-1}(B))$ , for each  $B \in \mathcal{B}(\mathbb{R}^k)$ . This implies by differentiation at the point  $u = 0$ ,

$$\mathbb{E} \left( \int_A f(x) N(t, dx) \right) = t \int_A f(x) \mu(dx)$$

and

$$\text{Var} \left( \int_A f(x) N(t, dx) \right) = t \int_A |f(x)|^2 \mu(dx),$$

where  $\mu(x) = \mathbb{E}(N(1, \{x\}))$ . We call to the process  $\tilde{N}(t, A) = N(t, A) - t\mu(A)$  the *compensated Poisson random measure*.

Now, defining  $Y_t = \int_A f(x)N(t, dx)$ , we see that  $\mathbb{E}(Y_t|\mathcal{F}_s) = \mathbb{E}(Y_{t-s} + Y_s|\mathcal{F}_s) = E(Y_{t-s}) + Y_s$ . Since  $\mathbb{E}(Y_t) = t \int_A f(x)\mu(dx)$ , one concludes that the *compensated Poisson integral* defined by

$$\int_A f(x)\tilde{N}(t, dx) = \int_A f(x)N(t, dx) - t \int_A f(x)\mu(dx)$$

is a martingale.

The following result is a fundamental block on the study of Lévy Processes. We refer the reader to [2] (p. 126) for a proof.

**Theorem 2.1.4** (Lévy-Itô decomposition). *If  $L_t$  is an  $\mathbb{R}^k$ -valued Lévy process, then there exists a vector  $b \in \mathbb{R}^k$ , a Brownian motion  $B_A$  with covariance matrix  $A \in \mathbb{R}^k$ , and an independent Poisson measure  $N$  on  $\mathbb{R}_+ \times (\mathbb{R}^k \setminus \{0\})$ , such that for each  $t \geq 0$  the following representation holds,*

$$L_t = bt + B_A(t) + \int_{|x|<1} x\tilde{N}(t, dx) + \int_{|x|\geq 1} xN(t, dx).$$

## 2.2 Stochastic integration. Itô's formula

Here, we further develop the machinery introduced in the previous section. We follow closely the treatment in [2] (see §4).

Let  $E \in \mathcal{B}(\mathbb{R}^k)$  and  $\mathcal{I}$  be the ring containing finite unions of subsets  $I \times A \subset \mathbb{R}^+ \times E$ , where  $A \in \mathcal{B}(E)$  and  $I$  is a finite union of intervals. Let  $M$  be a random measure on  $(\mathbb{R}^+ \times \mathbb{R}^k, \mathcal{I})$ , and define the process  $M_A(t) = M([0, t] \times A)$ , which we also denote by  $M([0, t], A)$ . We say that  $M$  is a *martingale-valued measure* if for each  $A$  the process  $M_A(t)$  is a martingale. Consider the random function  $\varphi : [0, T] \times E \times \Omega \rightarrow \mathbb{R}$  and two associated mappings,  $\varphi_{x,\omega} : [0, T] \rightarrow \mathbb{R} : t \rightarrow \varphi(t, x, \omega)$  and  $\varphi_t : E \times \Omega \rightarrow \mathbb{R} : x \rightarrow \varphi(t, x, \omega)$ . The *predictable  $\sigma$ -algebra* is the  $\sigma$ -algebra generated by the set of random functions  $\varphi : [0, T] \times E \times \Omega \rightarrow \mathbb{R}$  such that  $\varphi_{x,\omega}$  is left-continuous for each  $(x, \omega)$ , and  $\varphi_t$  is  $\mathcal{B}(E) \times \mathcal{F}_{t-}$ -measurable for each  $0 \leq t \leq T$ .

We say that  $\varphi : [0, T] \times E \times \Omega \rightarrow \mathbb{R}$  is *predictable* if it is measurable with respect to the predictable  $\sigma$ -algebra.

Let then  $M$  be a martingale-valued measure and assume that for  $A \in \mathcal{B}(E)$  it holds  $M(0, A) = 0$ ,  $M((s, t], A) := M(t, A) - M(s, A)$  is independent of  $\mathcal{F}_s$ , and  $\mathbb{E}(M(t, A)^2) = t\mu(A)$ , where  $\mu$  is a  $\sigma$ -finite measure on  $E$ . Then we can define, for a predictable process  $F(t, x)$  such that  $\int_0^t \int_E \mathbb{E}(F(s, x)^2) d\mu(dx) < \infty$ , the *stochastic integral* of  $F$  with respect to  $M$

$$I_T(F) = \int_0^T \int_E F(s, x)M(ds, dx),$$

a random variable with first moment  $\mathbb{E}(I_T(F)) = 0$  and finite second moment

$\mathbb{E}(I_T(F)^2) = \int_0^T \int_E \mathbb{E}(F(s, x)^2) ds \mu(dx)$ . In particular, for  $E = \{0\}$  and  $M_t = B_t$ , we obtain the *Brownian integral*  $\int_0^T F(s) dB_s$ , while if  $E = \mathbb{R} \setminus \{0\}$  and  $M_t = \tilde{N}(s, x)$  we obtain the *Lévy integral*

$$\int_0^T \int_E F(s) \tilde{N}(ds, dx) = \int_0^T \int_E F(s, x) \tilde{N}(ds, dx) - \int_0^T \int_E F(s, x) \mu(dx).$$

We denote by  $\mathcal{P}_2(T, E)$  the set of all equivalence classes of mappings  $F : [0, T] \times E \times \Omega \rightarrow \mathbb{R}$  under almost sure equality with respect to  $\mu(x)t$ , such that  $F$  is predictable and  $P(\int_0^T \int_E |F(s, x)| dt \mu(dx) < \infty) = 1$ .

Let now  $G$  and  $F$  be respectively  $\mathbb{R}^n$  and  $\mathbb{R}^{n \times m}$  valued processes, such that  $|G^i|^{\frac{1}{2}}, F_j^i \in \mathcal{P}_2(T)$ ,  $i = 1, 2, \dots, n$ ,  $j = 1, 2, \dots, m$ . Let also  $Z^i \in \mathcal{P}_2(T, B_1 \setminus \{0\})$ , and  $Z$  be predictable.

The *Lévy-Type* stochastic integral is the  $\mathbb{R}^n$ -valued stochastic process

$$\begin{aligned} Y_s = Y(0) + \int_0^t G_s ds + \int_0^t F_s dB_s + \int_0^t \int_{|x| < c} \tilde{Z}(s, u) \tilde{N}(ds, du) \\ + \int_0^t \int_{|u| \geq c} Z(s, u) N(ds, du), \end{aligned} \quad (2.1)$$

where  $c$  is a constant and can also be infinity. In this latter case the last term in (2.1) is zero. Define  $Y_c = Y - Y_d - Y(0)$ , the *continuous part* of  $Y$ , where we use  $Y_d$  to denote the *discontinuous part* of  $Y$ . Clearly, for each  $t \geq 0$

$$Y_c(t) = \int_0^t G_s ds + \int_0^t F_s dB_s,$$

and

$$Y_d(t) = \int_0^t \int_{|x| < 1} \tilde{Z}(s, x) \tilde{N}(ds, dx) + \int_0^t \int_{|x| \geq 1} Z(s, x) N(ds, dx).$$

It will be seen that in order to obtain the existence of solutions to FBSDEs, a link between the forward equation and the backward equation will be needed. The following paramount result provides that link. Below, we use the notation  $\int_{0+}^t$  to denote the integral over the half-open interval  $(0, t]$ .

**Theorem 2.2.1** (Itô's formula). *Let  $X = (X_1, \dots, X_n)$  be an  $n$ -tuple of semimartingales and let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  have continuous second order partial derivatives.*

Then  $f(X)$  is a semimartingale and the following holds with probability 1,

$$\begin{aligned} f(X(t)) - f(X(0)) &= \sum_{i=1}^n \int_{0+}^t \partial_i f(X(s-)) dX_s^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \int_{0+}^t \partial_i \partial_j f(X(s-)) d[X_c^i, X_c^j](s) \\ &\quad + \sum_{0 < s \leq t} [f(X(s)) - f(X(s-)) - \sum_{i=1}^n \partial_i f(X(s-)) \Delta X^i(s)]. \end{aligned}$$

*Proof.* Refer to [12] (see p. 74).  $\square$

In this dissertation, we make use of a variation of this result, that can be proved with the help of the previous result. We let  $C_b^{1,2}([0, T] \times \mathbb{R}^n)$  denote the class of continuous functions  $u : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ , that are bounded together with their derivatives  $u_t$ ,  $u_x$ , and  $u_{xx}$ .

**Lemma 2.2.2.** *Let  $X_t$  be an  $\mathbb{R}^n$ -valued semimartingale with càdlàg paths taking the form*

$$X_t = x + \int_0^t F_s ds + \int_0^t G_s dB_s + \int_0^t \int_Z \Phi_s(y) \tilde{N}(ds, dy),$$

where the  $d$ -dimensional Brownian motion  $B_t$  and the compensated Poisson random measure  $\tilde{N}$  are defined as above. Further let  $Z \subset \mathbb{R}^l$  be such that  $\nu(Z) < \infty$ , and  $F_s$ ,  $G_s$ , and  $\Phi_s(y)$  be bounded. Then for any real-valued function  $\theta(t, x)$  in  $C_b^{1,2}([0, T] \times \mathbb{R}^n)$ , it holds a.s.,

$$\begin{aligned} \theta(t, X_t) &= \theta(0, x) + \int_0^t \partial_s \theta(s, X_s) ds + \int_0^t (\partial_x \theta(s, X_s), F_s) ds \\ &\quad + \int_0^t (\partial_x \theta(s, X_s), G_{s-} dB_s) + \frac{1}{2} \int_0^t \text{tr}(\partial_{xx}^2 \theta(s, X_s) G_s G_s^\top) ds \\ &\quad + \int_0^t \int_Z [\theta(s, X_{s-} + \Phi_s(y)) - \theta(s, X_{s-})] \tilde{N}(ds, dy) \\ &\quad + \int_0^t \int_Z [\theta(s, X_{s-} + \Phi_s(y)) - \theta(s, X_{s-}) - (\partial_x \theta(s, X_{s-}), \Phi_s(y))] \nu(dy) ds. \end{aligned} \tag{2.2}$$

**Remark 1.** In the above lemma we agree that  $X_{0-} = X_0 = x$ .

*Proof of Lemma 2.2.2.* Let us first assume that the function  $\theta$  does not depend

on  $t$ . Applying Itô's formula in Theorem 2.2.1 we obtain

$$\begin{aligned} \theta(X_t) - \theta(x) &= \int_0^t (\partial_x \theta(X_s), F_s) ds + \int_0^t (\partial_x \theta(X_{s-}), dX_s) \\ &+ \frac{1}{2} \int_0^t \text{tr}(\partial_{xx}^2 \theta(X_s) G_s G_s^\top) ds + \sum_{0 < s \leq t} (\theta(X_s) - \theta(X_{s-}) - (\partial_x \theta(X_{s-}), X_s - X_{s-})). \end{aligned} \quad (2.3)$$

Note that the last summand in (2.3) equals to  $\int_0^t \int_Z (\theta(X_{s-} + \Phi_s(y)) - \theta(X_{s-}) - (\partial_x \theta(X_{s-}), \Phi_s(y))) N(ds dy)$ . By the standard argument (see, e.g., [2], p. 256), we obtain formula (2.2) without the term containing  $\partial_s \theta(s, X_s)$ .

Now take a partition of the interval  $[0, t]$ . Then, for each pair of successive points,

$$\begin{aligned} \theta(t_{n+1}, X_{t_{n+1}}) - \theta(t_n, X_{t_n}) &= [\theta(t_{n+1}, X_{t_n}) - \theta(t_n, X_{t_n})] \\ &+ [\theta(t_{n+1}, X_{t_{n+1}}) - \theta(t_{n+1}, X_{t_n})]. \end{aligned} \quad (2.4)$$

The first difference on the right-hand side equals to  $\int_{t_n}^{t_{n+1}} \partial_s \theta(s, X_{t_n}) ds$ , while the second difference is computed by formula (2.3). Assume the mesh of the partition goes to zero as  $n \rightarrow \infty$ . Then, summing identities (2.4) and considering the limit as  $n \rightarrow \infty$  in the  $L_2(\Omega)$ -space, we pass to the limit under the expectation sign by Lebesgue's bounded convergence theorem. Taking into account that  $X_t$  has càdlàg paths, we arrive at formula (2.2).  $\square$

From Itô's formula, we can deduce an important corollary giving the product rule for SDEs.

**Corollary 2.2.3** (Itô's Product Formula). *Let  $Y^1$  and  $Y^2$  be two real valued stochastic processes of type (2.1). Then*

$$d(Y^1 Y^2)_t = Y_{t-}^1 dY_t^2 + Y_{t-}^2 dY_t^1 + d[Y^1, Y^2](t),$$

where

$$\begin{aligned} [Y^1, Y^2](t) &= \int_0^t F_s^1 F_s^2 ds + \int_0^t \int_{|x| < 1} \tilde{Z}^1(s, u) \tilde{Z}^2(s, u) \tilde{N}(ds, du) \\ &+ \int_0^t \int_{|x| \geq 1} Z^1(s, u) Z^2(s, u) N(ds, du). \end{aligned}$$

The next result allows the estimation of sup-norms of stochastic integrals in terms of their quadratic variation. (For a proof, see [6], see p.38)

**Proposition 2.2.4** (Burkholder-Davis-Gundy). *Let  $M_t$  be a local martingale for which  $\mathbb{E}[M_t]^{\frac{p}{2}} < \infty$  and  $p \geq 2$ . Then, there exist positive constants  $C_p, D_p$*

such that

$$C_p \mathbb{E}[M_t]^{\frac{p}{2}} \leq \mathbb{E} \sup_{s \in [0, t]} |M_s|^p \leq D_p \mathbb{E}[M_t]^{\frac{p}{2}}.$$

## 2.3 Itô's representation Theorem. Martingale representation Theorem

In the theory of FBSDEs, and specially when dealing with BSDEs, it is sometimes useful to represent certain processes with the help of alternative processes for which some useful properties hold. In this section, we give first a representation of random-variables based on Lévy-type integrals of type (2.1). Below,  $B_t = (B_t^1, \dots, B_t^d)$  is a  $d$ -dimensional Brownian-motion,  $L_t$  a  $d$ -dimensional Lévy process. For the case  $d = 1$  we have the following (see p. 303 in [2] for a proof)

**Theorem 2.3.1** (Itô's Representation Theorem). *Any square-integrable  $\mathcal{F}_T$ -adapted real-valued random variable  $X$ , admits, for a unique pair of  $\mathcal{F}_t$ -adapted square-integrable processes  $Z_t$  and predictable  $\tilde{Z}(t, \cdot)$ , the representation*

$$X = \mathbb{E}(X) + \int_0^T Z_s dB_s + \int_0^T \int_{\mathbb{R}} \tilde{Z}(s, x) \tilde{N}(ds, dx),$$

In fact, Itô's representation Theorem holds for complex-valued random variables, but that is beyond the scope of the present work. The following result holds for martingales in a multi-dimensional real space, and is fundamental in the formulation of well-specified FBSDEs.

**Theorem 2.3.2** (Martingale representation Theorem). *Any square-integrable,  $\mathcal{F}_t$ -adapted, martingale  $(M_t, t \geq 0)$  admits, for a unique pair of  $\mathcal{F}_t$ -adapted square-integrable processes  $Z_t$  and predictable  $\tilde{Z}(t, \cdot)$ , the representation*

$$M(t) = \mathbb{E}(M(0)) + \sum_{j=1}^d \int_0^t Z_s^j dB_s^j + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \tilde{Z}(s, x) \tilde{N}(ds, dx).$$

where  $Z_t = (Z_t^1, \dots, Z_t^d)$ .

*Proof.* We refer the reader to [5], p. 51. □

## Chapter 3

# A short review on Teugels martingales

Here, we first review the details of the construction of a family of strongly orthonormal martingales  $\{H_t^{(i)}\}_{i=1}^\infty$ , first advanced by Nualart and Schoutens ([16]). The elements of this family are somehow related to the power-jump processes of a one-dimensional Lévy process. We then give an additional result, helpful in the sequel.

### 3.1 Predictable representation property

Let then  $\ell_t$  be a Lévy process with associated Lévy measure  $\nu$  such that

$$\int_{(-\epsilon, \epsilon)^c} e^{\lambda|x|} \nu(dx) < \infty, \quad (3.1)$$

for every  $\epsilon > 0$  and some  $\lambda > 0$ . Define  $\ell_t^{(1)} = \ell_t$ , and consider the *power jump processes*  $Y_t^i = \sum_{0 \leq s \leq t} (\Delta \ell_t)^i$ , for  $i = 2, 3, \dots$ . We remark that the power-jump processes of the Lévy process  $\ell_t$  are themselves Lévy processes. Indeed, one can see that

$$\begin{aligned} \ell_t^i - \ell_s^i &= \sum_{0 \leq u \leq t} (\Delta \ell_u)^i - \sum_{0 \leq u \leq s} (\Delta \ell_u)^i \\ &= \sum_{s \leq u \leq t} (\Delta \ell_u)^i \stackrel{d}{=} \sum_{0 \leq u \leq t-s} (\Delta \ell_u)^i = \ell_{t-s}^i. \end{aligned} \quad (3.2)$$

Note that the stationarity of  $\ell_s$  implies the third equality on (3.2), since obviously  $\Delta \ell_u \stackrel{d}{=} \Delta \ell_{u+s}$ , for any  $u + s \geq 0$ . Moreover, taking a partition of the interval  $[0, t]$ ,  $\ell_{t_{j+1}}^{(i)} - \ell_{t_j}^{(i)} = \sum_{t_j \leq u \leq t_{j+1}} (\Delta \ell_u)^i$ , and hence the increments are independent since the intervals  $]t_j, t_{j+1}[$  are disjoint. Finally, (iii) from definition

2.2.1 is inherited by (iii) for the original process. We recall a useful lemma.

**Lemma 3.1.1.** *If  $\ell_t$  is an  $\mathcal{F}_t$ -adapted Lévy process, then the centered process  $\ell_t - \mathbb{E}[\ell_t]$  is a martingale.*

*Proof.* We just have to note that for  $s \leq t$ , we have  $\mathbb{E}(\ell_t - \mathbb{E}[\ell_t] - \ell_s + \mathbb{E}[\ell_s] | \mathcal{F}_s) = 0$ , since the increments are independent.  $\square$

As such the collection  $\{Y_t^{(i)}\}_{i=1}^\infty$ , where  $Y_t^{(i)} = Y_t^i - \mathbb{E}[Y_t^i]$  for  $i = 1, 2, \dots$ , is a family of martingales. In their work, Nualart and Schoutens derived from  $\{Y_t^{(i)}\}_{i=1}^\infty$  a family of strongly orthonormalized martingales  $\{H_t^{(i)}\}_{i=1}^\infty$ , that is a family where  $\langle H^{(i)}, H^{(j)} \rangle_t = \delta_{ij}t$ .

**Remark 2.** *The orthogonalization of the Teugels martingales is here understood as in the sense of strong orthogonality. This is to say, by definition, that for any two such martingales  $X, Y$ , one has  $\langle X, Y \rangle = 0$ , where  $\langle, \rangle$  is the angle bracket process (see [3] for details).*

Specifically,

$$H^i = c_{i,i}Y^i + c_{i,i-1}Y^{i-1} + \dots + c_{i,1}Y^1.$$

Set

$$\begin{aligned} q_{i-1}(x) &= c_{i,i}x^{i-1} + c_{i,i-1}x^{i-2} + \dots + c_{i,1}, \\ p_i(x) &= xq_{i-1}(x) = c_{i,i}x^i + c_{i,i-1}x^{i-1} + \dots + c_{i,1}x, \\ \tilde{p}_i(x) &= x(q_{i-1}(x) - q_{i-1}(0)) = c_{i,i}x^i + c_{i,i-1}x^{i-1} + \dots + c_{i,2}x^2. \end{aligned} \quad (3.3)$$

The following result gives the defining property of the family of orthonormalized Teugels martingales.

**Proposition 3.1.2.** *The orthonormalization of the monomials  $1, x, x^2, \dots$  with respect to the measure  $\nu(dx) = x^2\nu(dx) + \sigma^2\delta_0(dx)$  is given by the coefficients  $c_{i,j}$  and it is equivalent to the orthonormalization of the family of martingales  $\{Y_t^{(i)}\}_{i=1}^\infty$ .*

*Proof.* See [16] (p. 112).  $\square$

Henceforth, we define the family of *orthonormalized Teugels martingales*  $\{H_t^{(i)}\}_{i=1}^\infty$  as the set of orthonormalized martingales  $\{Y_t^{(i)}\}_{i=1}^\infty$ , where the orthonormalization is as prescribed by Proposition 3.1.2. The family  $\{H_t^{(i)}\}_{i=1}^\infty$  has then the following useful representation property. Below, we identify  $\mathcal{F}$  with  $\mathcal{F}_\infty$ .

**Theorem 3.1.3.** *Let  $X$  be a  $\mathcal{F}$ -measurable random variable. Then there exists a family of predictable processes  $\{\varphi_s^i\}_{i=1}^\infty$ , with  $\mathbb{E} \int_0^\infty \sum_{i=1}^\infty |\varphi_s^i|^2 ds < \infty$  and such that*

$$X = \mathbb{E}[X] + \sum_{i=1}^\infty \int_0^\infty \varphi_s^i dH_s^{(i)}.$$

*Proof.* See [16] (p. 118). □

We have the following important corollary.

**Corollary 3.1.4.** *Let  $M_t$  be an  $\mathcal{F}_t$ -adapted square-integrable martingale such that  $\sup_t \mathbb{E}|M_t|^2 < \infty$ . Then there exists a family of processes  $\{\varphi_t^i\}_{i=1}^\infty$ , such that the following predictable representation holds,*

$$M_t = \mathbb{E}[M_0] + \sum_{i=1}^{\infty} \int_0^t \varphi_s^i dH_s^{(i)}.$$

From here, the authors proved the existence and uniqueness of a solution holding for  $t \in [0, T]$  to the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_{s-}, Z_{s-}) ds - \sum_{i=1}^{\infty} \int_t^T Z_s^i dH_s^{(i)}, \quad (3.4)$$

where  $\xi$  is an  $\mathcal{F}_T$ -measurable random variable, with  $\mathcal{F}_t$  being the filtration generated by the family of random variables  $\{\ell_s, 0 \leq s \leq t\}$ . (see [17] for additional details).

## 3.2 An auxiliary result

In this section, we introduce a lemma that will be useful in the application developed in Chapter 5. Namely, Lemma 3.2.1 below provides a useful representation for the orthonormalized Teugels martingales  $H_t^{(i)}$ . Here,  $\ell_t$  is an arbitrary one-dimensional Lévy process with Lévy triple  $(b, a, \nu)$ ,  $a = (a_1, \dots, a_M)$ , where the Lévy measure  $\nu$  satisfies (3.1).

We remark that one can prove that  $E[L_t^{(i)}] = tE[X_1^{(1)}]$  (see [12], p. 29). Then, from (3.3) we can derive the following identity,

$$H_t^{(i)} = q_{i-1}(0)\ell_t + \sum_{0 < s \leq t} \tilde{p}_i(\Delta \ell_s) - t\mathbb{E}\left[\sum_{0 < s \leq 1} \tilde{p}(\Delta \ell_s)\right] - tq_{i-1}(0)\mathbb{E}[\ell_1]. \quad (3.5)$$

**Lemma 3.2.1.** *Let  $\ell$  be a one-dimensional Lévy process with the Lévy-Itô decomposition*

$$\ell_t = bt + \sum_{i=1}^M a_i \beta_i(t) + \int_{|x| \leq 1} x \tilde{\mu}(t, dx) + \int_{|x| > 1} x \mu(t, dx),$$

where  $\{\beta_i(t)\}_{i=1}^M$  are independent real-valued standard Brownian motions. Then, it holds that

$$H_t^{(i)} = q_{i-1}(0) \sum_{j=1}^M a_j \beta_j(t) + \int_{\mathbb{R}} p_i(x) \tilde{\mu}(t, dx).$$

In particular, if  $\ell_t$  is purely discontinuous, then  $H_t^{(i)} = \int_{\mathbb{R}} p_i(x) \tilde{\mu}(t, dx)$ .

*Proof.* Define  $\tilde{p}_i(x) = p_i(x) - xq_{i-1}(0)$ . We will use the representation (3.5) for  $H_t^{(i)}$ . Since  $\ell_t = \ell_t^c + \sum_{0 \leq s \leq t} \Delta \ell_s$ , where  $\ell_t^c$  is the continuous part of  $\ell_t$ , we obtain:

$$\begin{aligned} H_t^{(i)} &= q_{i-1}(0)\ell_t^c + \sum_{0 < s \leq t} p_i(\Delta \ell_s) - \mathbb{E} \left[ \sum_{0 < s \leq t} \tilde{p}_i(\Delta \ell_s) \right] - q_{i-1}(0)\mathbb{E}[\ell_t] \\ &= q_{i-1}(0) \left[ \ell_t^c - \mathbb{E}[\ell_t^c] \right] + \sum_{0 < s \leq t} p_i(\Delta \ell_s) - \mathbb{E} \left[ \sum_{0 < s \leq t} p_i(\Delta \ell_s) \right] \\ &= q_{i-1}(0) \sum_{j=1}^M a_j \beta_j(t) + \int_{\mathbb{R}} p_i(x) \tilde{\mu}(t, dx). \end{aligned} \quad \square$$

## Chapter 4

# FBSDEs driven by a Brownian motion

In this chapter, and in light of the work to be presented in the next chapter, we review a well-known method, known as the four-step scheme ([24]), with the objective to obtain explicitly a solution to FBSDEs of the type

$$\begin{cases} X_t = x + \int_0^t f(s, X_s, Y_s, Z_s) ds + \int_0^t \sigma(s, X_s, Y_s) dB_s, \\ Y_t = h(X_T) + \int_t^T g(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \end{cases} \quad (4.1)$$

with the help of a solution to a related second order parabolic PDE. We then introduce an asset pricing model to find optimal portfolios for contingent claims in Brownian markets, and in the presence of a large investor. Such a model allows an investor in the stock market to select a portfolio that is able to hedge an option from any exposure arising from the movement of the underlying stocks (see §4.2 for details).

While the purpose of this dissertation is the investigation of FBSDES with jumps, we have decided to give the current chapter the aforementioned structure due to a reason of two-fold nature : First, some steps of the method to be developed in Chapter 5 to solve FBSDEs with jumps are somehow related to those here presented. Furthermore, although the literature has provided applications of FBSDEs to the problem of hedging contingent claims under Brownian environments, it is still relatively sparse when the problem is considered in a framework in which markets are assumed to have jumps. Indeed, to the best of our knowledge we are the first to present an hedging model that can account for asynchronous jumps. Such feature is indeed a significant improvement, since in practice, the price of different stocks frequently jumps at different times. Here, by way of introduction to the problem of hedging contingent claims, we consider a simplified version, much in the lines of an already existing hedging model, introduced by Cvitanic and Ma ([9]).

In the first section, as a motivation to the main contribution of this work,

the existence and uniqueness of solutions to FBSDEs with jumps on intervals of arbitrary duration and for a large class of coefficients, we review the main elements of the method developed by Ma. et al ([24]), used to find the existence of an explicit solution to FBSDEs driven by Brownian motions. It will be then evident how the derivation of this method heavily relies on results on the existence and uniqueness results for second order PDEs ([21]). In the second section we derive a system of FBSDEs, whose solution can be used to obtain a hedging portfolio for a contingent claim. The results of the first section will then assure the existence of such a solution. As mentioned, the application here presented will be the starting point for further development in the final section of the next chapter.

## 4.1 FBSDEs driven by a Brownian motion and parabolic quasilinear PDEs

Let  $(\Omega, \mathcal{F}, P)$  be a probability space, where we define a  $d$ -dimensional Brownian motion  $B_t$ . Define the filtration  $\mathcal{F}_t = \sigma\{B_s, 0 \leq s \leq t\} \vee \mathcal{N}$ , where  $\mathcal{N}$  is a collection of subsets of all  $P$ -null sets. In this section, we go back to the fully coupled FBSDEs

$$\begin{cases} X_t = x + \int_0^t f(s, X_s, Y_s, Z_s) ds + \int_0^t \sigma(s, X_s, Y_s) dB_s, \\ Y_t = h(X_T) + \int_t^T g(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \end{cases} \quad (4.2)$$

where  $f, \sigma, h, g$  are functions taking values in  $\mathbb{R}^n, \mathbb{R}^{n \times d}, \mathbb{R}^m$ , and  $\mathbb{R}^m$  respectively, and measurable with respect to their corresponding borelian  $\sigma$ -algebras. We define after Ma et al., an *ordinary adapted solution* to (4.2) as a triple of  $\mathcal{F}_t$ -adapted and square-integrable  $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$ -valued processes  $(X_t, Y_t, Z_t)$  satisfying (4.2) a.s. ([24]). As mentioned in the introduction, the process  $Z_s$  is linked with the representation given by Theorem 2.3.2. We now show how the theory of Ladyzhenskaya et al. on parabolic second order PDEs can be used to build explicitly ordinary adapted solutions to (4.2). We follow closely the original work of [24], and base our explanation on a heuristic argument.

Let us assume that  $X_t$  and  $Y_t$  in (4.2) are related by  $Y_t = \theta(t, X_t)(a.s)$ , where  $\theta$  is a  $C^{1,2}(\mathbb{R}^n)$  real-valued function. An application of Itô's formula to  $\theta(t, X_t)$  then gives

$$\begin{aligned} h(X_T) - Y_t &= \int_t^T \theta_x(t, X_t) \sigma(s, X_s, \theta(s, X_s)) dB_s \\ &+ \int_t^T [\theta_s + \frac{1}{2} \text{tr}(\theta_{xx}(t, X_t) \sigma \sigma^T(s, X_s, \theta(s, X_s))) + \theta_x(s, X_s) f(s, X_s, \theta(s, X_s), Z_s)] ds, \end{aligned}$$

whereby a comparison with the BSDE in (4.2) implies

$$\begin{cases} \theta_t + \frac{1}{2}\text{tr}(\theta_{xx}\sigma\sigma^T(t, X_t, \theta(t, X_t))) + \theta_x f(t, X_t, \theta(t, X_t), Z_t) \\ = -g(t, X_t, \theta(t, X_t), Z_t), \\ Z_t = \theta_x \sigma(t, X_t, \theta(t, X_t)), \quad \theta(T, X_T) = h(X_T), \end{cases} \quad (4.3)$$

where  $\text{tr}(\theta_{xx}\sigma\sigma^T(t, x, \theta(t, x)))$  is a vector function in  $\mathbb{R}^n$ , with  $k$ -component given by  $\text{tr}(\theta_{xx}^k\sigma\sigma^T(t, x, \theta(t, x)))$ . Here,  $\theta_t, \theta_x$  and  $\theta_{xx}$  are understood to be evaluated in  $(t, X_t)$ , and we omit the parameters for simplicity of notation.

Assume for now that  $\theta(t, x)$  is a classical solution to the final value problem

$$\begin{cases} \theta_t + \frac{1}{2}\text{tr}(\theta_{xx}\sigma\sigma^T(t, x, \theta)) + \theta_x f(t, x, \theta, \theta_x \sigma(t, x, \theta)) = -g(t, x, \theta, \theta_x \sigma(t, x, \theta)), \\ \theta(T, x) = h(x), \end{cases} \quad (4.4)$$

where by *classical* solution we mean a solution that is bounded together with its derivatives  $\theta_t, \theta_x$  and  $\theta_{xx}$ . Next, let  $X_t$  satisfy the FSDE

$$X_t = x + \int_0^t \tilde{f}(s, X_s) ds + \int_0^t \tilde{\sigma}(s, X_s) dB_s, \quad (4.5)$$

where for  $0 \leq t \leq T$ ,  $\tilde{f}(t, x) = f(t, x, \theta(t, x), \theta_x(t, x)\sigma(t, x, \theta(t, x)))$ , and  $\tilde{\sigma}(t, x) = \sigma(t, x, \theta(t, x))$ . Then, by (4.3), the triple  $(X_t, Y_t, Z_t)$ , where

$$\begin{cases} Y_t = \theta(t, X_t), \\ Z_t = \theta_x(t, X_t)\sigma(t, X_t, \theta(t, X_t)) \end{cases} \quad (4.6)$$

is an ordinary solution to (4.2). From what was seen above, we have obtained the solution to (4.2) as follows:

- Solve the parabolic system of second order PDEs

$$\begin{cases} \theta_t + \frac{1}{2}\text{tr}(\theta_{xx}\sigma\sigma^T(t, x, \theta)) + \theta_x f(t, x, \theta, \theta_x \sigma(t, x, \theta)) = \\ -g(t, x, \theta, \theta_x \sigma(t, x, \theta)) \theta(T, x) = h(x). \end{cases} \quad (4.7)$$

- Solve (4.5) with the help of  $\theta$ .
- Obtain an adapted solution to (4.2) by the prescription (4.6).

**Remark 3.** In their paper, the authors use a function  $\hat{\sigma}(X_t, Y_t, Z_t)$  instead of  $Z_t$  as the integrand process. In that case, the scheme needs to be updated with an additional assumption: There exists a smooth map  $z : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \rightarrow \mathbb{R}^{m \times d}$  such that for all  $(t, x, y, p) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$ ,

$$p\sigma(t, x, y) + \hat{\sigma}(t, x, y, z(t, x, y, p)) = 0.$$

This alternative presentation does not add to the generality of the result.

We now recall some results of the theory on parabolic second order PDEs ex-

pounded in [21]. Let  $\mathbb{F}$  be a bounded domain in  $\mathbb{R}^n$  with piecewise-smooth boundary  $\partial\mathbb{F}$ . Then, Theorem 7.1 (p. 596) asserts the conditions under which the initial-boundary value problem

$$\begin{cases} -\sum_{i,j=1}^n a_{ij}(t, x, u(t, x)) u_{x_i x_j}(t, x) + \sum_{i=1}^n a_i(t, x, u(t, x), u_x(t, x)) u_{x_i}(t, x) \\ + a(t, x, u(t, x), u_x(t, x)) + u_t \\ u(0, x) = \psi(x), \\ u_{\partial\mathbb{F}} = 0, \end{cases} \quad (4.8)$$

has a unique classical solution  $u : [0, T] \times \mathbb{F} \rightarrow \mathbb{R}^m$ . Here,  $a_{ij} : [0, T] \times \mathbb{F} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ ,  $a_i : [0, T] \times \mathbb{F} \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$  are scalar functions for  $i, j = 1, \dots, n$ , and  $a : [0, T] \times \mathbb{F} \times \mathbb{R}^m \times \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^m$ .

We recall that a *uniformly parabolic* PDE satisfies

$$\mu(|u|)\xi^2 \leq \sum_{i,j=1}^n a_{ij}(s, x, u)\xi_i \xi_j \leq \nu(|u|)\xi^2,$$

with  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ ,  $\nu(s)$  and  $\mu(s)$  positive continuous functions defined for  $s \geq 0$ , with  $\nu(s)$  *non-increasing* and  $\mu(s)$  *non-decreasing* (see [21]). Note that we obtain the so-called *Cauchy problem* (4.8)-(4.9), if we replace the initial condition in (4.8) by

$$\begin{cases} \theta(0, x) = h(x), \end{cases} \quad (4.9)$$

and consider the domain of the coefficients in (4.8) to be  $\mathbb{R}^n$ .

Now, we can make the change of coordinates  $\tilde{\theta}(t, x) = \theta(T - t, x)$  and the system (4.4) takes the form (4.8)-(4.9). Then, by means of a diagonal argument and some other results we can obtain a classical solution to (4.8)-(4.9) (see Lemma 3.2 and Prop. 3.3 in [24] and Theorem 7.5 in [21]).

We now outline the proof of the existence and uniqueness of a classical solution to (4.2) (see Theorem 4.1 in [24]). For completeness we list the required assumptions.

(A1)  $n=d$ , the functions  $f, g, \sigma$ , are smooth, and take values in  $\mathbb{R}^n, \mathbb{R}^m$  and  $\mathbb{R}^{m \times n}$  respectively. Furthermore, their first order derivatives with respect to  $x, y$  and  $z$  are bounded by a constant  $L > 0$ .

(A2) (uniform-parabolic condition) The function  $\sigma$  satisfies

$$\sigma(t, x, y)\sigma^T(t, x, y) \geq \nu(|y|)I, \quad \forall (t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m,$$

where  $\nu(s)$  is a positive continuous function defined for  $s \geq 0$ .

(A3) The function  $h$  is bounded in  $C^{2+\alpha}(\mathbb{R})$  for  $\alpha \in (0, 1)$  and for all  $(t, x, y, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times n}$ ,

$$|\sigma(t, x, y), f(t, x, y, 0)| \leq \mu(|y|), \quad |g(t, x, 0, z)| \leq C,$$

where the constant  $C > 0$  and  $\mu$  is a positive non-decreasing function.

We now state the result, and proceed with an illustration of the proof.

**Theorem 4.1.1.** *Let (A1)-(A3) hold. Then the forward-backward SDE (4.2) admits a unique adapted solution  $(X, Y, Z)$  where  $Y$  and  $Z$  are given by (4.6).*

Assume that (A1)-(A3) hold, and let the function  $\theta(t, x)$  be the unique solution to (4.4). Recall the functions  $\tilde{f}(t, X_t)$  and  $\tilde{g}(t, X_t)$ , defined in (4.5). In addition, it is known that if  $b$  and  $\sigma$  are uniformly Lipschitz, it is possible to obtain a unique  $\mathcal{F}_t$ -adapted continuous solution  $X_t$  to (4.5) (see Proposition 4.1.2 below). Finally, we can use  $X_t$  in the prescription (4.6) to obtain an adapted solution  $(X, Y, Z)$  to (4.2).

The uniqueness of solution is obtained as follows: First, an application of Itô's formula together with Gronwall's Lemma guarantees that any solution to (4.2) has to be of the form given by (4.6). Finally, two solutions of type (4.6) have to be almost surely identical.

The next result is needed to find the solution of the FSDE, and we refer the reader to [2] (p. 367) for a proof.

**Proposition 4.1.2.** *If the coefficients  $\tilde{f}_i(t, x)$ , and  $\tilde{\sigma}(t, x)$  are uniformly Lipschitz, then the FSDE (4.5) has a unique solution which is  $\mathcal{F}_t$ -adapted and continuous.*

The last theorem of this section is useful to compare solutions of BSDEs. Consider then the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s^i, Z_s^i)(\omega) ds - \int_t^T Z_s dB_s, \quad (4.10)$$

where the *generator* function  $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^{m \times d} \times \omega \rightarrow \mathbb{R}^m$  is such that process  $(t, \omega) \rightarrow f(t, y, z)(\omega)$  is  $\mathcal{F}_t$ -progressively measurable, and  $\xi$  is an  $\mathcal{F}_T$ -adapted,  $\mathbb{R}^m$ -valued random variable.

A solution to (4.10) is a pair of stochastic processes  $(Y, Z)$  such that  $Y_t$  is  $\mathbb{R}^m$ -valued, continuous and  $\mathcal{F}_t$ -adapted, while  $Z_t$  is  $\mathbb{R}^{m \times d}$ -valued,  $\mathcal{F}_t$ -predictable, and satisfies  $\int_0^T \|Z_s\|^2 ds < \infty$  a.s. We say that  $(f, \xi)$  are *standard* parameters, if in addition  $f(t, 0, 0)(\omega)$  is square-integrable predictable, and  $f$  is uniformly Lipschitz, i.e. there exists  $C > 0$  such that  $dP \times dt$ -a.s.,

$$|f(t, y_1, z_1)(\omega) - f(t, y_2, z_2)(\omega)| \leq C(|y_1 - y_2| + |z_1 - z_2|), \forall (y_1, z_1), \forall (y_2, z_2).$$

In the following,  $m = 1$ .

**Theorem 4.1.3** (Comparison Theorem for BSDEs). *Let  $(f^1, \xi_1)$  and  $(f^2, \xi_2)$  be two standard parameters of BSDEs of type (4.10), and  $(X_t^1, Z_t^1), (X_t^2, Z_t^2)$  their associated solutions. If  $\xi^1 \geq \xi^2$  a.s. and  $f^1(t, Y_t^2, Z_t^2) \geq f^2(t, Y_t^2, Z_t^2)$ ,  $dP \times dt$ -a.s., then  $Y_t^1 \geq Y_t^2$  a.s., for all  $t \in [0, T]$ .*

*Proof.* See [13], p. 23. □

Now, since the pair of null processes  $(0, 0)$  is a solution to the null BSDE, i.e. that associated to standard parameters that are both null, we have the following corollary.

**Corollary 4.1.4.** *Let  $(f, \xi)$  be standard parameters of a given BSDE of type (4.10) which admits a solution  $(Y_s, Z_s)$ . Then, if  $\xi \geq 0$  a.s., and  $f(t, 0, 0) \geq 0$   $dP \times dt$ -a.s., it holds that  $Y_t \geq 0$ .*

In case the FBSDEs contain Poisson-type stochastic integrals, the associated PDE becomes a partial-integral differential equation (PIDE), and so the four-step scheme cannot be applied to these type of FBSDEs. The next chapter is dedicated to overcome such difficulties, and will allow obtaining explicit solutions to FBSDEs with jumps. We now give an application of the theory developed in this section to the field of asset pricing, more specifically, to the problem of hedging options in Brownian markets.

## 4.2 Hedging options for a large investor in Brownian markets

In Finance, in particular in the sub-field of asset pricing, SDEs arise as natural objects to model the evolution of stock prices. Here, we show how FBSDEs can be used to find a hedging portfolio for contingent claims, such as an option in the present case. Recall that a contingent claim is an asset whose value is indirectly related to another asset (the *underlying*) and may depend explicitly on the value of the underlying at a pre-specified time. An example of such a contingent claim is the *European call (put) option* with *expiry date*  $T$  and *exercise* (or *strike*) price  $K$ . This is a contract that gives the right to its holder at a certain time  $T$  to buy (sell) an asset at a pre-specified price  $K$ . The *payoff* to the holder of the call is  $C = (S_T - K)^+$ , whereas that of the put is  $P = (K - S_T)^+$ .

An investor holding options is often interested in *hedging* their position against movements in the underlying. Such type of hedging is usually called *delta-hedging* and consists in holding a running position on the underlying asset that is modified according to the changes in the price of the option.

The reasons for delta-hedging may be of various nature: For instance, the holder may be interested only in exploiting a view on the volatility of the underlying, engaging in the so called volatility trading, or market conditions may be expected to go temporarily against their position and an investor is looking to buy insurance on their portfolio, by, for instance, acquiring a put option against the fall of the price of a certain stock. In actual business practice, delta-hedging is effected by adding or removing to their portfolio a certain quantity of the stock such that the variations in the price of the option due to movements in the underlying price are minimised, and ideally neutralised<sup>1</sup>. The first attempt

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<sup>1</sup>Note however that the price of European calls and put options depends not only on moves in the underlying asset, but also the volatility of the underlying, the time to expiry, the risk-free interest rate, and the proportion of dividend paid (*dividend yield*). Thus *delta-hedging*, i.e. hedging against movements in the underlying, still leaves the investor exposed to other risks, mainly the volatility and the time to expiry. In theory, an investor can hedge all the remaining risks, but the cost of doing so may have a significant impact on the overall performance of the strategy. As such, delta-hedging strategies, specially those that do not need injection of outside capital in order to be implemented, remain useful in portfolio management.

to address the problem of finding a replicating portfolio was the seminal work of Black and Scholes ([15]), where the authors propose a hedging strategy that translate into the eponymous pricing formula for a call option. The derivation of the formula is based on the existence of a replicating portfolio, that is a portfolio consisting of the underlying stock and a risk-free deposit, which is worth the payoff of the option at the expiry date. Furthermore it is fundamental that such a portfolio is self-financing or what is the same, that the associated replicating strategy can be maintained without the need of outside money. The fact that the replication property holds necessarily only at the expiry date is one of the defining characteristics of the original Black-Scholes, which implies that the option to be priced has to be European, or what is the same, can only be exercised at the expiry date.

In the classical Black-Scholes framework, some assumptions are made: The stock does not pay dividends, the market is efficient in the sense that prices do reflect all the known information at a given time, the distribution of the returns is normally distributed, and both the volatility and drift coefficients are known and constant.

It is well known that options are often used by individuals or firms as a hedging tool, or insurance against certain risks. As such, given the importance that the price paid for such an insurance reflects the likelihood of the materialisation of the risks it purports to mitigate, it is worth to analyse the importance of the last two assumptions of the Black-Scholes model, since the formula can be easily adapted to options on dividend-paying stocks.

One of the assumptions of the classical Black-Scholes model is that no individual investor action is able to influence market prices. The importance of accounting for the existence of large investors, however, has been increasing, given the prevalence of electronic trading and, in particular, high frequency trading which makes possible the issuance of thousands of orders over short periods of time. Other manifestations of this type of influence can be felt when central banks are looking to add to or dispose of their vast assets. The issue of the presence of a large investor was addressed by Cvitanić and Ma [9], where an FBSDE hedging model that can account for the presence of a large investor is proposed. Here, we propose a similar model, but under a different set of assumptions.

#### 4.2.1 The FBSDE hedging model

Consider a market composed of  $d$  stocks, and one risk-free asset, that we assume to be an interest-bearing risk-free deposit. We also assume that there are no transaction costs and the number of stocks is infinitely divisible.

Fix a time horizon  $T > 0$  and let  $(\Omega, \mathcal{F}, P)$  be a complete filtered probability space, where  $\mathcal{F}_t = \sigma\{B_s, 0 \leq s \leq t\} \vee \mathcal{N}$ ,  $B_t$  is a  $d$ -dimensional Brownian

motion, and  $\mathcal{N}$  is a collection of subsets of all  $P$ -null sets. Consider the FSDE

$$\begin{cases} dD_t = D_t r(t, S_t, \pi_t) dt, \\ dS_t^i = S_t^i [\tilde{f}_i(t, S_t, V_t, \pi_t) dt + \sum_{j=1}^d \sigma_{ij}(t, S_t, V_t) dB_t^j], \\ D_0 = 1, \quad S_0^i = S_i > 0, \quad i = 1, \dots, d, \end{cases} \quad (4.11)$$

where  $r : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $\tilde{f} : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\sigma : [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^{d \times d}$ ,  $D_t$  is the deposit process,  $S = \{S_t^i\}_{i=1}^d$  is the  $d$ -dimensional stock price process,  $V_t$  is the (real-valued) *value* process, and  $\pi_t = \{\pi_t^i\}_{i=1}^d$  is the portfolio process, where  $\pi^i(t)$  is the amount invested in stock  $i$ . Note that  $V_t - \sum_{i=1}^d \pi^i(t)$  is the amount that the investor allocates to the deposit at time  $t$ .

The prescription given by (4.11) allows the investor actions to have an impact on the price of the stocks and on the interest rate, but here, we chose to make the volatility independent of the investor actions. This is so, since it is reasonable to suppose that the actions of a single investor have a negligible impact on the volatility of an asset over short time duration. Let us formulate the BSDE for the value process. For the sake of motivation, we remark that for  $\delta > 0$ ,  $\frac{\pi^i(t)}{S^i(t)}(S^i(t+\delta) - S^i(t))$  is approximately the change in the value of the position associated to the stock  $i$  over the period  $[t, t+\delta]$ . It is therefore quite natural to assume that the value process has differential

$$dV_t = \sum_{i=1}^d \frac{\pi_t^i}{S_t^i} dS_t^i + \left( V_t - \sum_{i=1}^d \pi_t^i \right) D_t^{-1} dD_t \quad (4.12)$$

with  $V_0 = x \geq 0$ . Substituting (4.11) into (4.12) we obtain

$$dV_t = \tilde{g}(t, S_t, V_t, \pi_t) dt + \sum_{i=1}^n \sum_{j=1}^n \sigma_{ij}(t, S_t, V_t) \pi_t^i dB_t^j, \quad (4.13)$$

with  $V_T = h(S_T)$ , where for  $(t, x, u, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$

$$\tilde{g}(t, x, u, p) = \sum_{i=1}^d a_i \tilde{f}_i(t, x, u, p) + \left( u - \sum_{i=1}^d p_i \right) r(t, x, p). \quad (4.14)$$

Now, in order to obtain a realistic model for the replication of a contingent claim, we need the following two considerations, the first for the hedging model, and the second for the strategy itself. First, we require for the obvious reasons of a well-functioning market, that the prices  $S_t^i$  have to be always positive. Furthermore, if at any point in time  $s \in [0, T]$ , the value  $V_s$  would be negative, there would be the need of an injection of outside capital to maintain the strategy, and thus the portfolio would not be self-financing. In view of these last points,

we say that the strategy is *admissible* if

$$V_t \geq 0, \quad 0 \leq t \leq T \quad i = 1, \dots, d. \quad (4.15)$$

The problem of hedging a contingent claim in the presence of a large investor is equivalent to finding an admissible portfolio that is also replicating.

Let  $c$  be the contingent-claim with payoff function  $h(x)$ , which we model as an  $\mathcal{F}_T$ -adapted real-valued random variable such that  $c = h(S_T)$ , a.s. In order to use the results of the last section, we define the process  $Z_t = \sigma^T(t, S_t, V_t)\pi(t)$ , where  $\det(\sigma(t, x, y)) \neq 0$  for all  $(t, x, y)$ . Define for  $(t, x, u, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$  the functions

$$\begin{cases} f_i(t, x, u, p) = x_i \tilde{f}_i(t, x, u, p(\sigma^T)^{-1}(t, x, u)), \\ g(t, x, u, p) = \tilde{g}(t, x, u, p(\sigma^T)^{-1}(t, x, u)). \end{cases} \quad (4.16)$$

Then, if  $(S_t, V_t, Z_t)$  is an ordinary solution to the system

$$\begin{cases} S_t^i = S_0^i + \int_0^t f_i(s, S_s, V_s, Z_s) ds + \int_0^t \sum_{j=1}^n S_s^i \sigma_{ij}(s, S_s, V_s) dB_s^j, \\ V_t = h(S_T) - \int_t^T g(s, S_s, V_s, Z_s) ds - \int_t^T \sum_{j=1}^n Z_s^j dB_s^j, \\ V_0, S_0^i > 0, \quad i = 1, \dots, d, \end{cases} \quad (4.17)$$

such that  $S_t$  and  $V_t$  satisfy (4.15), then  $P_t = \sigma^{-1}(t, S_t, V_t)Z_t$  is a replicating portfolio under admissible conditions. We now prove that (4.17) admits an (ordinary) solution.

Now, note that in practice, the prices of stocks trade within a range. This assumption is realistic, since the price of a stock trading in one of the main exchanges going to zero would mean that the company is bankrupt. In practice, however the stock of such a company would be suspended from trading before it goes below a certain value. Moreover, stocks whose normal trading range is in the fraction of cents (the so called "pink sheets"), are not suitable to hedging purposes, not the least due to the associated low levels of liquidity. On the other hand, with the exception of the class A stock of Berkshire Hathaway that currently trades in the hundreds of thousands of dollars, virtually all the remaining stocks tend to trade in levels well below those.<sup>2</sup> We can thus assume that stocks trade inside the interval  $[a, b]$ , where  $a$  and  $b$  can be easily picked based on historical data.

Let  $e = b - a$ , and consider the  $d$ -dimensional cube  $C'_e$  centred at the point  $\left(a + \frac{e}{2}, \dots, a + \frac{e}{2}\right)$  with edge  $e$  and a copy  $C_e$  with smooth corners and edges<sup>3</sup>.

Let  $\varepsilon < a$  and let  $C_{e+\varepsilon}$ , be a cube sharing the center with  $C$  with edge  $e + \varepsilon$  and

<sup>2</sup>Indeed, after the price of their stocks reach a certain level, companies usually engage in stock-splitting, whereby the price of a stock is reduced by a certain proportion, while at the same time creating a number of new stock into the inverse proportion.

<sup>3</sup>We can more generally assume that all the ranges of all the prices are independent, which results in a parallelepiped, that can also be included in a smooth parallelepiped.

with the corners and edges already smoothed out. Finally, consider a smooth function  $\eta(x)$  that takes values in  $[0, 1]$ , is zero outside  $C_{e+\varepsilon}$ , and is equal to 1 in  $C_e$ . It is clear that under these circumstances  $h(S_T)\eta(S_T) = c_T$ , and that any solution to

$$\begin{cases} S_t^i = S_0^i + \int_0^t f_i(s, S_s, V_s, Z_s) ds + \int_0^t S_s^i \sum_{j=1}^d \sigma_{ij}(s, S_s, V_s) dB_s^j, \\ V_t = \eta(S_T)h(S_T) - \int_t^T g(s, S_s, V_s, Z_s) ds - \int_t^T \sum_{j=1}^d Z_s^j dB_s^j, \\ V_0, S_0^i > 0, \quad i = 1, \dots, d. \end{cases} \quad (4.18)$$

is a solution to (4.17). In order to find a solution to (4.18), we assume, as in the previous section,  $V_t = \theta(t, S_t)$ , with  $\theta \in C^{1,2}(\mathbb{R})$ . Similarly to the derivation of (4.3), an application of Itô's formula to the FSDE in (4.18) yields

$$\begin{aligned} \eta(S_T)\theta(T, S_T) - V_t &= \int_t^T \left[ \sum_{i=1}^n \partial_i \theta(s, S_s) f_i(s, S_s, \theta(s, S_s), Z_s) \right. \\ &\quad \left. + \theta_s + \frac{1}{2} \sum_{i,j=1}^n \partial_{ij}^2 \theta(s, S_s) S_s^i S_s^j \sigma^i(\sigma^j)^T(s, S_s, \theta(s, S_s)) \right] ds \\ &\quad + \sum_{i,j=1}^n \int_t^T S_s^i \partial_i \theta(s, S_s) \sigma_{ij}(s, S_s, \theta(s, S_s)) dB_s^j, \end{aligned} \quad (4.19)$$

where  $\sigma^i$  is the  $i$ -th row of the matrix  $\sigma$ . Next, a comparison with the BSDE of (4.18) implies

$$\begin{cases} \theta_t + \frac{1}{2} \sum_{i,j=1}^n S_t^i S_t^j (\sigma^i, \sigma^j)(t, S_t, \theta(t, S_t)) \theta_{x_i x_j}(t, S_t) \\ + \sum_{i=1}^n f_i(t, S_t, \theta(t, S_t), Z_t) \theta_{x_i}(t, S_t) = g(t, S_t, \theta(t, S_t), Z_t), \\ Z_t^i = \sum_{j=1}^n S_t^i \partial_i \theta(t, S_t) \sigma_{ij}(t, S_t, \theta(t, S_t)), \\ \theta(T, S_T) = h(S_T) \eta(S_T), \end{cases} \quad (4.20)$$

where we remark that  $f_i(t, S_t, \theta(t, S_t), Z_t) = S_t^i \tilde{f}_i(t, S_t, \theta(t, S_t), (S_t, \partial_x \theta(t, S_t)))$ .

Define  $\hat{f}_i(t, x, u, p) = f_i(t, x, u, (x, p))$ ,  $\hat{g}(t, x, u, p) = g(t, x, u, (x, p))$ , and  $\hat{\sigma}_{ij}(t, x, y) = x_i \sigma_{ij}(t, x, y)$ . With these definitions, if  $\theta(t, x)$  is a solution to the final value problem

$$\begin{cases} \sum_{i=1}^n \hat{f}_i(t, x, \theta(t, x), \theta_x) \theta_{x_i} + \frac{1}{2} \sum_{i,j=1}^n (\hat{\sigma}^i, \hat{\sigma}^j)(t, x, \theta(t, x)) \theta_{x_i x_j} + \theta_t \\ - \hat{g}(t, x, \theta(t, x), \theta_x) = 0, \\ \theta(T, x) = \eta(x) h(x), \end{cases} \quad (4.21)$$

we can use it to solve with the help of Proposition 4.1.2 the FSDE

$$S_t^i = S_0^i + \int_0^t f_i^1(s, S_s) ds + \int_0^t S_s^i \sum_{j=1}^d \sigma_{ij}^1(s, S_s) dB_s^j, \quad (4.22)$$

with  $f_i^1(t, x) = \hat{f}_i(t, x, \theta(t, x), (x, \theta_x(t, x)))$  and  $\sigma^1(t, x) = \sigma(t, x, \theta(t, x))$ . From here we conclude that the triple of adapted processes  $(S_t, \theta(t, S_t), (S_t, \theta_x(t, S_t)))$  is a solution to (4.20).

Next, define  $\hat{\theta}(t, x) = \theta(T - t, x)$ , and consider the initial-boundary value problem

$$\begin{cases} -\sum_{i=1}^n \hat{f}_i(t, x, \hat{\theta}(t, x), \hat{\theta}_x) \hat{\theta}_{x_i} - \frac{1}{2} \sum_{i,j=1}^n (\hat{\sigma}^i, \hat{\sigma}^j)(t, x, \hat{\theta}(t, x)) \hat{\theta}_{x_i x_j} \\ + \hat{\theta}_t + \hat{g}(t, x, \hat{\theta}(t, x), \hat{\theta}_x) = 0, \\ \hat{\theta}(0, x) = \eta(x)h(x), \quad \hat{\theta}(t, x)|_{\partial C_r} = 0, \end{cases} \quad (4.23)$$

where  $\partial C_r$  is the boundary of  $C_r$ . This problem is equivalent to (4.21), and so we now introduce the standing assumptions needed to obtain the existence and uniqueness of a solution to (4.23).

- (A1) For all  $t, x, u \in [0, T] \times C_r \times \mathbb{R}$ ,  $\hat{\sigma}(t, x, u) \hat{\sigma}^T(t, x, u)$  is positive definite.  
(A2) For all  $(t, x, u, p) \in [0, T] \times C_r \times \mathbb{R} \times \mathbb{R}^d$ , there exist nonnegative constants  $c_1$  and  $c_2$  such that

$$\hat{g}(t, x, u, p)u \geq -c_1 - c_2 u^2. \quad (4.24)$$

**Remark 4.** Under assumptions (A1)-(A2) there exists a constant  $M > 0$  such that for any classical solution  $\theta(t, x)$  of (4.23),  $\sup_{[0, T] \times C_r} |\theta| \leq M$ .

- (A3) The functions  $\hat{\sigma}^i \hat{\sigma}^j(t, x, u)$ ,  $\hat{f}_i(t, x, u, p)$ ,  $\hat{g}(t, x, u, p)$ , and the derivatives  $\partial_u \hat{\sigma}^i \hat{\sigma}^j(t, x, u)$ ,  $\partial_x \hat{\sigma}^i \hat{\sigma}^j(t, x, u)$ , are continuous in the region  $\mathcal{R} = [0, T] \times \overline{C_r} \times [-M, M] \times \mathbb{R}^d$ ,  $i, j = 1, 2, \dots, n$ . Moreover, in  $\mathcal{R}$ , the functions  $\hat{f}_i$  and  $\hat{g}$  satisfy

$$|\hat{f}_i(t, x, u, p)| \leq \mu(|u|)(1 + |p|), \quad (4.25)$$

$$|\hat{g}(t, x, u, p)| \leq \mu(|u|)(1 + |p|^2), \quad (4.26)$$

$$|\partial_x \hat{\sigma}_{ij}, \partial_u \hat{\sigma}_{ij}| \leq \mu(|u|), \quad (4.27)$$

while  $\sigma$  satisfies

$$\nu(|u|)\xi^2 \leq \sum_{i,j=1}^d \hat{\sigma}^i \hat{\sigma}^j(t, x, u) \xi_i \xi_j \leq \mu(|u|)\xi^2,$$

where  $\mu(x)$  is a non-decreasing function,  $\nu(x)$  a non-decreasing function, both defined for  $x \geq 0$  and taking positive values, and  $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$ ,  $i, j \in \{1, \dots, d\}$ .

**Remark 5.** Under (A1)-(A3) there exists a constant  $M_1 > 0$  such that for any classical solution  $\theta(t, x)$  of (4.23),  $\sup_{[0, T] \times C_r} |\theta_x| \leq M_1$ .

- (A4) The derivatives of the functions  $\hat{\sigma}^i \hat{\sigma}^j(t, x, u)$ ,  $\hat{f}_i(t, x, u, p)$ ,  $\hat{g}(t, x, u, p)$ , in  $t, x, u$  and  $p$ , the first derivatives  $\partial_u(\hat{\sigma}^i \hat{\sigma}^j)(t, x, u)$ ,  $\partial_x(\hat{\sigma}^i \hat{\sigma}^j)(t, x, u)$ , and the second derivatives  $\partial_u^2(\hat{\sigma}^i \hat{\sigma}^j)$ ,  $\partial_x^2(\hat{\sigma}^i \hat{\sigma}^j)$ ,  $\partial_{xu}^2(\hat{\sigma}^i \hat{\sigma}^j)$ ,  $\partial_{tx}^2(\hat{\sigma}^i \hat{\sigma}^j)$ ,  $\partial_{tu}^2(\hat{\sigma}^i \hat{\sigma}^j)$

are continuous in the region  $\hat{\mathcal{R}} = [0, T] \times \overline{C_r} \times [-M, M] \times \{|p| \leq M_1\}$ .

**(A5)** The function  $h$  is bounded in the Hölder norm on  $C^{2+\alpha}(\mathbb{R}^d)$  for  $\alpha \in (0, 1)$ .

**Theorem 4.2.1.** *Under (A1)-(A5), the initial-boundary value problem (4.23) has a unique solution that is bounded together with its derivatives  $\theta_t, \theta_x$ , and  $\theta_{xx}$ .*

*Proof.* See Theorem 7.1, §6, [21].  $\square$

In order to apply this last Theorem, we give the assumptions on the original coefficients of FBSDE (4.18).

**(B1)** For all  $t, x, u \in [0, T] \times C_e \times \mathbb{R}$ ,  $\sigma(t, x, u)\sigma^T(t, x, u)$  is positive definite.

**(B2)** There exist  $c_1, c_2 > 0$  such that

$$g(t, x, u, p)u > -c_1 - c_2|u|^2,$$

for all  $(t, x, u, p) \in [0, T] \times C_{r+\varepsilon} \times \mathbb{R} \times \mathbb{R}^d$ .

**(B3)** The functions  $\sigma^i \sigma^j(t, x, u)$ ,  $\tilde{f}_i(t, x, u, p)$ ,  $r(t, x, p)$ , and the derivatives  $\partial_x(x_i \sigma_{ij})$ ,  $\partial_u(x_i \sigma_{ij})$ ,  $\partial_u(\sigma^i \sigma^j)$ ,  $\partial_x(\sigma^i \sigma^j)$ , are continuous in the region  $\mathcal{R} = [0, T] \times \overline{B_r} \times [-M, M] \times \mathbb{R}^d$ ,  $i, j \in \{1, 2, \dots, n\}$ . Moreover, in  $\mathcal{R}$ , the functions  $f_i$  and  $\sigma$  satisfy

$$|\tilde{f}_i(t, x, u, p)| \leq \mu(|u|)(1 + |p|), \quad (4.28)$$

$$|\partial_x \sigma_{ij}(t, x, u), \partial_u \sigma_{ij}(t, x, u)| \leq \mu(|u|) \quad (4.29)$$

and  $\sigma$  satisfies

$$\nu(|u|)\xi^2 \leq \sum_{i,j=1}^d \sigma^i(t, x, u) \sigma^j(t, x, u) \xi_i \xi_j \leq \mu(|u|)\xi^2, \quad (4.30)$$

where  $\mu(x)$  is a non-decreasing function,  $\nu(x)$  a non-decreasing function, both defined for  $x \geq 0$  and taking positive values, and  $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$ ,  $i, j \in \{1, \dots, d\}$ .

**(B4)** The derivatives of the functions  $\sigma^i \sigma^j(t, x, u)$ ,  $\tilde{f}_i(t, x, u, p)$ ,  $\tilde{g}(t, x, u, p)$ , in  $t, x, u$  and  $p$ , the first derivatives  $\partial_u(\sigma^i \sigma^j)(t, x, u)$ ,  $\partial_x(\sigma^i \sigma^j)(t, x, u)$ , and the second derivatives  $\partial_u^2(\sigma^i \sigma^j)$ ,  $\partial_x^2(\sigma^i \sigma^j)$ ,  $\partial_{xu}^2(\sigma^i \sigma^j)$ ,  $\partial_{tx}^2(\sigma^i \sigma^j)$ ,  $\partial_{tu}^2(\sigma^i \sigma^j)$  are continuous in the region  $\mathcal{R} = [0, T] \times \overline{B_r} \times [-M, M] \times \{|p| \leq M_1\}$ .

**Theorem 4.2.2.** *Assume (B1)-(B4) and (A5) hold. Then, the initial-boundary value problem (4.23) has a unique solution that is bounded together with its derivatives  $\theta_s, \theta_x$ , and  $\theta_{xx}$ .*

*Proof.* We verify that (B1)-(B4) imply (A1)-(A4), and apply Theorem 4.2.1.

First, since  $B_r \subset (\mathbb{R}^n)^+$ , (B1) implies (A1). Recall that

$$\begin{aligned}\hat{g}(t, x, u, p) &= \sum_{i=1}^d p_i \hat{f}_i(t, x, u, p) + (u - (x, p))r(t, x, (x, p)) \\ &= \sum_{i=1}^d p_i x_i \tilde{f}_i(t, x, u, (x, p)) + u r(t, x, (x, p)) - \sum_{i=1}^d p_i x_i r(t, x, (x, p)).\end{aligned}$$

Now, the continuity of  $\hat{f}_i$ ,  $\hat{\sigma}$ ,  $\hat{g}$  and their derivatives in (A3) is a consequence of (B3) and the fact that these functions are obtained as the composition of polynomials in the coordinates of the functions with the original functions  $\tilde{f}_i$ ,  $\sigma$  and  $r$ , composed with  $(x, p)$ . Moreover, since

$$\partial_{x_k}(\hat{\sigma}^i \hat{\sigma}^j) = (x_j \delta_k^i + x_i \delta_k^j)(\sigma^i, \sigma^j) + x_i x_j \partial_{x_k}(\sigma^i, \sigma^j), \quad \partial_u(\hat{\sigma}^i \hat{\sigma}^j) = x_i x_j \partial_u(\sigma^i, \sigma^j),$$

(4.29) implies the inequality

$$\max_{i,j,k} \sup_{[0,T] \times C_e} |\partial_{x_k} \hat{\sigma}^i \hat{\sigma}^j(t, x, u), \partial_u \hat{\sigma}^i \hat{\sigma}^j(t, x, u)| \leq \tilde{\mu}(|u|), \quad (4.31)$$

where  $\tilde{\mu}(x) = 2(b + b^2)\mu^2(x)$ . Moreover,

$$\hat{f}_i(s, x, u, p) \leq x_i \mu(|u|)(1 + |(x, p)|) \leq \mu^1(|u|)(1 + |p|),$$

with  $\mu^1(x) = b(1 + b)\mu(x)$ . In either case we are left with

$$\hat{g}(s, x, u, p) \leq |p|(1 + |p|)\mu^1(|u|) + (u + |(x, p)|)(R + L) \leq \hat{\mu}(|u|)(1 + |p|^2),$$

where  $\hat{\mu}(x) = 2(\mu^1(x) + (x + 1 + b)(R + L))$ .

Finally, from (4.30), we have for any  $(t, x, u) \in [0, T] \times C_e \times [-M, M]$ ,

$$\hat{\nu}(|y|)|\xi^2| \leq \sum_{i,j=1}^n (\hat{\sigma}^i(t, x, u) \hat{\sigma}^j(t, x, u) \xi_i \xi_j) \leq \hat{\mu}(|y|)|\xi|^2,$$

where  $\hat{\nu}(x) = da^2\nu(x)$  and  $\hat{\mu}(x) = db^2\mu(x)$ . Thus, (A3) holds.

In the same way as we derived the first part of (B3) from (A3), we obtain (B4) from (A4). Thus (B1)-(B4) together with (A5) satisfy the hypotheses of Theorem 4.2.1.  $\square$

Clearly the solution to (4.23) is a solution to (4.21).

If now  $(S_t, V_t, Z_t)$  is a solution to (4.18), the price  $S^i$  from (4.11) is given

explicitly via Itô's formula as the *Doleans-Dade exponential*

$$\begin{aligned} S_t^i &= S_0^i \exp \left( V_t - \frac{1}{2} \langle V_t \rangle \right) \\ &= S^i(0) \exp \left( \int_0^t [f_i(s, S_s, V_s, \pi_s) - \frac{1}{2} \|\sigma_i(s, S_s, V_s)\|^2] ds \right. \\ &\quad \left. + \int_0^t \sigma_i(s, S_s, V_s) dB_s \right), \end{aligned}$$

and thus  $S_t^i$  is a.s. finite and  $S_t^i > 0$  for all  $t \geq 0$ . It remains to prove the admissibility of the hedging strategy. We can rewrite the SDE for the value process

$$dV_t = g(t, S_t, V_t, \pi_t)dt + Z_t dB_t, \quad (4.32)$$

in the integral form as a BSDE of type (4.10) with generator  $-G$

$$V_t = \eta(S_T)h(S_T) + \int_t^T G(s, V_s, Z_s) ds - \int_t^T Z_s dB_s,$$

where  $Z_t = \sigma^T(t, S_t, V_t)\pi_t$ , and

$$\begin{aligned} G(t, y, z) &= yr(t, S_t, (\sigma^T)^{-1}(t, S_t, y)z) \\ &+ \left( (\sigma^T)^{-1}(t, S_t, y)z, \tilde{f}(t, S_t, y, (\sigma^T)^{-1}(t, x, y)z) - r(t, S_t, (\sigma^T)^{-1}(t, x, y)z)1 \right) \end{aligned}$$

Since  $-G(t, 0, 0) = 0$ , Corollary 4.1.4 implies that if  $h(S_T) \geq 0$ , then  $V_t \geq 0$ , for all  $t \geq 0$ , a.s.

Now, Theorem 4.2.2, Proposition 4.1.2, and Corollary 4.1.4 altogether, allow the fulfilment of all the steps in the scheme (4.6) for the FBSDE (4.18). As such, we have proved the following result.

**Theorem 4.2.3.** *Let  $c$  be an option with payoff function  $h$ . Assume the relationship between the price process  $S_t$  and the value process  $V_t$  is governed by the FBSDE (4.18). Assume moreover that  $h(S_T) \geq 0$ . Then,  $c$  admits an admissible self-financing replicating strategy.*

We have finally obtained a model for the replication of contingent claims under the presence of a large investor in Brownian markets. Now, it has been observed that the statistical moments of the historical distribution of the returns of actual stock prices deviate frequently, and sometimes significantly, from those of the normal distributions (see §5.2.2 for more details). Therefore, as the normality of the distributions is one of the defining characteristics of a Brownian motion process, the hedging model presented in this chapter is not able to account for "deviations from normality", and thus its practical use will become impaired. The FBSDE scheme here presented is, however, useful in the development of our new hedging model in the next chapter. Since it will hold on a

market with jumps, where the stocks can have asynchronous jumps, we believe the model we present in the next chapter renders a more accurate representation of the actual dynamics in the stock markets, and as such is more reliable.

## Chapter 5

# FBSDEs with jumps

### 5.1 Introductory remarks

Let  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$  be a filtered probability space with the augmented filtration  $\mathcal{F}_t$  satisfying the usual conditions. In this section we investigate systems of FBSDEs with jumps of type

$$\begin{cases} X_t = x + \int_0^t f(s, X_s, Y_s, Z_s, \tilde{Z}_s(\cdot)) ds + \int_0^t \sigma(s, X_s, Y_s) dB_s \\ \quad + \int_0^t \int_{\mathbb{R}^l} \varphi(s, X_{s-}, Y_{s-}, u) \tilde{N}(ds, du), \\ Y_t = h(X_T) + \int_t^T g(s, X_s, Y_s, Z_s, \tilde{Z}_s(\cdot)) ds - \int_t^T Z_s dB_s \\ \quad - \int_t^T \int_{\mathbb{R}^l} \tilde{Z}_s(u) \tilde{N}(ds, du), \end{cases} \quad (5.1)$$

where the functions  $f, g, \sigma, h$  and  $\varphi$  are of appropriate dimensions,  $B_t$  is a  $d$ -dimensional standard  $\mathcal{F}_t$ -Brownian motion,  $N(t, A)$  an  $\mathcal{F}_t$ -adapted Poisson random measure on  $\mathbb{R}_+ \times \mathcal{B}(\mathbb{R}^l)$ , where  $\mathcal{B}(\mathbb{R}^l)$  is the  $\sigma$ -algebra of Borel sets on  $\mathbb{R}^l$ , and  $\tilde{N}(t, A) = N(t, A) - t\nu(A)$  is the associated compensated Poisson random measure on  $\mathbb{R}_+ \times \mathcal{B}(\mathbb{R}^l)$  with the intensity  $\nu(A)$  being a Lévy measure. We are interested in strong solutions to FBSDE (5.1), i.e., an  $\mathcal{F}_t$ -adapted quadruplet  $(X_t, Y_t, Z_t, \tilde{Z}_t)$  taking values in  $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{d \times n} \times L_2(\nu, \mathbb{R}^l \rightarrow \mathbb{R}^m)$ , satisfying (5.1) a.s. and such that the pair  $(X_t, Y_t)$  is càdlàg and the pair  $(Z_t, \tilde{Z}_t)$  is predictable.

In the first part of section 5.2 we review the existence and uniqueness result to (5.1) obtained by Wu ([23]). The method used to derive this result for FBSDEs with jumps is an extension of Peng's continuation method (see §4.1). It requires additional conditions on the coefficients: First, there is the introduction of a full-rank matrix. Second, and of greater importance, certain monotonicity conditions, the so-called weak monotonicity conditions, have to be satisfied. These conditions, however, are not natural, in the sense that for instance a certain class of coefficients useful for applications is not admissible. The asset pricing model we present in a later stage of the chapter is an example of such a potential application. For this reason, we apply our new result rather than this

method. In the second part of section 5.1, we propose an extension of the short-time interval method by Delarue (see Chapter 4) that allows solving (5.1) for a larger class of coefficients than Wu. Indeed, the type of conditions required are less restrictive than Wu. Moreover, we will see that under certain conditions, it is possible to obtain  $X$  and  $Y$  as càdlàg processes. We stress that this method is valid only in a short-time interval.

In section 5.3 we present the main contribution of this dissertation, the existence and uniqueness of a strong solution to (5.1) on intervals of arbitrary duration. The result is based on a connection between classical solutions to non-local PDEs and (strong) solutions to FBSDEs with jumps. In fact, since the FBSDE contains Lévy-type stochastic integrals, the associated PDE becomes a PIDE and the theory of Ladyzenskaya is not applicable anymore. Existence and uniqueness results for non-local PDEs were only recently obtained (see [11]). The main advantage of our method is to overcome the requirement of the weak monotonicity assumptions, and, consequently, enrich significantly the range of potential applications for our method. Moreover, our result allows the explicit construction of a solution to FBSDEs with jumps, with the help of the unique classical solution to the associated non-local PDE.

## 5.2 Other FBSDE techniques

### 5.2.1 Peng's method of continuation for FBSDEs with jumps

In this subsection, we review the extension of Peng's method continuations for the case where FBSDEs have jumps.

While the result holds for intervals of arbitrary duration, some of the required assumptions are not natural in the sense that they are not verified by functions that appear in concrete applications. In particular, condition iv) below is not suited for applications where coefficients are bounded below by certain positive constants. We will see later how our result to be presented in the next section overcomes naturally this shortcoming of Wu's result. Let  $\mathcal{M}^2(0, T; \mathbb{R}^k) = \{\mathbb{R}^k\text{-valued martingales } M_t \text{ such that } \sup_{t \in [0, T]} \mathbb{E}|M_t|^2 < \infty\}$ .

A solution is then a quadruplet  $(X_t, Y_t, Z_t, \tilde{Z}(t, \cdot))$  of  $\mathcal{F}_t$ -adapted processes satisfying

$$\begin{cases} X_t = x + \int_0^t f(s, X_s, Y_s, Z_s, \tilde{Z}(s, \cdot)) ds + \int_0^t \sigma(s, X_s, Y_s) dB_s \\ \quad + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \psi(s, X_{s-}, Y_s, Z_{s-}, \tilde{Z}(s-, u)) \tilde{N}(ds, du), \\ Y_t = h(X_T) + \int_t^T g(s, X_s, Y_s, Z_s, \tilde{Z}(s, \cdot)) ds - \int_t^T Z_s dB_s \\ \quad - \int_t^T \int_{\mathbb{R}^d \setminus \{0\}} \tilde{Z}(s-, u) \tilde{N}(ds, du), \end{cases} \quad (5.2)$$

such that  $(X_t, Y_t, Z_t) \in \mathcal{M}^2(0, T; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times d})$  and  $\tilde{Z}(t, \cdot) \in \mathcal{P}(0, T; \mathbb{R}^n)$ , where  $\mathcal{P}(0, T; \mathbb{R}^n) = \{\mathbb{R}^n\text{-valued, } \mathcal{F}_t\text{-adapted, predictable processes } \tilde{Z}(t, \cdot) \text{ such that } \mathbb{E} \int_0^T \int_{\mathbb{R}^d} |\tilde{Z}(s, u)|^2 \mu(du) ds < \infty\}$ .

In this section,  $P$  is a full-rank matrix of dimension  $m \times n$ ,  $v$  is the vector  $v = (x, y, z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times d}$ , and we define the matrix

$$A(t, v, w) = (-Pf(t, v, w), Pg(t, v, w), P\sigma(t, x, y))^T.$$

The following assumptions are required.

i) The coefficient functions

$$\begin{aligned} f &: [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n, \\ g &: [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^m \times \Omega \rightarrow \mathbb{R}^m, \\ \psi &: [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times \mathbb{R}^m \times \Omega \rightarrow \mathbb{R}^n, \\ \sigma &: [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \Omega \rightarrow \mathbb{R}^{n \times d}, \\ h &: \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^m, \end{aligned}$$

are uniformly Lipschitz continuous with respect to  $x, y, z$  and  $w$ .

- ii) For each  $x$ ,  $h(x)$  is square-integrable and  $\mathcal{F}_T$  adapted.
- iii) The processes  $f(t, 0, 0, 0, 0, \omega)$ ,  $g(t, 0, 0, 0, \omega)$ ,  $\sigma(t, 0, 0, \omega)$ , are square-integrable martingales, and  $\psi(t, 0, 0, 0, 0, \omega)$  is square-integrable and predictable.
- iv) (weak monotonicity) For all  $(x, y, z, w(\cdot))$  and  $(x', y', z', w'(\cdot))$  in  $\mathbb{R}^n \times \mathbb{R}^m \times (\mathbb{R}^{n \times d}) \times L^2_{\mathcal{F}}(\mathbb{R}^d)$ , and  $t \in [0, T]$  define  $\bar{x} = x - x'$ ,  $\bar{y} = y - y'$ ,  $\bar{z} = z - z'$ , and  $\bar{w}(\cdot) = w(\cdot) - w'(\cdot)$ . Then, the following monotonicity conditions hold

$$\begin{aligned} &(A(t, v, z) - A(t, v', z'), v - v') + \int_{\mathbb{R}^d} (P(g(v, w) - g(v, w')), \bar{w}(u)) \mu(du) \\ &\leq -\beta_1 |P\bar{x}|^2 - \beta_2 (|P\bar{y}|^2 + |P\bar{z}|^2 + \int_{\mathbb{R}^d \setminus \{0\}} |P^T \bar{w}(u)|^2 \mu(du)), \\ &(h(x) - h(x'), P\bar{x}) \geq \beta_3 |P\bar{x}|^2, \end{aligned}$$

where  $\beta_1, \beta_2, \beta_3$  are nonnegative constants,  $\beta_1 + \beta_2 > 0$ , and  $\beta_2 + \beta_3 > 0$ . Moreover, if  $n \neq m$ ,  $\beta_1 > 0$  and  $\beta_3 > 0$  if  $m > n$ , and  $\beta_2 > 0$  if  $n > m$ .

**Theorem 5.2.1.** *Assume conditions i) - iv) hold for the coefficients of (5.2). Then, the FBSDE (5.2) has a unique  $\mathcal{F}_t$ -adapted solution  $(X_t, Y_t, Z_t, \tilde{Z}(t, \cdot)) \in \mathcal{M}^2(0, T; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times d}) \times \mathcal{P}(0, T; \mathbb{R}^n)$ .*

*Proof.* We refer the reader to [23], p. 463. □

We note that the class of admissible coefficients, which is too restrictive for our application, is not necessarily small.

### 5.2.2 Delarue's short-time interval method: an extension to the Lévy-case

In this section, we extend the result on the existence of a solution on a short-time interval from Delarue [4] to FBSDEs with jumps of the type (5.1)

In this context, a solution  $(X_t, Y_t, Z_t, \tilde{Z}(t, \cdot))$  is understood as an element of  $\mathcal{F}(0, T; \mathbb{R}^n) \times \mathcal{F}(0, T; \mathbb{R}^m) \times \mathcal{F}(0, T; \mathbb{R}^{n \times m}) \times \mathcal{P}(0, T; \mathbb{R}^m)$ , where  $\mathcal{F}(0, T; \mathbb{R}^n) = \{\mathbb{R}^n\text{-valued, } \mathcal{F}_t\text{-adapted processes}\}$ .

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space. Consider a  $d$ -dimensional Brownian motion  $B_t$  and an  $l$ -dimensional Lévy process  $L_t$  with measure  $\nu$  such that  $\nu(\{0\}) = 0$  and  $\int_{\mathbb{R}^l} (1 \wedge |x|^2) \nu(dx) < D$ .

The following result is useful in the sequel.

**Proposition 5.2.2.** *Let  $\xi$  be an  $\mathcal{F}_0$  square-integrable adapted process. Assume  $f : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\sigma : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$ , and  $\psi : [0, T] \times \mathbb{R}^n \times \mathbb{R}^l \rightarrow \mathbb{R}^n$  are measurable functions with respect to their respective borelian  $\sigma$ -algebras. Moreover, assume that there exists a constant  $L > 0$  such that*

$$|f(t, x) - f(t, x')| + |\sigma(t, x) - \sigma(t, x')| + \|\psi(t, x, \cdot) - \psi(t, x', \cdot)\|_{L_2} \leq L|x - x'|. \quad (5.3)$$

Then the FSDE

$$X_t = \xi + \int_0^t f(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s + \int_0^t \int_{\mathbb{R}^l} \psi(s, X_{s-}, u) \tilde{N}(ds, du) \quad (5.4)$$

has a pathwise unique  $\mathcal{F}_t$ -adapted solution which admits a càdlàg modification.

*Proof.* Define for  $X_t \in \mathcal{F}(0, T; \mathbb{R}^n)$  the map  $\Psi(X)_t = \xi + \int_0^t f(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s + \int_0^t \int_{\mathbb{R}^l} \psi(s, X_{s-}, u) \tilde{N}(ds, du)$ . We prove that  $\Psi$  is a contraction, and apply the fixed-point Theorem. Let  $\hat{X}_t \in \mathcal{F}(0, T; \mathbb{R}^n)$ . First, we have

$$\begin{aligned} \mathbb{E}|\Psi(X)_t - \Psi(\hat{X})_t|^2 &\leq 3 \left[ \mathbb{E} \left| \int_0^t (f(s, X_s) - f(s, \hat{X}_s)) ds \right|^2 \right. \\ &\quad + \mathbb{E} \left| \int_0^t (\sigma(s, X_s) - \sigma(s, \hat{X}_s)) dB_s \right|^2 \\ &\quad \left. + \mathbb{E} \left| \int_0^t \int_{\mathbb{R}^l} (\psi(s, X_{s-}, u) - \psi(s, \hat{X}_{s-}, u)) \tilde{N}(ds, du) \right|^2 \right]. \end{aligned}$$

Now, an application of Burkhölder-Davis-Gundy inequality (see Proposition 2.2.4) to the martingale  $\int_0^t \int_{\mathbb{R}^l} (\psi(s, X_{s-}, u) - \psi(s, \hat{X}_{s-}, u)) \tilde{N}(ds, du)$ , together with

the Lipschitz condition (5.3) yields a constant  $C > 0$  such that

$$\begin{aligned} & \mathbb{E} \sup_{r \in [0, t]} \left| \int_0^r \int_{\mathbb{R}^l} [\psi(s, X_{s-}, u) - \psi(s, \hat{X}_{s-}, u)] \tilde{N}(ds, du) \right|^2 \\ & \leq C \mathbb{E} \left[ \int_0^\bullet \int_{\mathbb{R}^l} [\psi(s, X_{s-}, u) - \psi(s, \hat{X}_{s-}, u)] \tilde{N}(ds, du) \right]_t \\ & \leq CL^2 \mathbb{E} \int_0^t \int_{\mathbb{R}^l} |X_s - \hat{X}_s|^2 \nu(dx) ds. \end{aligned}$$

This and the above stochastic integral estimate imply there exists a constant  $K > 0$  depending on  $C$ ,  $\nu(\mathbb{R}^l)$ , and  $L$  from the statement of the Theorem, such that

$$\begin{aligned} \mathbb{E} \sup_{s \in [0, t]} |\Psi(X)_s - \Psi(\hat{X})_s|^2 & \leq K \mathbb{E} \int_0^t |X_s - \hat{X}_s|^2 ds \\ & \leq K \mathbb{E} \int_0^t \sup_{r \in [0, s]} |X_r - \hat{X}_r|^2 ds. \quad (5.5) \end{aligned}$$

But this implies, iterating one time more and using Fubini's to change the order of the integration and expectation,

$$\begin{aligned} & \mathbb{E} \sup_{s \in [0, t]} |\Psi^2(X)_s - \Psi^2(\hat{X})_s| \leq \\ & K \int_0^t \mathbb{E} |\Psi(X)_s - \Psi(\hat{X})_s|^2 ds \leq K^2 \int_0^t ds \int_0^s \mathbb{E} |X_r - \hat{X}_r|^2 dr \\ & \leq K^2 \left( \int_0^t s ds \right) \mathbb{E} \sup_{r \in [0, s]} |X_r - \hat{X}_r|^2 \leq \frac{K^2 t^2}{2} \mathbb{E} \sup_{s \in [0, t]} |X_s - \hat{X}_s|^2, \end{aligned}$$

where we used (5.5) in the inequality of the second line on the last chain of inequalities. If we iterate  $n - 1$  times we obtain

$$\mathbb{E} \sup_{s \in [0, t]} |\Psi^n(X)_s - \Psi^n(\hat{X})_s| \leq \frac{K^n t^n}{n!} \sup_{s \in [0, t]} \mathbb{E} |X_s - \hat{X}_s|^2.$$

So, we choose  $n$  such that  $\frac{C^n t^n}{n!} < 1$ , and conclude that  $\Psi^n$  is a contraction, and thus  $\Psi$  has a fixed point, which is the solution. In order to obtain the solution as the limit of a sequence, define  $X_n$ , setting  $X_0 = \xi$ , and recursively  $X_{n+1} = \Psi(X_n)$ . We prove that each  $X_n$  is càdlàg. For this purpose we note that  $X_0$  is càdlàg since it is constant. Then, if  $X_n$  is càdlàg, as  $X_n$  is defined as a continuous term plus a martingale, which admits a càdlàg modification, the process  $X_{n+1}$  will hence be càdlàg. The fact that we have  $\sup_{s \in [0, t]} \mathbb{E} |X_n(s) - X(s)|^2 \xrightarrow{n} 0$  implies the series is almost surely uniformly convergent, and so contains a subsequence which converges almost surely and uniformly to  $X$ .

As every term is càdlàg,  $X$  is the uniform limit of a sequence whose members

have càdlàg paths, and thus is càdlàg. This càdlàg solution is unique in the domain of the function  $\Psi$  and as such is pathwise unique.  $\square$

In order to proof the existence and uniqueness of the solution to (5.1), we will make use of the following assumptions. We take for elements  $w$  of  $L_2(\mu, \mathbb{R}^l \rightarrow \mathbb{R}^m)$  the finite norm

$$\|w\|_{L_2} = \left( \int_{\mathbb{R}^l} |w(x)|^2 \mu(dx) \right)^{\frac{1}{2}}.$$

(C1) The functions  $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times L_2(\mu, \mathbb{R}^l \rightarrow \mathbb{R}^m) \rightarrow \mathbb{R}^n$ ,  $g : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times L_2(\mu, \mathbb{R}^l \rightarrow \mathbb{R}^m) \rightarrow \mathbb{R}^m$ ,  $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{m \times d}$  are measurable with respect to their corresponding  $\sigma$ -algebras.

(C2) For all  $t \in [0, T]$ ,  $(x, y, z, w), (x', y', z', w') \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times L_2(\mu, \mathbb{R}^l \rightarrow \mathbb{R}^m)$ , it holds

$$\begin{aligned} |f(t, x, y, z, w) - f(t, x, y', z', w')| &\leq L(|y - y'| + |z - z'| + \|w - w'\|_{L_2}), \\ |g(t, x, y, z, w) - g(t, x', y, z', w')| &\leq L(|x - x'| + |z - z'| + \|w - w'\|_{L_2}), \\ |\sigma(t, x, y) - \sigma(t, x', y')|^2 &\leq L(|x - x'|^2 + |y - y'|^2), \\ \|\psi(t, x, y, \cdot) - \psi(t, x', y', \cdot)\|_{L_2}^2 &\leq L(|x - x'|^2 + |y - y'|^2), \\ |h(x) - h(x')| &\leq L|x - x'|. \end{aligned}$$

(C3) For all  $t \in [0, T]$ ,  $(x, y, z, w) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times L_2(\mu, \mathbb{R}^l \rightarrow \mathbb{R}^m)$ , and  $(x', y') \in \mathbb{R}^n \times \mathbb{R}^m$ ,

$$\begin{aligned} \langle x - x', f(t, x, y, z, w) - f(t, x', y, z, w) \rangle &\leq L|x - x'|^2, \\ \langle y - y', g(t, x, y, z, w) - g(t, x, y', z, w) \rangle &\leq L|y - y'|^2. \end{aligned}$$

For all  $t \in [0, T]$ ,  $(x, y, z, w) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \times L_2(\mu, \mathbb{R}^l \rightarrow \mathbb{R}^m)$ ,

$$\begin{aligned} |f(t, x, y, z, w)| + |g(t, x, y, z, w)| &\leq L(1 + |x| + |y| + |z| + \|w\|_{L_2}), \\ |\sigma(t, x, y)| &\leq L(1 + |x| + |y|), \\ |h(x)| &\leq L(1 + |x|). \end{aligned}$$

Suppose now  $(Y_t, Z_t, \tilde{Z}(t, \cdot)) \in \mathcal{F}(0, T; \mathbb{R}^m) \times \mathcal{F}(0, T; \mathbb{R}^{m \times d}) \times \mathcal{P}(0, T; \mathbb{R}^m)$ . Define the process

$$\tilde{Y}_t = E \left[ h(Y_T) + \int_t^T g(s, \hat{X}_s, Y_s, Z_s, \tilde{Z}(s, \cdot)) ds | \mathcal{F}_t \right],$$

where, with the help of the previous lemma,  $\hat{X}_t$  is the unique solution to

$$\begin{aligned}\hat{X}_t = \xi + \int_0^t f(s, \hat{X}_s, Y_s, Z_s, \tilde{Z}(s, \cdot)) ds + \int_0^t \sigma(s, \hat{X}_s, Y_s) dB_s \\ + \int_0^t \int_{\mathbb{R}^l} \psi(s, \hat{X}_{s-}, Y_s, u) \tilde{N}(ds, du),\end{aligned}$$

where we evaluate the integrals in  $\hat{X}_{s-}$  so that we get a predictable process, and the processes  $\hat{Z}_t, \hat{\tilde{Z}}(t, \cdot)$  are given by Theorem 2.3.2 applied to the square-integrable random variable

$$h(\hat{X}_T) + \int_0^T g(s, \hat{X}_s, Y_s, Z_s, \tilde{Z}(s, \cdot)) ds,$$

that is

$$\begin{aligned}h(\hat{X}_T) + \int_0^T g(s, \hat{X}_s, Y_s, Z_s, \tilde{Z}(s, \cdot)) ds = \\ \mathbb{E} \left[ h(\hat{X}_T) + \int_0^T g(s, \hat{X}_s, Y_s, Z_s, \tilde{Z}(s, \cdot)) ds \right] \\ + \int_0^T \hat{Z}_s dB_s + \int_0^T \int_{\mathbb{R}^l} \hat{\tilde{Z}}(s, \cdot) \tilde{N}(ds, du),\end{aligned}$$

after which we can take the conditional expectation with respect to  $\mathcal{F}_t$  and obtain

$$\begin{aligned}\tilde{Y}_t + \int_0^t g(s, \hat{X}_s, Y_s, Z_s, \tilde{Z}(s, \cdot)) ds = \tilde{Y}_0 + \int_0^t \hat{Z}_s dB_s \\ + \int_0^t \int_{\mathbb{R}^l} \hat{\tilde{Z}}(s, \cdot) \tilde{N}(ds, du).\end{aligned}$$

If we substitute in the expression  $t$  by  $T$ , and subtract the former to the latter, we obtain the identity

$$\begin{aligned}\tilde{Y}_T - \tilde{Y}_t + \int_t^T g(s, \hat{X}_s, Y_s, Z_s, \tilde{Z}(s, \cdot)) ds \\ = \int_t^T \hat{Z}_s dB_s + \int_t^T \int_{\mathbb{R}^l} \hat{\tilde{Z}}(s, \cdot) \tilde{N}(ds, du),\end{aligned}$$

that we can rearrange into

$$\begin{aligned}\tilde{Y}_t = h(\hat{X}_T) &+ \int_t^T g(s, \hat{X}_s, Y_s, Z_s, \tilde{Z}(s, \cdot)) ds \\ &- \int_t^T \hat{Z}_s dB_s - \int_t^T \int_{\mathbb{R}^l} \hat{\tilde{Z}}(s, \cdot) \tilde{N}(ds, du)\end{aligned}$$

Hence, we can take a càdlàg version of  $\tilde{Y}_t$  and define the left continuous process

$$\begin{aligned}\hat{Y}_t = h(\hat{X}_T) &+ \int_{t-}^T g(s, \hat{X}_s, Y_s, Z_s, \tilde{Z}(s, \cdot)) ds \\ &- \int_{t-}^T \hat{Z}_s dB_s - \int_{t-}^T \int_{\mathbb{R}^l} \hat{\tilde{Z}}(s, \cdot) \tilde{N}(ds, du).\end{aligned}\quad (5.6)$$

Thus,  $\hat{Y}_s$  is càdlàg. We have shown we can define the map  $\Theta : \mathcal{F}(0, T; \mathbb{R}^m) \times \mathcal{F}(0, T; \mathbb{R}^{n \times d}) \times \mathcal{P}(0, T; \mathbb{R}^m) \rightarrow \mathcal{F}(0, T; \mathbb{R}^m) \times \mathcal{F}(0, T; \mathbb{R}^{n \times d}) \times \mathcal{P}(0, T; \mathbb{R}^m)$  by

$$\Theta(Y_t, Z_t, \tilde{Z}(t, \cdot)) := (\hat{Y}, \hat{Z}_t, \hat{\tilde{Z}}(t, \cdot)). \quad (5.7)$$

We have the following.

**Proposition 5.2.3.** *Assume (C1)-(C3) hold. Then, there exists  $0 \leq \tau \leq T$  such that for every  $t \leq \tau$ ,  $\Theta$  has a fixed point.*

*Proof.* We prove that  $\Theta$  is contractive, and apply Banach's fixed-point Theorem.

Let us fix  $(Y_t, Z_t, \tilde{Z}(t, \cdot)), (V, W, \tilde{W}(t, \cdot)) \in \mathcal{F}(0, T; \mathbb{R}^m) \times \mathcal{F}(0, T; \mathbb{R}^{n \times d}) \times \mathcal{P}(0, T; \mathbb{R}^m)$ . Let  $\hat{X}_t$  be the solution (see Proposition 5.2.2) of

$$\begin{aligned}\hat{X}_t = \xi &+ \int_0^t f(s, \hat{X}_s, Y_s, Z_s, \tilde{Z}(s, \cdot)) ds + \int_0^t \sigma(s, \hat{X}_s, Y_s) dB_s \\ &+ \int_0^t \int_{\mathbb{R}^l} \psi(s, \hat{X}_{s-}, Y_s, u) \tilde{N}(ds, du)\end{aligned}$$

and  $\hat{U}_t$  the solution of

$$\begin{aligned}\hat{U}_t = \xi &+ \int_0^t f(s, \hat{U}_s, V_s, W_s, \tilde{W}(s, \cdot)) ds + \int_0^t \sigma(s, \hat{U}_s, V_s) dB_s \\ &+ \int_0^t \int_{\mathbb{R}^l} \psi(s, \hat{U}_{s-}, V_s, u) \tilde{N}(ds, du).\end{aligned}$$

Now, an application of Itô's product formula to  $|\hat{X}_t - \hat{U}_t|^2$  gives

$$|\hat{X}_t - \hat{U}_t|^2 = 2 \int_0^t \left( (\hat{X}_{s-} - \hat{U}_{s-}), d(\hat{X}_s - \hat{U}_s) \right) + \int_0^t d[\hat{X} - \hat{U}, \hat{X} - \hat{U}]_s. \quad (5.8)$$

Since

$$d[\hat{X} - \hat{U}, \hat{X} - \hat{U}]_s = |\sigma(s, \hat{X}_s, Y_s) - \sigma(s, \hat{U}_s, V_s)|^2 ds \\ + |\psi(s, \hat{X}_s, Y_s, x) - \psi(s, \hat{U}_s, V_s, x)|^2 \mu(dx) ds,$$

we can take expectations in (5.8) and obtain

$$\mathbb{E}|\hat{X}_t - \hat{U}_t|^2 = \\ + \mathbb{E} \int_0^t |\sigma(s, \hat{X}_s, Y_s) - \sigma(s, \hat{U}_s, V_s)|^2 ds \\ + \mathbb{E} \int_0^t \int_{\mathbb{R}^l} |\psi(s, \hat{X}_s, Y_s, x) - \psi(s, \hat{U}_s, V_s, x)|^2 \mu(dx) ds \\ + 2\mathbb{E} \int_0^t \left( \hat{X}_s - \hat{U}_s, f(s, \hat{X}_s, Y_s, Z_s, \tilde{Z}(s, \cdot)) - f(s, \hat{U}_s, V_s, W_s, \tilde{W}(s, \cdot)) ds \right),$$

which implies with the help of (C2) and (C3) the existence of a positive constant  $C_L$  depending only on  $L$  such that

$$\mathbb{E}|\hat{X}_t - \hat{U}_t|^2 \leq C_L \left[ \mathbb{E} \int_0^t |\hat{X}_s - \hat{U}_s| (|\hat{X}_s - \hat{U}_s| + |Y_s - V_s| + |Z_s - W_s|) ds \right. \\ \left. + \mathbb{E} \int_0^t \int_{\mathbb{R}^l} |\hat{X}_s - \hat{U}_s| |\tilde{Z}(s, x) - \tilde{W}(s, x)| \mu(dx) ds \right. \\ \left. + \mathbb{E} \int_0^t (|\hat{X}_s - \hat{U}_s|^2 + |Y_s - V_s|^2) ds \right]. \quad (5.9)$$

Now, by Hölder's inequality we obtain

$$2\mathbb{E} \int_0^t \int_{\mathbb{R}^k} |\tilde{Z}(s, x) - \tilde{W}(s, x)| |\hat{X}_s - \hat{U}_s| \mu(dx) ds \leq \\ \leq 2\mathbb{E} \left( \int_0^t \int_{\mathbb{R}^l} |\tilde{Z}(s, x) - \tilde{W}(s, x)|^2 \mu(dx) ds \right)^{\frac{1}{2}} \left( \hat{\mu} \int_0^t |\hat{X}_s - \hat{U}_s|^2 ds \right)^{\frac{1}{2}} \\ \leq d\hat{\mu} \mathbb{E} \int_0^t |\hat{X}_s - \hat{U}_s|^2 \mu(dx) ds + \frac{1}{d} \mathbb{E} \int_0^t \int_{\mathbb{R}^l} |\tilde{Z}(s, x) - \tilde{W}(s, x)|^2 \mu(dx) ds,$$

where the expression in the third line of the above chain follows from Cauchy's inequality,  $\hat{\mu} = \mu(\mathbb{R}^l)$ , and  $d \geq 0$ . With the help of the last estimate and using Cauchy inequality on the first term of the right-hand side of (5.9), we can take

supremums on both sides of (5.9) and transform the expression into

$$\begin{aligned}
& \sup_{s \in [0, T]} \mathbb{E} |\hat{X}_s - \hat{U}_s|^2 (1 - TC_{L, d, \hat{\mu}}) \\
& \leq C_{L, d, \hat{\mu}} \left( \mathbb{E} \int_0^T |Y_s - U_s|^2 ds + \mathbb{E} \int_0^T |Z_s - W_s|^2 ds \right. \\
& \quad \left. + \mathbb{E} \int_0^T \int_{\mathbb{R}^l} |\tilde{Z}(s, x) - \tilde{W}(s, x)|^2 \mu(dx) ds \right) \\
& \leq C_{L, d, \hat{\mu}} \left( T \sup_{s \in [0, T]} \mathbb{E} |Y_s - U_s|^2 + T \sup_{s \in [0, T]} \mathbb{E} |Z_s - W_s|^2 \right. \\
& \quad \left. + \mathbb{E} \int_0^T \int_{\mathbb{R}^l} |\tilde{Z}(s, x) - \tilde{W}(s, x)|^2 \mu(dx) ds \right), \tag{5.10}
\end{aligned}$$

where we changed the order of integration in the integrals, and  $C_{L, d, \hat{\mu}}$  is a constant depending only on  $C_L, d$  and  $\hat{\mu}$ . Henceforth, we assume  $d$  is such that  $(1 - TC_{L, d, \hat{\mu}}) > 0$ .

Now, by (5.7) the triple  $(\hat{Y}_t, \hat{Z}_t, \hat{\hat{Z}}(t, \cdot)) (= \Theta(Y_t, Z_t, \tilde{Z}(t, \cdot)))$  satisfies the BSDE

$$\begin{aligned}
\hat{Y}_t = h(\hat{X}_T) & - \int_{t-}^T g(s, \hat{X}_s, Y_s, Z_s, \tilde{Z}(s, \cdot)) ds - \int_{t-}^T \hat{Z}_s dB_s \\
& - \int_{t-}^T \int_{\mathbb{R}^l} \hat{\hat{Z}}(s, \cdot) \tilde{N}(ds, du), \tag{5.11}
\end{aligned}$$

while  $(\hat{V}_t, \hat{W}_t, \hat{\hat{W}}(t, \cdot))$  satisfy the BSDE

$$\begin{aligned}
\hat{V}_t = h(\hat{U}_T) & + \int_{t-}^T g(s, \hat{U}_s, V_s, W_s, \tilde{W}(s, \cdot)) ds - \int_{t-}^T \hat{W}_s dB_s \\
& - \int_{t-}^T \int_{\mathbb{R}^l} \hat{\hat{W}}(s, \cdot) \tilde{N}(ds, du).
\end{aligned}$$

Hence, and similarly to (5.8), we obtain by an application of Itô's product formula to  $|\hat{Y}_t - \hat{V}_t|^2$  the expression

$$\begin{aligned}
|\hat{Y}_t - \hat{V}_t|^2 & = |h(\hat{X}_T) - h(\hat{U}_T)|^2 - 2 \int_{t-}^T \left( \hat{Y}_{s-} - \hat{V}_{s-}, d(\hat{Y}_s - \hat{V}_s) \right) \\
& \quad + [\hat{Y} - \hat{V}, \hat{Y} - \hat{V}]_{t-} - [\hat{Y} - \hat{V}, \hat{Y} - \hat{V}]_T,
\end{aligned}$$

that, after plugging (5.11), taking expectations, and rearranging terms, can be

transformed into

$$\begin{aligned}
& \mathbb{E}|\hat{Y}_t - \hat{V}_t|^2 + \mathbb{E} \int_t^T |\hat{Z}_s - \hat{W}_s|^2 ds + \mathbb{E} \int_t^T \int_{\mathbb{R}^l} |\hat{\tilde{Z}}(s, x) - \hat{\tilde{W}}(s, x)|^2 \mu(dx) ds \\
&= \mathbb{E}|h(\hat{X}_T) - h(\hat{U}_T)|^2 \\
&+ 2\mathbb{E} \int_t^T \left( \hat{Y}_s - \hat{V}_s, g(s, \hat{X}_s, Y_s, Z_s, \tilde{Z}(s, \cdot)) - g(s, \hat{U}_s, V_s, W_s, \tilde{W}(s, \cdot)) \right) ds.
\end{aligned}$$

Next, using (C2) and (C3) in the last identity, there exists a constant  $C'_{L, \hat{\mu}}$  depending only on  $L$  and  $\hat{\mu}$  such that

$$\begin{aligned}
& \mathbb{E}|\hat{Y}_t - \hat{V}_t|^2 + \mathbb{E} \int_t^T |\hat{Z}_s - \hat{W}_s|^2 ds + \mathbb{E} \int_t^T \int_{\mathbb{R}^l} |\hat{\tilde{Z}}(s, x) - \hat{\tilde{W}}(s, x)|^2 \mu(dx) ds \\
&\leq C'_{L, \hat{\mu}} \mathbb{E}|\hat{X}_T - \hat{U}_T|^2 \\
&+ C'_{L, \hat{\mu}} \mathbb{E} \int_t^T |\hat{Y}_s - \hat{V}_s| (|\hat{X}_s - \hat{U}_s| + |Y_s - V_s| + |Z_s - W_s|) ds \\
&+ \mathbb{E} \int_t^T \int_{\mathbb{R}^l} |\hat{Y}_s - \hat{V}_s| |\tilde{Z}(s, x) - \tilde{W}(s, x)| \mu(dx) ds \leq C'_{L, \hat{\mu}} \mathbb{E}|\hat{X}_T - \hat{U}_T|^2 \\
&+ C'_{L, \hat{\mu}} \left[ \frac{1}{c} \left( \mathbb{E} \int_t^T (|\hat{X}_s - \hat{U}_s|^2 + |Y_s - V_s|^2 + |Z_s - W_s|^2) ds \right. \right. \\
&\left. \left. + \mathbb{E} \int_t^T \int_{\mathbb{R}^l} |\tilde{Z}(s, x) - \tilde{W}(s, x)|^2 \mu(dx) ds \right) + c \mathbb{E} \int_t^T |\hat{Y}_s - \hat{V}_s|^2 ds \right],
\end{aligned}$$

where the last inequality holds by Cauchy inequality, for any constant  $c > 0$ . We can now plug (5.10) in the last inequality, modify  $C'_{L, \hat{\mu}}$  if needed (note that we may choose  $c$  such that  $1 - TC'_{L, \hat{\mu}} > 0$ ), regroup similar terms and obtain after taking supremums

$$\begin{aligned}
& (1 - C'_{L, \hat{\mu}} T) \left( \sup_{s \in [0, T]} \mathbb{E}|\hat{Y}_s - \hat{V}_s|^2 + \sup_{s \in [0, T]} \mathbb{E}|\hat{Z}_s - \hat{W}_s|^2 ds \right) \\
&+ \mathbb{E} \int_0^T \int_{\mathbb{R}^l} |\hat{\tilde{Z}}(s, x) - \hat{\tilde{W}}(s, x)|^2 \mu(dx) ds \\
&\leq C'_{L, \hat{\mu}} T \left( \sup_{s \in [0, T]} \mathbb{E}|Y_s - V_s|^2 + \sup_{s \in [0, T]} \mathbb{E}|Z_s - W_s|^2 \right) \\
&+ \mathbb{E} \int_0^T \int_{\mathbb{R}^l} |\tilde{Z}(s, x) - \tilde{W}(s, x)|^2 \mu(dx) ds,
\end{aligned}$$

which, since we picked  $c$  so that  $1 - C'_{L,\hat{\mu}}T > 0$ , implies the estimate

$$\begin{aligned} & \sup_{s \in [0, T]} \mathbb{E} |\hat{Y}_s - \hat{V}_s|^2 + \sup_{s \in [0, T]} \mathbb{E} |\hat{Z}_s - \hat{W}_s|^2 \\ & + \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^l} |\hat{\tilde{Z}}(s, x) - \hat{\tilde{W}}(s, x)|^2 \mu(dx) ds \right] \leq \\ & \frac{C'_{L,\hat{\mu}} \max(T, 1)}{(1 - C'_{L,\hat{\mu}}T)} \left[ \sup_{s \in [0, T]} \mathbb{E} |Y_s - V_s|^2 + \sup_{s \in [0, T]} \mathbb{E} |Z_s - W_s|^2 \right. \\ & \left. + \mathbb{E} \int_0^T \int_{\mathbb{R}^l} |\tilde{Z}(s, x) - \tilde{W}(s, x)|^2 \mu(dx) ds \right]. \end{aligned}$$

Hence, picking  $T = \tau$  sufficiently small, we can find a constant  $D < 1$  such that

$$\begin{aligned} & \|\hat{Y} - \hat{V}\|_{\mathcal{F}(0, \tau; \mathbb{R}^m)}^2 + \|\hat{Z} - \hat{W}\|_{\mathcal{F}(0, \tau; \mathbb{R}^m)}^2 + \|\hat{\tilde{Z}} - \hat{\tilde{W}}\|_{\mathcal{P}(0, \tau; \mathbb{R}^m)}^2 \\ & \leq D \left( \|Y - V\|_{\mathcal{F}(0, \tau; \mathbb{R}^m)}^2 + \|Z - W\|_{\mathcal{F}(0, \tau; \mathbb{R}^{m \times d})}^2 + \|\tilde{Z} - \tilde{W}\|_{\mathcal{P}(0, \tau; \mathbb{R}^m)}^2 \right). \end{aligned}$$

Thus,  $\Theta$  is a contraction for sufficiently small  $\tau$ , and the fixed-point Theorem guarantees the existence of a unique fixed point.  $\square$

**Theorem 5.2.4.** *Assume (C1)-(C3) hold. Then there exists  $\tau \in [0, \tau]$  such that, for  $t \leq \tau$ , the FBSDE (5.1) has a unique solution  $(\bar{X}_t, \bar{Y}_t^0, \bar{Z}_t, \bar{\tilde{Z}}_t)$ . Furthermore, the couple  $(\bar{X}_t, Y_t^0)$  admits a càdlàg version.*

*Proof.*

Let, by an application of Proposition 5.2.3,  $(\bar{Y}_t, \bar{Z}_t, \bar{\tilde{Z}}(t, \cdot))$  be a fixed point of  $\Theta$ . Then, pick  $X_t$  with the help of Proposition 5.2.2 as the unique solution of

$$\begin{aligned} X_t = \xi & + \int_0^t f(s, X_s, \bar{Y}_s, \bar{Z}_s, \bar{\tilde{Z}}(s, \cdot)) ds + \int_0^t \sigma(s, X_s, \bar{Y}_s) dB_s \\ & + \int_0^t \int_{\mathbb{R}^l} \psi(s, X_{s-}, \bar{Y}_s, u) \tilde{N}(ds, du) \end{aligned}$$

and, let  $\bar{X}_t$  be its càdlàg version. Then the triple  $(\bar{Y}_t, \bar{Z}_t, \bar{\tilde{Z}}(t, \cdot))$  satisfies

$$\begin{aligned} \bar{Y}_t = h(\bar{X}_T) & + \int_{t-}^T g(s, \bar{X}_s, \bar{Y}_s, \bar{Z}_s, \bar{\tilde{Z}}(s, \cdot)) ds - \int_{t-}^T \bar{Z}_s dB_s \\ & - \int_{t-}^T \int_{\mathbb{R}^l} \bar{\tilde{Z}}(s, u) \tilde{N}(ds, du). \end{aligned}$$

We can now choose a càdlàg version of the process

$$h(\bar{X}_T) + \int_{t-}^T g(s, \bar{X}_s, \bar{Y}_s, \bar{Z}_s, \bar{\bar{Z}}(s, \cdot)) ds - \int_{t-}^T \bar{Z}_s dB_s \\ - \int_{t-}^T \int_{\mathbb{R}^l} \bar{\bar{Z}}(s, u) \tilde{N}(ds, du),$$

and consequently we can choose a left-continuous version with right limits of  $\bar{Y}_t$ , say  $\bar{Y}_t^{l.c.}$ .

Define now  $Y_t^0$  to be the càdlàg version of the process  $\bar{Y}_{t+}^{l.c.}$  that satisfies

$$\bar{Y}_{t+}^{l.c.} = h(\bar{X}_T) + \int_t^T g(s, \bar{X}_s, \bar{Y}_s^{l.c.}, \bar{Z}_s, \bar{\bar{Z}}(s, \cdot)) ds - \int_t^T \bar{Z}_s dB_s \\ - \int_t^T \int_{\mathbb{R}^l} \bar{\bar{Z}}(s, u) \tilde{N}(ds, du) = h(\bar{X}_T) + \int_t^T g(s, \bar{X}_s, Y_s^0, \bar{Z}_s, \bar{\bar{Z}}(s, u)) ds \\ - \int_t^T \bar{Z}_s dB_s - \int_{t-}^T \int_{\mathbb{R}^l} \bar{\bar{Z}}(s, u) \tilde{N}(ds, du),$$

where the last identity is valid since  $Y_t^0$  and  $\bar{Y}_t^{l.c.}$  differ only at a countable number of points which implies by (C2)

$$\mathbb{E} \left| \int_t^T g(s, \bar{X}_s, Y_s^0, \bar{Z}_s, \bar{\bar{Z}}(s, \cdot)) - g(s, \bar{X}_s, \bar{Y}_s^{l.c.}, \bar{Z}_s, \bar{\bar{Z}}(s, \cdot)) ds \right|^2 \\ \leq L^2 \int_t^T \mathbb{E} |Y_s^0 - \bar{Y}_s^{l.c.}|^2 ds = 0.$$

Moreover, similarly we obtain with the help of (C2) the identities

$$\mathbb{E} \left| \int_0^t [f(s, \bar{X}_s, Y_s^0, \bar{Z}_s, \bar{\bar{Z}}(s, \cdot)) - f(s, \bar{X}_s, \bar{Y}_s, \bar{Z}_s, \bar{\bar{Z}}(s, \cdot))] ds \right|^2 = 0,$$

$$\mathbb{E} \left| \int_0^t [\sigma(s, \bar{X}_s, Y_s^0) - \sigma(s, \bar{X}_s, \bar{Y}_s)] dB_s \right|^2 = 0,$$

and, since the measure  $\mu$  is finite,

$$\mathbb{E} \left| \int_t^T \int_{\mathbb{R}^l} \psi(s, \bar{X}_s, Y_s^0, u) - \psi(s, \bar{X}_s, \bar{Y}_s, u) \tilde{N}(ds, du) \right|^2 = 0.$$

This implies

$$\begin{aligned}\bar{X}_t = \xi + \int_0^t f(s, \bar{X}_s, Y_s^0, \bar{Z}_s, \bar{\tilde{Z}}(s, \cdot)) ds + \int_0^t \sigma(s, \bar{X}_s, Y_s^0) dB_s \\ + \int_0^t \int_{\mathbb{R}^l} \psi(s, \bar{X}_{s-}, Y_{s-}^0, u) \tilde{N}(du, ds)\end{aligned}$$

Thus,  $(\bar{X}_t, Y_t^0, \bar{Z}_t, \bar{\tilde{Z}}(t, \cdot))$  is a solution to the FBSDE in the interval  $[0, \tau]$ , where  $\tau$  is given by Proposition 5.2.3. We already have seen that  $(\bar{X}_t, Y_t^0)$  admits a càdlàg version. The uniqueness follows from the uniqueness of the fixed point of the function  $\Theta$ . □

### 5.3 Strong solution to FBSDEs with jumps via non-local parabolic PDEs

Here, we prove the existence and uniqueness of a solution to FBSDEs with jumps (5.1).

In the first section, we give an outline of how the following existence and uniqueness result for non-local PDEs was obtained recently (see [11]). In the second section, we present the method to obtain our main result. In fact, the construction of a solution to fully coupled FBSDEs driven by a Brownian motion and a compensated Poisson random measure hinges on a solution to a Cauchy problem for quasilinear parabolic partial integro-differential equations (PIDEs). PIDEs are a special type of non-local PDEs and as such the results of the first section can be applied to prove the existence of a unique solution. We remark once again that our result holds for a class of coefficients that is more natural to applications than previously known results.

Let  $(\Omega, \mathcal{F}, P)$  be a probability space,  $B_t$  be a  $d$ -dimensional standard Brownian motion,  $N(t, A)$  be a Poisson random measure on  $\mathbb{R}_+ \times \mathcal{B}(\mathbb{R}^l)$ , where  $\mathcal{B}(\mathbb{R}^l)$  is the  $\sigma$ -algebra of Borel sets on  $\mathbb{R}^l$ , and  $\tilde{N}(t, A) = N(t, A) - t\nu(A)$  be the corresponding compensated Poisson random measure on  $\mathbb{R}_+ \times \mathcal{B}(\mathbb{R}^l)$  with  $\nu(A)$  being the associated Lévy measure. Define the filtration

$$\mathcal{F}_t = \sigma\{B_s, 0 \leq s \leq t\} \vee \sigma\{N(s, A), 0 \leq s \leq t, A \in \mathcal{B}(\mathbb{R}^l)\} \vee \mathcal{N}$$

where  $\mathcal{N}$  is a collection of subsets of all  $P$ -null sets.

Fix  $T > 0$ , and consider FBSDEs driven by the Brownian motion  $B_t$  and

the compensated Poisson random measure  $\tilde{N}(t, \cdot)$ :

$$\begin{cases} X_t = x + \int_0^t f(s, X_s, Y_s, Z_s, \tilde{Z}_s(\cdot)) ds + \int_0^t \sigma(s, X_s, Y_s) dB_s \\ \quad + \int_0^t \int_{\mathbb{R}^l} \varphi(s, X_{s-}, Y_{s-}, u) \tilde{N}(ds, du), \\ Y_t = h(X_T) + \int_t^T g(s, X_s, Y_s, Z_s, \tilde{Z}_s(\cdot)) ds - \int_t^T Z_s dB_s \\ \quad - \int_t^T \int_{\mathbb{R}^l} \tilde{Z}_s(u) \tilde{N}(ds, du). \end{cases} \quad (5.12)$$

The forward SDE in (5.12) is  $\mathbb{R}^n$ -valued while the BSDE is  $\mathbb{R}^m$ -valued. The solution to (5.12) is an  $\mathcal{F}_t$ -adapted càdlàg quadruplet  $(X_t, Y_t, Z_t, \tilde{Z}_t(\cdot))$  satisfying (5.12) a.s.

Together with FBSDEs (5.12), we consider the associated final value problem for the following PIDE:

$$\begin{aligned} & \partial_x \theta \left\{ f(t, x, \theta, \partial_x \theta \sigma(t, x, \theta), \vartheta_\theta(t, x)) - \int_{\mathbb{R}^l} \varphi(t, x, \theta, y) \nu(dy) \right\} \\ & + \frac{1}{2} \text{tr}(\partial_{xx}^2 \theta \sigma(t, x, \theta) \sigma(t, x, \theta)^\top) + g(t, x, \theta, \partial_x \theta \sigma(t, x, \theta), \vartheta_\theta(t, x)) \\ & + \int_{\mathbb{R}^l} \vartheta_\theta(t, x)(y) \nu(dy) + \partial_t \theta = 0; \quad \theta(T, x) = h(x). \end{aligned} \quad (5.13)$$

In (5.13),  $\theta$ ,  $\partial_x \theta$ ,  $\partial_t \theta$ , and  $\partial_{xx}^2 \theta$  are everywhere evaluated at  $(t, x)$  (we omit the arguments to simplify the equation). Further,  $\partial_x \theta$  is understood as a matrix whose  $(ij)$ th component is  $\partial_{x_i} \theta^j$ , and the first term in (5.13) is understood as the multiplication of the matrix  $\partial_x \theta$  by the vector-valued function following after it. Furthermore,  $\text{tr}(\partial_{xx}^2 \theta \sigma(t, x, \theta) \sigma(t, x, \theta)^\top)$  is the vector whose  $i$ th component is the trace of the matrix  $\partial_{xx}^2 \theta^i \sigma \sigma^\top$ . Finally, for any  $v \in C_b([0, T] \times \mathbb{R}^n)$ , we define the function

$$\vartheta_v(t, x) = v(t, x + \varphi(t, x, v(t, x), \cdot)) - v(t, x). \quad (5.14)$$

By introducing the time-changed function  $u(t, x) = \theta(T - t, x)$ , we transform problem (5.13) to the following Cauchy problem:

$$\begin{aligned} & \partial_x u \left\{ \int_{\mathbb{R}^l} \hat{\varphi}(t, x, u, y) \nu(dy) - \hat{f}(t, x, u, \partial_x u \hat{\sigma}(t, x, u), \vartheta_u(t, x)) \right\} \\ & - \frac{1}{2} \text{tr}(\partial_{xx}^2 u \hat{\sigma}(t, x, u) \hat{\sigma}(t, x, u)^\top) - \hat{g}(t, x, u, \partial_x u \hat{\sigma}(t, x, u), \vartheta_u(t, x)) \\ & - \int_{\mathbb{R}^l} \vartheta_u(t, x)(y) \nu(dy) + \partial_t u = 0; \quad u(0, x) = h(x). \end{aligned} \quad (5.15)$$

In (5.15),  $\hat{f}(t, x, u, p, w) = f(T - t, x, u, p, w)$ , and the functions  $\hat{\sigma}$ ,  $\hat{\varphi}$ , and  $\hat{g}$  are defined via  $\sigma$ ,  $\varphi$ , and, respectively,  $g$  in the similar manner.

In order to obtain a solution to (5.15), we are going to make use of a result that was obtained by means of pure PDE techniques, and that as such, falls outside the scope of this dissertation ([11]).

Consider then the following Cauchy problem for a non-local quasilinear parabolic PDE for an  $\mathbb{R}^m$ -valued function  $u(t, x)$ ,

$$\begin{cases} -\sum_{i,j=1}^n a_{ij}(t, x, u) \partial_{x_i x_j}^2 u + \sum_{i=1}^n a_i(t, x, u, \partial_x u, \vartheta_u) \partial_{x_i} u \\ + a(t, x, u, \partial_x u, \vartheta_u) + \partial_t u = 0, \quad u(0, x) = \varphi_0(x) \end{cases} \quad (5.16)$$

where  $\vartheta_u : [0, T] \times \bar{\mathbb{F}} \rightarrow E$  is a function built by means of  $u$  and taking values in a normed space  $E$ .

In PDE (5.16),  $a_{ij} : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ ,  $a_i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times m} \times E \rightarrow \mathbb{R}$ ,  $i, j = 1, \dots, n$ ,  $a : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times m} \times E \rightarrow \mathbb{R}^m$ . Further,  $\partial_{x_i x_j}^2 u$ ,  $\partial_{x_i} u$ ,  $\partial_t u$ ,  $u$ , and  $\vartheta_u$  are evaluated at  $(t, x)$ . Below, the Hölder space  $C_b^{2+\beta}(\mathbb{R}^n)$ ,  $\beta \in (0, 1)$ , is understood as the (Banach) space with the norm

$$\|h\|_{C_b^{2+\beta}(\mathbb{R}^n)} = \|h\|_{C_b^2(\mathbb{R}^n)} + [\nabla^2 h]_\beta, \quad \text{where} \quad [\psi]_\beta = \sup_{\substack{x, y \in \mathbb{R}^n, \\ 0 < |x-y| < 1}} \frac{|\psi(x) - \psi(y)|}{|x - y|^\beta}.$$

The following assumptions are required.

(D1') There exist a non-decreasing function  $\mu(s)$  and a non-increasing function  $\nu(s)$ , both taking positive values, such that

$$\nu(|u|)|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(t, x, u) \xi_i \xi_j \leq \mu(|u|)|\xi|^2$$

for all  $(t, x, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m$ ,  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ .

(D2') There exists a constant  $L_E > 0$  such that for each  $u \in C_b^{1,2}([0, T] \times \mathbb{R}^n)$ , the function  $\vartheta_u : [0, T] \times \mathbb{R}^n \rightarrow E$  for all  $\lambda \geq 0$  satisfies the inequality  $\sup_{[0, T] \times \mathbb{R}^n} \|e^{-\lambda t} \vartheta_u(t, x)\|_E \leq L_E \sup_{[0, T] \times \mathbb{R}^n} |e^{-\lambda t} u(t, x)|$ .

(D3') There exist non-negative constants  $c_1$ ,  $c_2$ , and  $c_3$  such that

$$(a(t, x, u, p, w), u) \geq -c_1 - c_2 |u|^2 - c_3 \|w\|_E^2$$

for all  $(t, x, u, p, w) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times E \times \mathbb{R}^{n \times m}$ .

(D4') The initial condition  $\varphi_0 : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is of class  $C_b^{2+\beta}(\mathbb{R}^n)$ ,  $\beta \in (0, 1)$ .

(D5') There exists a function  $\eta(s, r)$ , defined for  $s, r \geq 0$ , such that

$$|a_i(t, x, u, p, w)| \leq \eta(|u|, \|w\|_E)(1 + |p|)$$

for all  $(s, x, u, p, w)$  belonging to the region  $\mathcal{R} = [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times m} \times E$  and  $i \in \{1, \dots, n\}$ . Furthermore,  $\eta(s, r)$  is increasing in each variable when the other variable is fixed.

(D6') There exist functions  $P(s, r, q)$ ,  $s, r, q \geq 0$ , and  $\varepsilon(s, r)$ ,  $s, r \geq 0$ , such that

$$|a(s, x, u, p, w)| \leq (\varepsilon(|u|, \|w\|_E) + P(|u|, \|w\|_E, |p|))(1 + |p|)^2$$

for all  $(s, x, u, p, w) \in \mathcal{R}$ . Furthermore,  $P$  and  $\varepsilon$  possess the following properties:  $P(s, r, q)$  is non-decreasing in  $r$  when  $(s, q)$  is fixed, and for all  $s$  and  $r$ ,  $\lim_{q \rightarrow \infty} P(s, r, q) = 0$ ;  $\varepsilon(s, r)$  is non-decreasing in  $r$  when  $s$  is fixed. Moreover, for all  $s, r \geq 0$ , it holds that

$$2(s+1)\varepsilon(s, r) \leq \nu(s). \quad (5.17)$$

(D7') In the region  $\mathcal{R}_1 = [0, T] \times \mathbb{R}^n \times \mathbb{R}^m$ , there exist continuous partial gradients  $\partial_t a_{ij}(t, x, u)$ ,  $\partial_{uu}^2 a_{ij}(t, x, u)$ ,  $\partial_{ux}^2 a_{ij}(t, x, u)$ ,  $\partial_{xt}^2 a_{ij}(t, x, u)$ , and  $\partial_{ut}^2 a_{ij}(t, x, u)$  for all  $i, j \in \{1, \dots, n\}$ . Moreover, it holds that

$$\max \{ |\partial_x a_{ij}(t, x, u)|, |\partial_u a_{ij}(t, x, u)| \} \leq \mu(|u|).$$

(D8') The functions  $a(t, x, u, p, w)$  or  $a_i(t, x, u, p, w)$ ,  $i \in \{1, \dots, n\}$ , possess continuous partial gradients  $\partial_t a$ ,  $\partial_x a$ ,  $\partial_u a$ ,  $\partial_p a$ ,  $\partial_t a_i$ ,  $\partial_x a_i$ ,  $\partial_u a_i$ ,  $\partial_p a_i$  and continuous Hadamard derivatives  $\partial_w a$  and  $\partial_w a_i$  in the region  $\mathcal{R}$ . Moreover, all the above derivatives are bounded in each region of the form  $[0, T] \times \{|x| \leq M_1\} \times \{|u| \leq M_2\} \times \{|p| \leq M_3\} \times \{\|w\|_E \leq M_4\}$ , where  $M_1, M_2, M_3, M_4$  are constants.

(D9') For each  $u \in C_b^{1,2}([0, T] \times \mathbb{R}^n)$ ,  $\vartheta_u : [0, T] \times \mathbb{R}^n \rightarrow E$  possesses bounded and continuous partial derivatives  $\partial_t \vartheta_u$  and  $\partial_x \vartheta_u$ .

We now state the result, and refer the reader to [11] for a proof.

**Theorem 5.3.1** (Existence and uniqueness for Cauchy problem). *Let (D1')–(D9') hold. Then, there exists a unique  $C_b^{1,2}([0, T] \times \mathbb{R}^n)$ -solution to non-local Cauchy problem (5.16).*

Let us observe that problem (5.15) is, in fact, non-local Cauchy problem (5.16) if we define the coefficients  $a_{ij}$ ,  $a_i$ , and  $a$  by expressions (5.18) below while setting the Banach space  $E$  to be  $L_2(\nu, \mathbb{R}^l \rightarrow \mathbb{R}^m)$ , and the function  $\vartheta_u(t, x)$  to be given by (5.14).

$$\begin{aligned} a_{ij}(t, x, u) &= \frac{1}{2} \sum_{k=1}^n \hat{\sigma}_{ik} \hat{\sigma}_{jk}(t, x, u), \\ a_i(t, x, u, p, w) &= \int_Z \hat{\varphi}_i(t, x, u, y) \nu(dy) - \hat{f}_i(t, x, u, p \hat{\sigma}(t, x, u), w), \\ a(t, x, u, p, w) &= -\hat{g}(t, x, u, p \hat{\sigma}(t, x, u), w) - \int_Z w(y) \nu(dy), \end{aligned} \quad (5.18)$$

where

$$Z = \begin{cases} \mathbb{R}^l, & \text{if } \nu(\mathbb{R}^l) < \infty, \\ \mathbb{R}^l \setminus U_0, & \text{otherwise,} \end{cases} \quad (5.19)$$

where the neighbourhood of the origin  $U_0$  is defined in Assumption (D2) below.

In order to guarantee the existence and uniqueness of the solution to problem (5.15), we assume (D1)–(D9) below. Namely, assumptions (D1)–(D9) imply (D1')–(D9').

- (D1) There exist a non-decreasing function  $\mu(s)$  and a non-increasing function  $\nu(s)$ , both taking positive values, such that for all  $(t, x, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m$

$$\nu(|u|) \leq |\sigma(t, x, u)| \leq \mu(|u|).$$

- (D2) For each  $(t, x, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m$ ,  $\varphi(t, x, u, \cdot)$  belongs to  $L_2(\nu, \mathbb{R}^l \rightarrow \mathbb{R}^m)$ . Moreover, either  $\nu(\mathbb{R}^l) < \infty$ , or there exists a neighborhood  $U_0 \subset \mathbb{R}^l$  of the origin such that for all  $(t, x, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m$ ,  $\varphi(t, x, u, \cdot)|_{U_0} = 0$ .
- (D3) There exist non-negative constants  $c_1, c_2$ , and  $c_3$  such that for all  $(t, x, u, p, w) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times E \times \mathbb{R}^{n \times m}$

$$(g(t, x, u, p, w), u) \leq c_1 + c_2|u|^2 + c_3\|w\|_\nu^2,$$

where  $\|\cdot\|_\nu$  is the norm in  $L_2(\nu, \mathbb{R}^l \rightarrow \mathbb{R}^m)$ .

- (D4) The initial condition  $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is of class  $C_b^{2+\beta}(\mathbb{R}^n)$ ,  $\beta \in (0, 1)$ .
- (D5) There exist a positive non-decreasing function  $\varsigma(r)$ ,  $r \geq 0$ , and a function  $\eta(r, s)$ ,  $r, s \geq 0$ , with same properties as in (D5') such that

$$\left| \int_Z \varphi(t, x, u, y) \nu(dy) \right| \leq \varsigma(|u|) \quad \text{and} \quad |f(t, x, u, p, w)| \leq \eta(|u|, \|w\|_\nu)(1 + |p|)$$

for all  $(s, x, u, p, w)$  belonging to the region  $\mathcal{R} = [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times m} \times L_2(\nu, \mathbb{R}^l \rightarrow \mathbb{R}^m)$ .

- (D6) There exist functions  $P(s, r, q)$  and  $\varepsilon(s, r)$ ,  $s, r, q \geq 0$  with the same properties as in (D6') (except (5.17)) such that for all  $(s, x, u, p, w) \in \mathcal{R}$ ,

$$|g(s, x, u, p, w)| \leq (\varepsilon(|u|, \|w\|_\nu) + P(|u|, \|w\|_\nu, |p|))(1 + |p|)^2.$$

Inequality (5.17) is replaced by the following:  $2(1 + s)^3 \varepsilon(s, r) < \nu(s)$ .

- (D7) In the region  $\mathcal{R}_1 = [0, T] \times \mathbb{R}^n \times \mathbb{R}^m$ , there exist continuous partial gradients  $\partial_t \sigma(t, x, u)$ ,  $\partial_{uu}^2 \sigma(t, x, u)$ ,  $\partial_{ux}^2 \sigma(t, x, u)$ ,  $\partial_{xt}^2 \sigma(t, x, u)$ , and  $\partial_{ut}^2 \sigma(t, x, u)$ . Moreover, it holds that

$$\max \{ |\partial_x \sigma(t, x, u)|, |\partial_u \sigma(t, x, u)| \} \leq \mu(|u|).$$

- (D8) The functions  $f(t, x, u, p, w)$  or  $g(t, x, u, p, w)$  possess continuous partial derivatives  $\partial_t f$ ,  $\partial_x f$ ,  $\partial_u f$ ,  $\partial_p f$ ,  $\partial_t g$ ,  $\partial_x g$ ,  $\partial_u g$ ,  $\partial_p g$ , and continuous Hadamard derivatives  $\partial_w f$  and  $\partial_w g$  in  $\mathcal{R}$ . Moreover, all the above derivatives are bounded in each region of the form  $[0, T] \times \{|x| \leq M_1\} \times \{|u| \leq M_2\} \times \{|p| \leq M_3\} \times \{\|w\|_E \leq M_4\}$ , where  $M_1, M_2, M_3, M_4$  are constants.

- (D9) The function

$$\varphi : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow L_2(\nu, \mathbb{R}^l \rightarrow \mathbb{R}^m), \quad (t, x, u) \mapsto \varphi(t, x, u, \cdot)$$

is of class  $C_b^{1,1,1}([0, T] \times \mathbb{R}^n \times \mathbb{R}^m)$ .

**Theorem 5.3.2.** *Let (D1)–(D9) hold. Then, final value problem (5.13) has a unique solution  $\theta$  of class  $C_b^{1,2}([0, T] \times \mathbb{R}^n)$ .*

*Proof.* Since problem (5.13) is equivalent to problem (5.15), it suffices to prove the existence and uniqueness for the latter. As we already mentioned, by introducing functions (5.18), setting the normed space  $E$  to be  $L_2(\nu, \mathbb{R}^l \rightarrow \mathbb{R}^m)$ , and defining  $\vartheta_u$  by (5.14), we rewrite Cauchy problem (5.15) in form (5.16).

Let us prove that (D1')–(D9') are implied by (D1)–(D9). Indeed, (D1) implies (D1'). Next, we note that by (D2), the measure  $\nu$  is supported by  $Z$ , defined by (5.19), and  $\nu(Z) < \infty$ . This implies that for any  $\lambda \geq 0$  and for any  $u \in C_b([0, T] \times \mathbb{R}^n)$ ,

$$\|e^{-\lambda t} \vartheta_u(t, x)\|_\nu \leq 2\nu(Z) \sup_{[0, T] \times \mathbb{R}^n} |e^{-\lambda t} u(t, x)|.$$

Further, (D3') follows from (D3) and (5.18) since for any  $u \in \mathbb{R}^m$ , we have  $\int_Z (w(y), u) \nu(dy) \leq \frac{1}{2} \|w\|_\nu^2 + \frac{\nu(Z)}{2} |u|^2$ . Next, by (D5) and (D1),

$$\begin{aligned} |\hat{f}(t, x, u, p \hat{\sigma}(t, x, u), w)| &\leq \eta(|u|, \|w\|_\nu) (1 + |p| |\hat{\sigma}(t, x, u)|) \\ &\leq \eta(|u|, \|w\|_\nu) (1 + \mu(|u|)) (1 + |p|), \end{aligned}$$

which, together with the inequality for  $\varphi$  in (D5), implies (D5'). Also, (D6') follows from (D6) and (D1) by virtue of the following estimates

$$\begin{aligned} |\hat{g}(t, x, u, p \hat{\sigma}(t, x, u), w)| &\leq \left( \varepsilon(|u|, \|w\|_\nu) + P(|u|, \|w\|_\nu, |p| \mu(|u|)) \right) (1 + |p| \mu(|u|))^2 \\ &\leq (\tilde{\varepsilon}(|u|, \|w\|_\nu) + \tilde{P}(|u|, \|w\|_\nu, |p|)) (1 + |p|)^2, \end{aligned}$$

$$\text{and } \left| \int_Z w(y) \nu(dy) \right| \leq \hat{P}(\|w\|_\nu, |p|) (1 + |p|)^2,$$

where  $\tilde{\varepsilon}(s, r) = \varepsilon(s, r)(1 + s)^2$ ,  $\tilde{P}(s, r, q) = P(s, r, p \mu(s))(1 + s)^2$ , and  $\hat{P}(s, r) = \nu(Z)^{\frac{1}{2}} s (1 + r)^{-2}$ . Further, (D7') is implied by (D7), and (D8') is implied by (D8) if we note that the function  $L_2(\nu, \mathbb{R}^l \rightarrow \mathbb{R}^m) \rightarrow \mathbb{R}^m$ ,  $w \mapsto \int_Z w(y) \nu(dy)$  is Hadamard differentiable. It remains to show (D9'). Note that by (D2),  $\vartheta_u(t, x)$  takes values in  $L_2(\nu, Z \rightarrow \mathbb{R}^m)$  for any  $u \in C_b^{1,2}([0, T] \times \mathbb{R}^n)$ . Moreover, (D9) implies that  $\partial_t \vartheta_u(t, x)$  and  $\partial_x \vartheta_u(t, x)$  exist in  $L_2(\nu, Z \rightarrow \mathbb{R}^m)$  since the derivatives  $\partial_t u(t, x)$  and  $\partial_x u(t, x)$  are bounded and  $\nu(Z)$  is finite. Indeed, when computing the partial derivatives of  $\vartheta_u(t, x)$  in the  $L_2(\nu, Z \rightarrow \mathbb{R}^m)$ -norm, we can pass to the limit under the integral sign by the dominated convergence Theorem. Therefore, by Theorem 5.3.1, there exists a unique  $C^{1,2}([0, T] \times \mathbb{R}^n)$ -solution to problem (5.15).  $\square$

We are ready to state the main contribution of this dissertation.

**Theorem 5.3.3** (Existence). *Assume (D1)–(D9). Then, there exists an  $\mathcal{F}_t$ -adapted càdlàg solution  $(X_t, Y_t, Z_t, \tilde{Z}_t(\cdot))$  to FBSDEs (5.12), such that  $X_t$  is a*

solution to

$$\begin{aligned}
X_t = & x \\
& + \int_0^t f(s, X_s, \theta(s, X_s), \partial_x \theta(s, X_s) \sigma(s, X_s, \theta(s, X_s)), \vartheta_\theta(s, X_s)) ds \\
& + \int_0^t \sigma(s, X_s, \theta(s, X_s)) dB_s + \int_0^t \int_{\mathbb{R}^l} \varphi(s, X_{s-}, \theta(s, X_{s-}), y) \tilde{N}(ds dy), \quad (5.20)
\end{aligned}$$

where  $\theta(t, x)$  is the unique  $C_b^{1,2}([0, T], \mathbb{R}^n)$ -solution to problem (5.13) whose existence was established by Theorem 5.3.2, and  $\vartheta_\theta$  is given by (5.14). Furthermore,

$$Y_t = \theta(t, X_t), \quad Z_t = \partial_x \theta(t, X_{t-}) \sigma(t, X_{t-}, \theta(t, X_{t-})), \quad (5.21)$$

$$\tilde{Z}_t = \vartheta_\theta(t, X_{t-}). \quad (5.22)$$

*Proof.* First we prove that SDE (5.20) has a unique càdlàg solution. Introducing  $\tilde{f}(t, x) = f(t, x, \theta(t, x), \partial_x \theta(t, x) \sigma(t, x, \theta(t, x)), \vartheta_\theta(t, x))$ ,  $\tilde{\sigma}(t, x) = \sigma(t, x, \theta(t, x))$ , and  $\tilde{\varphi}(t, x, y) = \varphi(t, x, \theta(t, x), y)$ , we obtain the SDE

$$X_t = x + \int_0^t \tilde{f}(t, X_s) ds + \int_0^t \tilde{\sigma}(s, X_s) dB_s + \int_0^t \int_{\mathbb{R}^l} \tilde{\varphi}(s, X_{s-}, y) \tilde{N}(ds dy). \quad (5.23)$$

Note that since  $\theta$  is of class  $C_b^{1,2}([0, T] \times \mathbb{R}^n)$ , Assumptions (D1) and (D5) imply that  $\tilde{f}(t, x)$ ,  $\tilde{\sigma}(t, x)$ ,  $\int_Z \tilde{\varphi}(t, x, y) \nu(dy)$  are bounded. Furthermore, (D7) implies the boundedness of  $\partial_x \tilde{\sigma}(t, x)$ , while (D1), (D7), and (D9) imply the boundedness of  $\partial_x \tilde{f}(t, x)$ . Finally, (D1) and (D8) imply the boundedness of  $\partial_x \int_Z \tilde{\varphi}(t, x, y) \nu(dy)$ . Therefore, the Lipschitz condition and the linear growth conditions required for the existence and uniqueness of a càdlàg adapted solution to (5.23) (see [2], p. 375) are fulfilled. By Theorem 2.6.9 in [2] (more precisely, by its time-dependent version considered in Exercise 2.6.10, p. 375), there exists a unique  $\mathcal{F}_t$ -adapted càdlàg solution  $X_t$  to SDE (5.23).

Further, define  $Y_t$ ,  $Z_t$ , and  $\tilde{Z}_t$  by formulas (5.21). Applying Itô's formula

(Lemma 2.2.2) to  $\theta(t, X_t)$ , we obtain

$$\begin{aligned}
\theta(t, X_t) = & \theta(T, X_T) - \int_t^T \theta_x(s, X_{s-}) \sigma(s, X_{s-}, \theta(s, X_{s-})) dB_s \\
& - \int_t^T \left\{ \partial_x \theta(s, X_s) \cdot \right. \\
& f(s, X_s, \theta(s, X_s), \partial_x \theta(s, X_s) \sigma(s, X_s, \theta(s, X_s)), \vartheta_\theta(s, X_s)) \\
& + \partial_x \theta(s, X_s) \int_{\mathbb{R}^l} \varphi(s, X_s, \theta(s, X_s), y) \nu(dy) + \partial_s \theta(s, X_s) \\
& + \frac{1}{2} \text{tr} [\theta_{xx}(s, X_s) \sigma(s, X_s, \theta(s, X_s)) \sigma(s, X_s, \theta(s, X_s))^\top] + \int_{\mathbb{R}^l} \vartheta_\theta(s, X_s)(y) \nu(dy) \Big\} ds \\
& - \int_0^t \int_{\mathbb{R}^l} \vartheta_\theta(s, X_{s-})(y) \tilde{N}(ds dy). \quad (5.24)
\end{aligned}$$

Now PIDE (5.13) imply that  $Y_t$ ,  $Z_t$ , and  $\tilde{Z}_t$ , defined by (5.21), solve the BSDE in (5.12). Furthermore,  $Y_t$  is càdlàg, while  $Z_t$ , and  $\tilde{Z}_t$  are predictable since  $\theta \in C_b^{1,2}([0, T], \mathbb{R}^n)$  and  $X_t$  is càdlàg.  $\square$

To prove the uniqueness, we need to replace Assumption (D9) with the following stronger assumption:

**(D9')** The functions  $f(t, x, u, p, w)$  or  $g(t, x, u, p, w)$  possess bounded and continuous partial derivatives  $\partial_t f$ ,  $\partial_x f$ ,  $\partial_u f$ ,  $\partial_p f$ ,  $\partial_t g$ ,  $\partial_x g$ ,  $\partial_u g$ ,  $\partial_p g$ , as well as bounded and continuous Hadamard derivatives  $\partial_w f$  and  $\partial_w g$  in the region  $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n \times m} \times L_2(\nu, \mathbb{R}^l \rightarrow \mathbb{R}^m)$ .

**Theorem 5.3.4** (Uniqueness). *Assume (D1)–(D8) and (D9'). Then, the solution to FBSDEs (5.12) whose existence was established in Theorem 5.3.3 is unique in the class of processes satisfying the condition*

$$\sup_{t \in [0, T]} \{ \mathbb{E}|X_t|^2 + \mathbb{E}|Y_t|^2 \} + \int_0^T (\mathbb{E}|Z_t|^2 + \mathbb{E}\|\tilde{Z}_t\|_\nu^2) dt < \infty. \quad (5.25)$$

*Proof.* Assume  $(X'_t, Y'_t, Z'_t, \tilde{Z}'_t)$  is another solution satisfying (5.25). Let  $\theta(t, x)$  be the  $C_b^{1,2}([0, T], \mathbb{R}^n)$ -solution whose existence was established by Theorem 5.3.2. Define  $(Y''_t, Z''_t, \tilde{Z}''_t)$  by formulas (5.21) via  $\theta(t, x)$  and  $X'_t$ . Therefore,  $(Y'_t, Z'_t, \tilde{Z}'_t)$  and  $(Y''_t, Z''_t, \tilde{Z}''_t)$  are two solutions to the BSDE in (5.12) with the process  $X'_t$  being fixed. By the results of [8] (Lemma 2.4, p.1455), the solution to the BSDE in (5.12) is unique in the class of processes  $(Y_t, Z_t, \tilde{Z}_t)$  satisfying condition (5.25) (with  $X_t = 0$ ). Then, there exists a set  $\Omega'$  of full  $P$ -measure, such that for all  $\omega \in \Omega'$ ,  $Y'_t = \theta(t, X'_t)$  (remark that both  $Y'_t$  and  $Y''_t$  are càdlàg). Furthermore, for all  $\omega \in \Omega'$  and for almost all  $t \in [0, T]$ ,  $Z'_t$  and  $\tilde{Z}'_t$  are expressed via  $X'_t$  by formulas (5.21). Since the values of  $X'_t$  will not change if we consider the forward SDE in (5.12) with  $Z''_t$  and  $\tilde{Z}''_t$  instead of  $Z'_t$  and, respectively,  $\tilde{Z}'_t$ , then  $X'_t$  is the solution to SDE (5.20). By what was proved, the solution to (5.20) is unique. This proves the theorem.  $\square$

## 5.4 Hedging options in a market with jumps

In the last chapter, we presented a hedging model offering an improvement over the original Black-Scholes framework. In particular the coefficients of the FBSDEs are now not necessarily linear and depend not only on the replicating portfolio that can now contain several stocks, but also on the presence of a "large" investor. Even though this model yields a more robust pricing approach, it still assumes that stock prices follow geometric Brownian motions.

Observation of the prices, however, point to an array of statistically significant features, such as skewness, kurtosis, or the existence of "heavy tails" in the distribution of the log-returns. In particular, the documented evidence of "jumps" in the distribution of the returns (see, e.g., Eberlein and Keller [7]) suggests that a geometric Brownian motion is not entirely suited to model the evolution of stock prices in real markets. Indeed, in periods of heavy market turbulence, such as the "flash crash" in May 2010, when the main US indexes temporarily dropped by more than 9 per cent, hedging strategies resulting from Brownian models leave investors exposed to significant downside risk, not the least due to the mispricing of the hedging instruments.

It is also known that markets where stock prices are modelled involving Lévy processes are, in general, incomplete, so contingent claims may not admit self-financing replicating strategies. The first attempt to define optimal replicating strategies in the context of incomplete markets was made by Fölmer and Schweizer ([18]), where the authors propose an optimal strategy as the one that minimizes, in a certain sense, the injection of capital needed. As such, any model based on continuous diffusions will tend at times to misrepresent the behaviour of stocks over time, with increased unreliability in periods of heavy turbulence. Thus, in this section we propose a model where the stocks are allowed to "jump" at certain times independently from one another, and that can still account for the presence of a large investor.

In the present FBSDE hedging model, the evolution of the  $d$ -dimensional stock price  $S_t = \{S_t^i\}_{i=1}^d$  is governed by an SDE driven by independent Brownian motions and orthonormalized Teugels martingales  $H_t^{(ij)}$ ,  $j = 1, 2, \dots$  where the latter are associated to independent purely discontinuous Lévy processes  $\{L_t^i\}_{i=1}^d$ ,  $l < d$ , so different stock prices  $S_t^i$  may jump at different times. The value process  $V_t$  and the portfolio process  $\pi_t = \{\pi_t^i\}_{i=1}^d$  evolve according to a backward SDE with the final condition  $h(S_T)$  which is the payoff at maturity  $T$ . Our model involves the martingales  $H_t^{(ik)}$  since they are independent, strongly orthonormal, purely discontinuous, but most importantly, the system  $\{H_t^{(ik)}\}_{i=1, k \in \mathbb{N}}$ , completed with the Brownian motions  $\{B_t^i\}_{i=1}^m$ , possesses the predictable representation property. The latter permits a decomposition of the discounted value process into a sum of the value of the hedging portfolio and a strongly orthogonal martingale. Therefore, our model allows finding a hedging strategy which is optimal in the sense of [19].

Moreover, due to the presence of  $H_t^{(ik)}$ 's, the SDEs representing the evolution of stocks become, in fact, driven by the power-jump martingales built

on the basis of the underlying Lévy processes (see Chapter 3). The presence of these “power-jump” terms may reflect “skewness”, “kurtosis”, and other volatile behaviour or extremal movements of the market.

Thus, in this section we introduce a new model in asset pricing which can account for asynchronous jumps in stock prices and allows finding an optimal hedging strategy in markets with jumps.

#### 5.4.1 FBSDE model for hedging in a market with jumps

Let  $(\Omega, \mathcal{F}, P)$  be a complete probability space,  $\{B_t^i\}_{i=1}^m$  be independent real-valued Brownian motions,  $\{L_t^i\}$  independent purely discontinuous real-valued Lévy-processes with Lévy measures  $\nu^i$  satisfying

$$\int_{|x| \geq \varepsilon} e^{\lambda|x|} \nu^i(dx) < C, \quad (5.26)$$

for some  $\varepsilon, \lambda, C > 0$ . Define the augmented filtration

$$\mathcal{F}_t = \sigma\{B_s^i, 0 \leq s \leq t, 1 \leq i \leq m\} \vee \sigma\{N(s, A), 0 \leq s \leq t, A \in \mathcal{B}(\mathbb{R}^l)\} \vee \mathcal{N},$$

where  $\mathcal{N}$  is the collection of all  $P$ -null sets. Note that the third inequality of (5.26) implies that  $L^i$  has finite moments of all orders for  $i = 1, \dots, d$ .

Let, for each  $i \in \{1, \dots, l\}$ ,  $\{H_t^{(ij)}\}_{j=1}^\infty$  be the family of orthonormalized Teugels martingales associated to the Lévy process  $L_t^i$ .

Now, we make a similar argument to section 4.3.2 (see p.24), where  $b$  is an upper bound for the stock prices, and  $a$  the lower bound for the stock prices. Let  $e = b - a$ , and consider the  $d$ -dimensional cube  $C_e'$  centred at the point  $\underbrace{\left(a + \frac{e}{2}, \dots, a + \frac{e}{2}\right)}_d$  with edge  $e$  and a copy  $C_e$  with smooth corners and edges<sup>1</sup>.

Let  $\varepsilon < a$  and let  $C_{e+\varepsilon}$ , be a cube sharing the center with  $C$  with edge  $e + \varepsilon$  and with the corners and edges already smoothed out. Finally, consider a smooth function  $\eta(x)$  that takes values in  $[0, 1]$ , is zero outside  $C_{e+\varepsilon}$ , and is equal to 1 in  $C_e$ . Functions  $\tilde{f}_i, \sigma_i, \psi_i, \tilde{r}$  and  $h$  are assumed to be multiplied by  $\eta$ , i.e., they are of the form  $\eta \cdot \phi$  for some function  $\phi$ . This implies that they are zero outside of the price cube, and non-zero within the realistic range of stock prices.

Fix a finite time horizon  $T > 0$ . Consider a market consisting of  $d$  risky assets (stocks) and risk-free money on a deposit. We assume that the price process of the risk-free deposit evolves according to the equation

$$dD_t = \tilde{r}(t, S_t, V_t, \pi_t)D_t, \quad D_0 = 1, \quad (5.27)$$

where  $\tilde{r} : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$  is the interest rate,  $S_t = \{S_t^i\}_{i=1}^d$  is the  $d$ -dimensional risky asset price process,  $V_t$  is the (real-valued) hedger value process, and  $\pi_t = \{\pi_t^i\}_{i=1}^d$  is the portfolio process with  $\pi_t^i$  being the number

<sup>1</sup>We can more generally assume that all the ranges of all the prices are independent, which results in a parallelepiped, that can also be included in a smooth parallelepiped.

assets of the  $i$ th stock. We assume that  $d = m + l$ . Further, the evolution of  $S_t^i$  is governed by the SDE

$$dS_t^i = S_t^i \left\{ \tilde{f}_i(t, S_t, V_t, \pi_t) dt + \sum_{j=1}^d \sigma_i^{\alpha_j}(t, S_{t-}, V_{t-}) dM_t^{\alpha_j} \right\}, \quad (5.28)$$

with a deterministic initial condition  $S_0^i = s_i > 0$ . In (5.28),  $\tilde{f}_i$  and  $\sigma_i^{\alpha_j}$  are real-valued functions of appropriate dimensions. Further, for  $j = 1, 2, \dots, m$ ,  $\alpha_j = j$  and  $M_t^{\alpha_j} = B_t^j$ , while for  $j = (m+1), \dots, d$ ,  $(\alpha_j)$ 's are arbitrary picked multi-indexes from the set  $\{(ik), i = 1, \dots, l, k = 2, 4, \dots\}$  and  $M_t^{(\alpha_j)} = H_t^{(\alpha_j)}$ . We remark that the index  $k$  takes only even values.

The value process  $V_t$  represents the wealth of a “large” investor who holds  $d$  stocks and money on a deposit. The investor is assumed large, so the coefficients in our model would depend on  $V_t$ ,  $S_t$ , and  $\pi_t$ .

We define an *admissible hedging strategy* as a pair of predictable processes  $(\pi_t, \pi_t^0)$  such that  $V_t = \sum_{i=1}^d \pi_t^i S_t^i + \pi_t^0 D_t$  and  $V_T = h(S_T)$ , where  $h(S_T)$  is the payoff at maturity  $T$ . Note that the solution of (5.27) takes the form  $D_t = \exp\{\int_0^t r_s ds\}$ , where  $r_s = \tilde{r}(s, S_s, V_s, \pi_s)$ . Let  $A_t = \exp\{-\int_0^t r_s ds\}$ . Define  $\hat{S}_t^i = A_t S_t^i$  and  $\hat{V}_t = A_t V_t$  to be the *discounted stock price* and *discounted value process*, respectively. Furthermore, we define the *cumulative cost process* as  $C_t = \hat{V}_t - \sum_{i=1}^d \int_0^t \pi_s^i d\hat{S}_s^i$ . We say that the strategy is *optimal*, if it is admissible and  $C_t$  is a square-integrable martingale strongly orthogonal to the martingale part of each  $\hat{S}_t^i$ .

**Lemma 5.4.1.** *The representations  $\hat{V}_t = \sum_{i=1}^d \int_0^t \pi_s^i d\hat{S}_s^i + C_t$  and*

$$V_t = V_0 + \sum_{i=1}^d \int_0^t \pi_s^i dS_s^i + \int_0^t \pi_s^0 dD_s + \int_0^t D_s dC_s \quad (5.29)$$

*are equivalent.*

*Proof.* Assume we have the representation for  $\hat{V}_t$  as above. Since  $D_t$  is a process given by a non-stochastic integral with no discontinuous part, one has  $\langle V, A \rangle_t = \langle S^i, A \rangle_t = 0$ , and  $[V, A]_t = 0$  and  $[S^i, A]_t = 0$ . Hence, Itô's product formula 2.2.3 implies  $d\hat{S}_t^i = A_t dS_t^i - A_t r_t S_t^i dt$ , and  $d\hat{V}_t = A_t dV_t - r_t A_t V_t dt$ . We now substitute in the expression for  $V_t$ , and obtain in differential form  $dV_t = \sum_{i=1}^d \pi_t^i dS_t^i + (V_t - \sum_{i=1}^d \pi_t^i S_t^i) r_t dt + D_t dC_t$ . Finally, since  $(V_t - \sum_{i=1}^d \pi_t^i S_t^i) = \pi_t^0 D_t$ , an integration gives (5.29). Assume now (5.29) holds. We can rewrite the above differentials for  $dV_t$  and  $dS_t^i$  and, after substituting in (5.29), obtain  $d\hat{V}_t = \sum_{i=1}^d \pi_t^i d\hat{S}_t^i + (\sum_{i=1}^d \pi_t^i \hat{S}_t^i - \hat{V}_t) r_t dt + \pi_t^0 r_t dt + dC_t$ . Since the second and third summands on the right-hand side of the last identity cancel each other, we are left with  $\hat{V}_t = \sum_{i=1}^d \int_0^t \pi_s^i d\hat{S}_s^i + C_t$ .  $\square$

Now we derive a backward SDE (BSDE) for the process  $V_t$  with represen-

tation (5.29). First, we substitute  $dD_t$  and  $dS_t^i$  with the right-hand sides of equations (5.27) and (5.28), respectively. Then, since  $V_T = h(S_T)$ , from (5.29) we can reorganise the resulting identity, take conditional expectations, and obtain

$$\begin{aligned} V_t - \mathbb{E}[h(S_T) - \int_t^T g(s, S_s, V_s, Z_s^{(\alpha)}) ds - \sum_{j=1}^d \int_0^T Z_s^{\alpha_j} dM_s^{\alpha_j} - \int_0^T D_s dC_s | \mathcal{F}_t] \\ = \sum_{j=1}^d \int_0^t Z_s^{\alpha_j} dM_s^{\alpha_j} + \int_0^t D_s dC_s = \sum_{k=1}^l \sum_{j=1}^\infty \int_0^t \hat{Z}_s^{(kj)} dH_s^{(kj)} + \sum_{j=1}^m \int_0^t Z_s^{\alpha_j} dB_s^j, \end{aligned} \quad (5.30)$$

where  $g(t, s, v, z) = \tilde{g}(t, s, v, \mathbf{s}^{-1} \hat{\sigma}(t, s, v)^{-1} z)$  with  $\hat{\sigma}(t, s, v)$  being the  $d \times d$  matrix with the element  $\sigma_i^{\alpha_j}$  in the  $j$ th line and the  $i$ th column,  $\tilde{g}(t, s, v, \pi) = \sum_{i=1}^d s_i \pi_i f_i(t, s, v, \pi) + (v - \sum_{i=1}^d s_i \pi_i) r(t, s, v, \pi)$ ,  $\pi = (\pi_i)_{i=1}^d$ ,  $s = (s_i)_{i=1}^d$ , and  $\mathbf{s} = \text{diag}\{s_1, \dots, s_n\}$ .

**Remark 6.** We introduce  $\hat{\sigma}$  for the matrix just above, not to be confused with the matrix  $\sigma$ .

Further,  $(\alpha)$  denotes the set of multiindexes  $(\alpha) = \{\alpha_1, \dots, \alpha_d\}$  and  $Z_t^{(\alpha)} = (Z_t^{\alpha_1}, \dots, Z_t^{\alpha_d})$ . The last identity in (5.30) (with  $\hat{Z}_t^{(kj)}$  being predictable processes) follows from the predictable representation property of the system of martingales  $\{H_t^{(kj)}\}_{j=1}^\infty$  for a fixed  $k$  and from the independence of the Lévy processes  $L_t^k$ ,  $k = 1, \dots, l$ . Equation (5.30) implies the following BSDE for  $V_t$

$$\begin{aligned} V_t = h(S_T) - \int_t^T g(s, S_s, V_s, Z_s^{(\alpha)}) ds - \sum_{j=1}^d \int_t^T Z_s^{\alpha_j} dM_s^{\alpha_j} \\ - \sum_{(kj) \notin (\alpha)} \int_t^T \hat{Z}_s^{(kj)} dH_s^{(kj)}. \end{aligned} \quad (5.31)$$

Making the change of variable  $\pi = \mathbf{s}^{-1} \hat{\sigma}(t, s, v)^{-1} z$  and introducing the functions  $f_i(t, s, v, z) = \tilde{f}_i(t, s, v, \mathbf{s}^{-1} \hat{\sigma}(t, s, v)^{-1} z)$ , equation (5.28) transforms to

$$dS_t^i = S_t^i \{ f_i(t, S_t, V_t, Z_t^{(\alpha)}) dt + \sum_{j=1}^d \sigma_i^{\alpha_j}(t, S_t, V_t) dM_t^{\alpha_j} \}. \quad (5.32)$$

**Lemma 5.4.2.** *FBSDEs (5.31)-(5.32) are equivalent to*

$$\begin{cases} S_t^i = S_0^i + \int_0^t S_s^i f_i(s, S_s, V_s, \mathcal{Z}_s, \hat{\mathcal{Z}}_s(\cdot)) ds + \sum_{j=1}^m \int_0^t S_s^i \sigma_i^{\alpha_j}(s, S_s, V_s) dB_s^j \\ \quad + \int_0^t \int_{\mathbb{R}^l} S_s^i \psi_i(s, S_{s-}, V_{s-}, u) \tilde{N}(ds, du), \quad i = 1, \dots, d, \\ V_t = h(S_T) - \int_t^T g(s, S_s, V_s, \mathcal{Z}_s, \hat{\mathcal{Z}}_s(\cdot)) ds - \int_t^T \mathcal{Z}_s dB_s \\ \quad - \int_t^T \int_{\mathbb{R}^l} \hat{\mathcal{Z}}_s(u) \tilde{N}(ds, du), \end{cases} \quad (5.33)$$

where for  $u = (u_1, \dots, u_l)$ ,  $\psi_i(t, s, v, u) = \sum_{q=m+1}^d \sigma_i^{\alpha_q}(t, s, v) p_{\alpha_q}(u_{\alpha_q})$  with  $u_{\alpha_q} = u_k$  if  $\alpha_q = (kj)$ . Further,  $\mathcal{Z}_t = (Z_t^{\alpha_1}, \dots, Z_t^{\alpha_m})$  and for each  $k \in \{1, \dots, l\}$ ,  $\hat{\mathcal{Z}}_t^{(kj)}$  are the components of the decomposition of  $\hat{\mathcal{Z}}_t(0, \dots, u_k, \dots, 0)$  with respect to the basis of polynomials  $p_{(kj)}(u_k)$  in the space  $L_2(\nu^k(du_k))$ , while  $(Z_t^{\alpha_{m+1}}, \dots, Z_t^{\alpha_d}) = \{\hat{\mathcal{Z}}_t^{(kj)}\}_{(kj) \in (\alpha)}$ . Finally,  $\tilde{N}$  is the compensated Poisson random measure for the Lévy process  $(L_t^1, \dots, L_t^l)$ .

**Remark 7.** With a slight abuse of notation, in the coefficients  $f$  and  $g$ , we write  $\mathcal{Z}_t$  instead of  $(Z_t^{\alpha_1}, \dots, Z_t^{\alpha_m})$  and  $\hat{\mathcal{Z}}_t(\cdot)$  instead of  $(Z_t^{\alpha_{m+1}}, \dots, Z_t^{\alpha_d})$ . The dependence on  $\hat{\mathcal{Z}}_t(\cdot)$  is understood as the dependence on its  $d - m$  components  $(Z_t^{\alpha_{m+1}}, \dots, Z_t^{\alpha_d})$ .

*Proof of Lemma 5.4.2.* Note that for each  $k$ , the system  $\{H_t^{(kj)}\}_{j=1}^\infty$  has the predictable representation property. Therefore,

$$\begin{aligned} \int_t^T \int_{\mathbb{R}^l} \hat{\mathcal{Z}}_s(u) \tilde{N}(ds, du) &= \sum_{k=1}^l \int_t^T \int_{R_k} \hat{\mathcal{Z}}_s(0, \dots, u_k, \dots, 0) \tilde{N}^k(ds, du_k) \\ &= \sum_{k=1}^l \sum_{j=1}^\infty \int_t^T \hat{\mathcal{Z}}_s^{(kj)} dH_s^{(kj)}. \end{aligned}$$

Here  $R_k = \{te_k, t \in \mathbb{R}\}$ , where  $\{e_k\}_{k=1}^l$  is an orthonormal basis in  $\mathbb{R}^l$ , and  $\tilde{N}^k(t, \cdot)$  is the compensated Poisson random measure for  $L_t^k$ , which, by the independence of  $L_t^k$ 's, is the restriction of  $\tilde{N}(t, \cdot)$  to  $R_k$ . Since, by Lemma 3.2.1,  $H_t^{(kj)} = \int_{R_k} p_{(kj)}(u_k) \tilde{N}^k(t, du_k)$ , we obtain that in  $L_2(\nu^k)$ ,  $\hat{\mathcal{Z}}_t(0, \dots, u_k, \dots, 0) = \sum_{j=1}^\infty \hat{\mathcal{Z}}_t^{(kj)} p_{(kj)}(u_k)$ . Further, for each  $k$ , the system of polynomials  $\{p_{(kj)}\}_{j=1}^\infty$  is orthonormal in  $L_2(\nu^k)$  by the orthonormality of  $H_t^{(kj)}$ 's. Finally, since for each  $(kj)$ ,  $p_{(kj)}(0) = 0$ , we obtain

$$\begin{aligned} \int_{\mathbb{R}^l} \sum_{k=m+1}^d \sigma_i^{\alpha_k}(t, x, y) p_{\alpha_k}(u_k) \tilde{N}(dt, du) \\ = \sum_{k=m+1}^d \sigma_i^{\alpha_k}(t, x, y) \int_{R_k} p_{\alpha_k}(u_k) \tilde{N}^k(dt, du_k) = \sum_{k=m+1}^d \sigma_i^{\alpha_k}(t, x, y) dH_t^{\alpha_k}. \end{aligned}$$

□

**Remark 8.** Since  $\{\alpha_j\}_{j=m+1}^d$  are multiindexes picked from the set  $\{(ik), i = 1, \dots, l, k = 2, 4, \dots\}$ , then each polynomial  $p_{\alpha_j}$  is of even degree, and, therefore, achieves a finite global minimum, which we denote by  $A_j$ .

With the equivalence we have by our last result, we now show how we can apply Theorem 5.3.3 to obtain the existence and uniqueness of solution to FB-SDEs (5.31)–(5.32). We are going to require the following assumptions.

- (E1) Functions  $f_i, \sigma_i, \psi_i, r$ , and  $h$  are assumed to be multiplied by  $\eta$ , i.e., they are of the form  $\eta \cdot \phi$  for some function  $\phi$ . Furthermore,  $f_i, \psi_i, \sigma_i$ , and  $h$  satisfy their respective conditions in (D1)–(D2), (D4) (D5) and (D7)–(D9).
- (E2) There exist  $c_1, c_2, c_3 > 0$  such that

$$(g(t, x, u, p, w), u) \leq c_1 + c_2|u|^2 + c_3\|w\|_\nu^2,$$

for all  $(t, x, u, p, w) \in [0, T] \times C_{r+\varepsilon} \times \mathbb{R} \times \mathbb{R}^d \times L_2(\nu, \mathbb{R}^l \rightarrow \mathbb{R}^d)$ . Further, the function  $r$  is continuous.

**Theorem 5.4.3.** Assume (E1)–(E2) holds. Then, there exists a unique  $\mathcal{F}_t$ -adapted solution  $(S_t, V_t, Z_t^{(\alpha)})$  to FBSDE (5.33). Moreover, there is a unique  $C_b^{1,2}([0, T] \times \mathbb{R}^n)$ -function  $C(t, x)$ , whose existence established by Theorem 5.3.3, such that  $C(t, S_t) = V_t$ .

*Proof.* Given the explicit dependence of  $g$  on  $r$ , and  $f$ , (D9) holds also for  $g$ , which has also the form  $\eta \cdot \phi$ , for a function  $\phi$ . Now, since by (E2)  $r$  is bounded, (D5) implies that (D3) hold for  $-g$ . (D6) can be verified similarly to the proof of Theorem 4.2.2. As such, (D1)–(D6) hold for the coefficients of (5.33), and Theorem 5.3.3 gives then a unique  $\mathcal{F}_t$ -adapted solution  $(S_t, V_t, Z_t, \tilde{Z}_t(\cdot))$  to FBSDEs (5.33). By Lemma 5.4.2, this is equivalent to the existence of a unique  $\mathcal{F}_t$ -adapted solution  $(S_t, V_t, Z_t^{(\alpha)})$  to FBSDEs (5.31)–(5.32).  $\square$

**Remark 9.** The function  $C$  above is usually called *price function*.

The following result, known as the comparison theorem, was obtained in [14] (p. 1362). It will be useful to obtain the non-negativity of the value process.

**Proposition 5.4.4.** Let  $(Y_t, Z_t, \tilde{Z}_t)$  be the adapted solution, with values on  $\mathbb{R} \times \mathbb{R}^d \times L_2(\mu, \mathbb{R}^l \rightarrow \mathbb{R})$ , to the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \tilde{Z}_s) ds - \int_t^T Z_s dB_s - \int_t^T \int_{\mathbb{R}^l} \tilde{Z}_s(u) \tilde{N}(ds, du).$$

where  $\xi$  is  $\mathcal{F}_T$ -adapted, and the function  $f$  satisfies (i)–(iii) below:

- i)  $\mathbb{E} \left( \int_0^T |f(s, 0, 0, 0)|^2 ds \right) < \infty$ ,
- ii)  $f$  is Lipschitz with respect to  $y$  and  $z$  (with constant denoted by  $K$ .)
- iii) There exist  $-1 < C_1 \leq 0$  and  $C_2 \geq 0$  such that for  $y \in \mathbb{R}$ ,  $z \in \mathbb{R}^d$ ,

$u, u \in L_2(\mu, \mathbb{R}^l \rightarrow \mathbb{R})$  we have

$$f(t, y, z, u) - f(t, y, z, u') \leq \int_{\mathbb{R}^l} (u(x) - u'(x)) \lambda_t^{y, z, u, u'}(x) \mu(dx),$$

where  $\lambda_t^{y, z, u, u'} : \Omega \times [0, T] \times \mathbb{R}^l \rightarrow \mathbb{R}$  is measurable and satisfies the conditions  $C_1 \min(1, |x|) \leq \lambda_t \leq C_2 \min(1, |x|)$ .

Let  $(Y'_t, Z'_t, \tilde{Z}'_t)$  be the adapted solution, with values on  $\mathbb{R} \times \mathbb{R}^d \times L_2(\mu, \mathbb{R}^l \rightarrow \mathbb{R})$  to the BSDE

$$Y_t = \xi' + \int_t^T f'(s, Y_s, Z_s, \tilde{Z}_s) ds - \int_t^T Z_s dB_s - \int_t^T \int_{\mathbb{R}^l} \tilde{Z}_s(u) \tilde{N}(ds, du),$$

where  $\xi'$  is  $\mathcal{F}_T$ -adapted and the function  $f'$  satisfies the following conditions (iv)-(vi) below:

iv) There exists a constant  $K \geq 0$  such that for  $t \in [0, T]$ ,  $y \in \mathbb{R}$ ,  $z, z' \in \mathbb{R}^d$  and  $u, u' \in L_2(\mu, \mathbb{R}^l \rightarrow \mathbb{R})$  it holds

$$|f'(t, y, z, u) - f'(t, y, z', u')| \leq K|z - z'| + K \left( \int_{\mathbb{R}^l} |u(x) - u'(x)|^2 \mu(dx) \right)^{\frac{1}{2}}$$

v) There exists an  $\mathbb{R}_+$ -valued adapted process  $\phi_t$ , such that  $\mathbb{E}(\int_0^T \phi_s^2 ds) < \infty$  and

$$|f'(t, y, z, u)| \leq \phi_t + K|z| + K \left( |y| + |z| + \int_{\mathbb{R}^l} |u(x)|^2 \mu(dx) \right)^{\frac{1}{2}}$$

vi) There exists  $\alpha \in \mathbb{R}$  such that for  $t \geq 0$ ,  $y, y' \in \mathbb{R}$ ,  $z \in \mathbb{R}^d$  and  $u, u' \in L_2(\mu, \mathbb{R}^l \rightarrow \mathbb{R})$  one has

$$(y - y')(f'(t, y, z, u) - f'(t, y', z, u)) \leq \alpha |y - y'|^2.$$

Then, if  $\xi' \leq \xi$  and  $f'(t, Y'_t, Z'_t, \tilde{Z}'_t) \leq f(t, Y'_t, Z'_t, \tilde{Z}'_t)$ , then for any  $t \in [0, T]$ , it holds  $Y'_t \leq Y_t$ , a.s.

Assumptions (E3)–(E5) below guarantee the positivity of the stock prices  $S_t^i$ , the non-negativity of the value process  $V_t$ , and the existence of the optimal strategy:

(E3)  $\det\{\hat{\sigma}(t, s, v)\} \neq 0$  for all  $(t, s, v) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$ .

(E4) For all  $(t, s, v) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}$ ,  $i \in \{1, \dots, d\}$ , and  $j \in \{m+1, \dots, d\}$ ,  $\sigma_i^{\alpha_j}(t, s, v) > 0$ . Moreover, if  $A_j < 0$ , then  $\sigma_i^{\alpha_j}(t, x, v)|A_j| < (d-m)^{-1}$ .

(E5) If  $(S_t, V_t, Z_t, \hat{Z}_t(\cdot))$  is the  $\mathcal{F}_t$ -adapted solution to FBSDEs (5.33), then the random function  $(\omega, t, y, z, \hat{z}) \mapsto g(t, S_t, y, z, \hat{z})$  satisfies iii) in Theorem 5.4.4. Moreover,  $h(S_T) \geq 0$  a.s.

The main result of this section is then the following.

**Theorem 5.4.5.** *Let (E1)–(E5) hold, and let  $(S_t, V_t, Z_t^{(\alpha)})$  be the solution to*

FBSDEs (5.31)-(5.32). Then,  $S_t^i > 0$ ,  $i = 1, \dots, d$ , and  $V_t \geq 0$  a.s. Moreover, the pair of stochastic processes  $(\pi_t, \pi_t^0)$ , where  $\pi_t = \hat{\sigma}^{-1}(t, S_t, V_t)Z_t^{(\alpha)}$  and  $\pi_t^0 = \hat{V}_t - \sum_{i=1}^d \pi_s^i \hat{S}_t^i$ , is the optimal hedging strategy.

*Proof.* Note that, by construction, it holds that  $\pi_t = \hat{\sigma}^{-1}(t, S_t, V_t)Z_t^{(\alpha)}$ . Let us prove the positivity of the prices  $S_t^i$ . By the representation for the function  $\psi_i$ , obtained in Lemma 5.4.2, and by (E4),  $\inf_{t>0} \psi_i(t, S_{t-}, V_{t-}, \Delta L_t) > -1$ . Therefore,  $S_t^i$  can be represented by the Doleans-Dade exponential which is finite a.s.:

$$\begin{aligned} S_t^i = S_0^i \exp & \left[ \int_0^t \left( \tilde{f}_i(s, S_s, V_s, \pi_s) - \frac{\|\sigma_i(s, S_s, V_s)\|^2}{2} \right) ds + \int_0^t \sigma_i(s, S_s, V_s) dB_s \right. \\ & \left. + \int_0^t \int_{\mathbb{R}^l} \psi_i(s, S_{s-}, V_{s-}, u) \tilde{N}(ds, du) \right] \\ & \times \prod_{0 \leq s \leq t} (1 + \psi_i(s, S_{s-}, V_{s-}, \Delta L_s)) e^{-\psi_i(s, S_{s-}, V_{s-}, \Delta L_s)}, \end{aligned}$$

where  $\sigma_i = (\sigma_i^{\alpha_j})_{j=1}^d$ . Therefore, for all  $i$ ,  $S_t^i > 0$  a.s. Let us prove the non-negativity of  $V_t$ . To this end, we apply Theorem 5.4.4 to the BSDE in (5.33), considered with respect to  $(V_t, \mathcal{Z}_t, \hat{\mathcal{Z}}_t(\cdot))$ , whereas the process  $S_t$  is fixed and assumed known from Lemma 5.4.3. Note that, by the definition,  $g(t, S_t, 0, 0, 0) = 0$ . Therefore, we compare the solution  $(V_t, \mathcal{Z}_t, \hat{\mathcal{Z}}_t(\cdot))$  with the identically zero solution to the BSDE whose generator is the same as in (5.33) but the final condition is zero. Remark that (E5) implies the assumptions of the comparison Theorem in [14]. Thus, by Theorem 2.5 in [14],  $V_t \geq 0$  for all  $t$  a.s.

Note that, by (5.30),  $C_t = V_0 + \sum_{(kj) \notin (\alpha)} \int_0^t A_s \hat{\mathcal{Z}}_s^{(kj)} dH_s^{(kj)}$ , and, therefore, it is a square-integrable martingale. Moreover,  $C_t$  is (weakly) orthogonal to the stable subspace  $\mathcal{S}$  generated by  $\{M_t^{\alpha_j}\}_{j=1}^d$ , which follows from Theorem 35 of [12] (p.149) and from the strong orthogonality of the martingales  $M_t^{\alpha_j}$ . Indeed, the independence of  $L^k$  and  $L^j$  implies that  $H_t^{\alpha_k}$  and  $H_t^{\alpha_l}$  are independent and, moreover, that they don't have jumps in common. Hence,  $H_t^{\alpha_k} H_t^{\alpha_l}$  is a martingale, and by Corollary 2 in [12] (p. 65), one has  $\langle H^{\alpha_k}, H^{\alpha_l} \rangle_t = 0$ , for all  $t > 0$ , and thus the martingales are strongly orthogonal. By Theorem 36 of [12] (p.150),  $C_t$  is strongly orthogonal to  $\mathcal{S}$ . It remains to note that the martingale parts of  $\{\hat{S}_t^i\}_{i=1}^d$  belong to  $\mathcal{S}$ .  $\square$

**Corollary 5.4.6.** *The Föllmer-Schweizer decomposition of the discounted contingent claim  $A_T h(S_T)$  takes the form*

$$A_T h(S_T) = V_0 + \sum_{i=1}^d \int_0^T \pi_t^i d\hat{S}_t^i + \sum_{(kj) \notin (\alpha)} \int_0^T A_t \hat{\mathcal{Z}}_t^{(kj)} dH_t^{(kj)}.$$

We recall that in our result on the existence and uniqueness of solution to FBSDEs with jumps, we obtained associated PIDE (5.13). To keep notation

similar to the previous section, we introduce the  $d \times m$  matrix  $\sigma(t, s, v)$ , which is obtained from  $\hat{\sigma}^\top(t, s, v)$  by removing the last  $d - m$  columns.

**Theorem 5.4.7.** *Let  $(S_t, V_t, Z_t^{(\alpha)})$  be the solution to FBSDEs (5.31)-(5.32). Then, the function  $C(t, x)$  (given by Theorem 5.4.3) satisfies the PIDE*

$$\begin{aligned}
& \sum_{i=1}^d C_{x_i} x_i f_i(t, x, C, C_x \sigma(t, x, C), C(t, x + \psi(t, x, C, \cdot)) - C(t, x)) \\
& - \sum_{i=1}^n C_{x_i} x_i \int_{\mathbb{R}^l} \psi_i(t, x, C, u) \nu(du) + \frac{1}{2} \text{tr} \left( C_{xx} \mathbf{x} \sigma(t, x, C) (\mathbf{x} \sigma(t, x, C))^\top \right) \\
& + g(t, x, C, C_x \sigma(t, x, C), C(t, x + \psi(t, x, C, \cdot)) - C(t, x)) \\
& + \int_{\mathbb{R}^l} (C(t, x + \psi(t, x, C, u)) - C(t, x)) \nu(du) + C_t = 0, \tag{5.34}
\end{aligned}$$

where  $\mathbf{x} = \text{diag}\{x^1, \dots, x^d\}$ , for  $u = (u_1, \dots, u_l)$ ,  $\psi_i(t, s, v, u) = \sum_{q=m+1}^d \sigma_i^{\alpha_q}(t, s, v) p_{\alpha_q}(u_{\alpha_q})$ , with  $u_{\alpha_q} = u_k$  if  $\alpha_q = (kj)$ , and  $\psi = (\psi_1, \dots, \psi_d)$ . Moreover  $C$ ,  $C_t$ ,  $C_x$ , and  $C_{xx}$  are evaluated at  $(t, x)$ .

*Proof.* The proof is an immediate corollary of Theorem 5.3.3.  $\square$

PIDE (5.34) can be regarded as an analogue of the Black-Scholes PDE for the price function  $C(t, x)$ .

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