Hölder conditions for endomorphisms of hyperbolic groups

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ABSTRACT

It is proved that an endomorphism φ of a hyperbolic group G satisfies a Hölder condition with respect to a visual metric if and only if φ is virtually injective and $G\varphi$ is a quasiconvex subgroup of G. If G is virtually free or torsion-free co-hopfian, then φ is uniformly continuous if and only if it satisfies a Hölder condition if and only if it is virtually injective.

1 Introduction

The concept of boundary of a free group has been for a number of years a major subject of research from geometric, topological, dynamical, algebraic or combinatorial viewpoints. The boundary of F_A, denoted by ∂F_A, can be defined as the set of all infinite reduced words on $\widetilde{A} = A \cup A^{-1}$, but the topological (metric) structure is of utmost importance. It can be defined through the *prefix metric*. Given $u, v \in F_A$, let $u \wedge v$ denote the longest common prefix of u and v. An ultrametric $p_A : F_A \times F_A \to \mathbb{R}_0^+$ is defined by

$$p_A(u,v) = \begin{cases} 2^{-|u\wedge v|} & \text{if } u \neq v \\ 0 & \text{otherwise} \end{cases}$$

The completion $(\widehat{F_A}, \widehat{p_A})$ can be described as

$$\widehat{F_A} = F_A \cup \partial F_A,$$

and the metric $\widehat{p_A}$ is nothing but the prefix metric defined for finite and infinite reduced words altogether.

The theory of hyperbolic groups generalizes many aspects of free groups, and we can endow the boundary of a hyperbolic group with a metric structure proceeding analogously. This can be achieved with the help of the Gromov product and the visual metrics $\sigma^A_{p,\gamma}$. If $G = F_A$, $p = 1$ and $\gamma = \ln 2$, then $\sigma^A_{p,\gamma}$ is precisely the prefix metric defined above.

The completion $(\widehat{G}, \widehat{\sigma^A_{p,\gamma}})$ of $(G, \sigma^A_{p,\gamma})$ produces the boundary $\partial G = \widehat{G} \setminus G$ and its metric structure, which induces the Gromov topology on ∂G. This same topology can be induced by any of the visual metrics $d \in V^A(p, \gamma, T)$. These are the metrics considered in this paper, and their extensions $\widehat{d}$ to $\widehat{G}$.

Since the completion $(\widehat{G}, \widehat{d})$ is also compact, the endomorphisms of G which admit a continuous extension to the boundary are precisely the uniformly continuous ones. It is thus a natural problem to determine which endomorphisms of G admit such a continuous extension. It is well known that automorphisms do.

Uniform continuity is implied by a Hölder condition. A mapping $\varphi : (X, d) \to (X', d')$ satisfies a *Hölder condition* of exponent $r > 0$ if there exists a constant $K > 0$ such that

$$d'(x\varphi, y\varphi) \leq K(d(x, y))^r$$

for all $x, y \in X$. A Hölder condition of exponent 1 is a *Lipschitz condition*.

In this paper, we are interested mainly on Hölder conditions for endomorphisms, with respect to visual metrics. Given the exponential in the definition of the visual metric, it is not surprising that this reduces to some Lispschitz type condition involving Gromov products. As a preliminary result, we show that all visual metrics on a hyperbolic group are Hölder equivalent.

In the main theorem of the paper (Theorem 4.3), we establish several equivalent conditions for a nontrivial endomorphism of a hyperbolic group to satisfy a Hölder condition. The most interesting is undoubtedly the last one: φ must be virtually injective and $G\varphi$ must be a quasiconvex subgroup of G. This second requirement may be removed if the group is virtually free or torsion-free co-hopfian, when we also show that all uniformly continuous endomorphisms satisfy a Hölder condition. The second author had proved in [17, Proposition 7.2] that a nontrivial endomorphism of a finitely generated virtually free group is uniformly continuous if and only if it is virtually injective. We ignore whether this is also true for hyperbolic groups in general.

One of the motivations for our work is the possibility of defining new pseudometrics on $\mathrm{Aut}(G)$ for every hyperbolic group G. Given $\varphi \in \mathrm{Aut}(G)$ and a visual metric d on G, write

$$||\varphi||_d = \ln(\inf\{r \geq 1 \mid \varphi \text{ satisfies a Hölder condition of exponent } r^{-1} \text{ with respect to } d\}).$$

Since

$$||\varphi\psi||_d \leq ||\varphi||_d + ||\psi||_d \qquad (1)$$

for all $\varphi, \psi \in \mathrm{Aut}(G)$, we call $|| \cdot ||_d$ a seminorm. All inner automorphisms have seminorm 0.

Now we define a pseudometric $\overline{d}$ on $\mathrm{Aut}(G)$ by

$$\overline{d}(\varphi, \psi) = \max\{||\varphi^{-1}\psi||_d, ||\psi^{-1}\varphi||_d\}.$$

The inequality (1) implies the triangular inequality for $\overline{d}$. Note that, since inner automorphisms have seminorm 0, it follows easily from (1) that the pseudometric $\overline{d}$ induces a pseudometric on the outer automorphism group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$. These pseudometrics are the object of ongoing work by the authors.

The paper is organized as follows. In Section 2 we present basic concepts and notation for hyperbolic groups. Visual metrics and some of their basic properties in connection with Hölder conditions are discussed in Section 3. The main results of the paper, characterizing which uniformly continuous endomorphisms satisfy a Hölder condition, are presented in Section 4. Simplifications for the case of virtually free or torsion-free co-hopfian hyperbolic groups are discussed in Section 5. Finally, some open problems are proposed in Section 6.

2 Hyperbolic groups

We present in this section well-known facts regarding hyperbolic spaces and hyperbolic groups. The reader is referred to [2, 5] for details.

A mapping $\varphi : (X, d) \to (X', d')$ between metric spaces is called an *isometric embedding* if $d'(x\varphi, y\varphi) = d(x, y)$ for all $x, y \in X$. A surjective isometric embedding is an *isometry*.

A metric space (X, d) is said to be *geodesic* if, for all $x, y \in X$, there exists an isometric embedding $\xi : [0, s] \to X$ such that $0\xi = x$ and $s\xi = y$, where $[0, s] \subset \mathbb{R}$ is endowed with the usual metric of $\mathbb{R}$. We call ξ a *geodesic* of (X, d). We shall often call $\mathrm{Im}(\xi)$ a geodesic as well. In this second sense, we may use the notation $[x, y]$ to denote an arbitrary geodesic connecting x to y. Note that a geodesic metric space is always (path) connected.

A *quasi-isometric embedding* of metric spaces is a mapping $\varphi : (X, d) \to (X', d')$ such that there exist constants $\lambda \geq 1$ and $K \geq 0$ satisfying

$$\frac{1}{\lambda} d(x, y) - K \leq d'(x\varphi, y\varphi) \leq \lambda d(x, y) + K$$

for all $x, y \in X$. We may call it a (λ, K)-quasi-isometric embedding if we want to stress the constants. If in addition

$$\forall x' \in X' \; \exists x \in X : \; d'(x', x\varphi) \leq K,$$

we say that φ is a *quasi-isometry*.

Two metric spaces (X, d) and (X', d') are said to be *quasi-isometric* if there exists a quasi-isometry $\varphi : (X, d) \to (X', d')$. Quasi-isometry turns out to be an equivalence relation on the class of metric spaces. A *quasigeodesic* of (X, d) is a quasi-isometric embedding $\xi : [0, s] \to X$ such that $0\xi = x$ and $s\xi = y$, where $[0, s] \subset \mathbb{R}$ is endowed with the usual metric of $\mathbb{R}$.

Let (X, d) be a geodesic metric space. Given $x_0, x_1, x_2 \in X$, a *geodesic triangle* $[[x_0, x_1, x_2]]$ is a collection of three geodesics $[x_0, x_1]$, $[x_1, x_2]$ and $[x_2, x_0]$ in X.

Given $\delta \geq 0$, we say that (X, d) is *δ-hyperbolic* if

$$\forall y \in [x_0, x_2] \quad d(y, [x_0, x_1] \cup [x_1, x_2]) \leq \delta \tag{2}$$

holds for every geodesic triangle $[[x_0, x_1, x_2]]$ in X. If this happens for some $\delta \geq 0$, we say that (X, d) is *hyperbolic*.

Given $Y, Z \subseteq X$ nonempty, the *Hausdorff distance* between Y and Z is defined by

$$\mathrm{Haus}(Y, Z) = \max\{\sup_{y \in Y} d(y, Z), \; \sup_{z \in Z} d(z, Y)\}.$$

If (X, d) is δ-hyperbolic and $\lambda \geq 1$, $K \geq 0$, it follows from [5, Theorem 5.4.21] that there exists a constant $R(\delta, \lambda, K)$, depending only on δ, λ, K, such that any geodesic and (λ, K)-quasigeodesic in X having the same initial and terminal points lie at Hausdorff distance $\leq R(\delta, \lambda, K)$ from each other. This constant will be used in the proof of several results.

Given a subset A of a group G, we denote by $\langle A \rangle$ the subgroup of G generated by A. We assume throughout the paper that generating sets are finite.

Given $G = \langle A \rangle$, we write $\widetilde{A} = A \cup A^{-1}$. The *Cayley graph* $\Gamma_A(G)$ has vertex set G and edges of the form $g \xrightarrow{a} ga$ for all $g \in G$ and $a \in \widetilde{A}$. The *geodesic metric* d_A on G is defined by taking $d_A(g, h)$ to be the length of the shortest path connecting g to h in $\Gamma_A(G)$.

Since $\mathrm{Im}(d_A) \subseteq \mathbb{N}$, then (G, d_A) is not a geodesic metric space. However, we can remedy that by embedding (G, d_A) isometrically into the *geometric realization* $\overline{\Gamma}_A(G)$ of $\Gamma_A(G)$, when vertices become points and edges become segments of length 1 in some (euclidean) space, intersections being determined by adjacency only. With the obvious metric, $\overline{\Gamma}_A(G)$ is a geodesic metric space, and the geometric realization is unique up to isometry. We denote also by d_A the induced metric on $\overline{\Gamma}_A(G)$.

We say that the group G is *hyperbolic* if the geodesic metric space $(\overline{\Gamma}_A(G), d_A)$ is hyperbolic.

If A' is an alternative finite generating set of G and

$$N_{A,A'} = \max(\{d_{A'}(1, a) \mid a \in A\} \cup \{d_A(1, a') \mid a' \in A'\}), \tag{3}$$

it is immediate that

$$\frac{1}{N_{A,A'}} d_{A'}(g, h) \leq d_A(g, h) \leq N_{A,A'} d_{A'}(g, h) \tag{4}$$

holds for all $g, h \in G$, hence the identity mapping $(G, d_A) \to (G, d_{A'})$ is a quasi-isometry. It follows easily that the concept of hyperbolic group is independent from the finite generating set considered, but the hyperbolicity constant δ may vary with the generating set.

Condition (2), which became the most popular way of defining hyperbolic group, is known as *Rips condition*. An alternative approach is given by the concept of Gromov product, which we now define. It can be defined for every metric space.

Given $g, h, p \in G$, we define

$$(g|h)^A_p = \frac{1}{2}(d_A(p, g) + d_A(p, h) - d_A(g, h)).$$

We say that $(g|h)^A_p$ is the *Gromov product* of g and h, taking p as basepoint.

The following result is well known:

Proposition 2.1 *The following conditions are equivalent for a group $G = \langle A \rangle$:*

(i) G is hyperbolic;

(ii) there exists some $\delta \geq 0$ such that

$$(g_0|g_2)^A_p \geq \min\{(g_0|g_1)^A_p, (g_1|g_2)^A_p\} - \delta \tag{5}$$

holds for all $g_0, g_1, g_2, p \in G$.

Let H be a subgroup of a hyperbolic group $G = \langle A \rangle$ and let $q \geq 0$. We say that H is *q-quasiconvex* with respect to A if

$$\forall x \in [h, h'] \quad d_A(x, H) \leq q$$

holds for every geodesic $[h, h']$ in $\overline{\Gamma}_A(G)$ with endpoints in H. We say that H is *quasiconvex* if it is q-quasiconvex for some $q \geq 0$. Like most other properties in the theory of hyperbolic groups, quasiconvex does not depend on the finite generating set considered [2, Section III.Γ.3].

A (finitely generated) subgroup of a hyperbolic group needs not be hyperbolic, but a quasiconvex subgroup of a hyperbolic group is always hyperbolic. The converse is not true in general. Quasiconvex subgroups occur quite frequently in the theory of hyperbolic groups. In view of [6, Theorem 2.1], non quasiconvex subgroups are statistically rare. See [9] for more details on non quasiconvex subgroups.

We present next a model for the boundary of G.

Given a mapping $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$, we write

$$\varliminf_{i,j\to+\infty} (i,j)\varphi = \lim_{n\to+\infty} (\inf\{(i,j)\varphi \mid i,j \geq n\}).$$

Fix a generating set A for G and $p \in G$. We say that a sequence $(g_n)_n$ on G is a *Gromov sequence* if

$$\varliminf_{i,j\to+\infty} (g_i|g_j)_p^A = +\infty.$$

This property is independent from both A and g. Two Gromov sequences $(g_n)_n$ and $(h_n)_n$ on G are *equivalent* if

$$\lim_{n\to+\infty} (g_n|h_n)_p^A = +\infty.$$

We denote by $[(g_n)_n]$ the equivalence class of the Gromov sequence $(g_n)_n$. The set of all such equivalence classes is one of the standard models for the boundary ∂G, and is adopted in this paper.

We can identify G with the set of all constant sequences $(g)_n$ on G, and consider

$$\widehat{G} = \partial G \cup \{\{(g)_n\} \mid g \in G\}.$$

The Gromov product is extended to $\widehat{G}$ by setting

$$(\alpha|\beta)_p^A = \sup\{\varliminf_{i,j\to+\infty} (g_i|h_j)_p^A \mid (g_n)_n \in \alpha,\ (h_n)_n \in \beta\}$$

for all $\alpha, \beta \in \widehat{G}$.

3 The visual metrics

Let $G = \langle A \rangle$ be a hyperbolic group. Assuming that $\Gamma_A(G)$ is δ-hyperbolic, let $\gamma > 0$ be such that $\gamma\delta \leq \ln 2$. Following Holopainen, Lang and Vähäkangas [8], we define

$$\rho_{p,\gamma}^A(g,h) = \begin{cases} e^{-\gamma(g|h)_p^A} & \text{if } g \neq h \\ 0 & \text{otherwise} \end{cases}$$

for all $p, g, h \in G$. In general, $\rho_{p,\gamma}^A$ fails to be a metric because of the triangular inequality. Let

$$\sigma_{p,\gamma}^A(g,h) = \inf\{\rho_{p,\gamma}^A(x_0,x_1) + \ldots + \rho_{p,\gamma}^A(x_{n-1},x_n) \mid n \geq 0,\ x_0 = g,\ x_n = h;\ x_1,\ldots,x_{n-1} \in G\}.$$

By [8] (cf. also [4, 18]), $\sigma_{p,\gamma}^A$ is a metric on G and the inequalities

$$\frac{1}{4}\rho_{p,\gamma}^A(g,h) \leq \sigma_{p,\gamma}^A(g,h) \leq \rho_{p,\gamma}^A(g,h) \tag{6}$$

hold for all $g, h \in G$.

The metric $\sigma_{p,\gamma}^A$ is an important example of a *visual metric*. Given $p \in G$, $\gamma > 0$ and $T \geq 1$, we denote by $V^A(p,\gamma,T)$ the set of all metrics d on G such that

$$\frac{1}{T}\rho_{p,\gamma}^A(g,h) \leq d(g,h) \leq T\rho_{p,\gamma}^A(g,h) \tag{7}$$

holds for all distinct $g, h \in G$. By (6), we have

$$\sigma_{p,\gamma}^A \in V^A(p, \gamma, 4)$$

whenever $\gamma\delta \leq \ln 2$. We shall refer to the metrics in some $V^A(p, \gamma, T)$ as the *visual metrics* on G.

Let $d \in V^A(p, \gamma, T)$ be a visual metric. In general, the metric space (G, d) is not complete. But its completion is essentially unique and also compact, and can be obtained by adding to G the elements of the boundary ∂G [2, 5, 8, 18]. We denote it by $(\widehat{G}, \widehat{d})$. It is well known that $\widehat{d}$ induces the *Gromov topology* on ∂G [2, Section III.H.3].

To understand the metric $\widehat{d}$, we must consider the extension of $\rho_{p,\gamma}^A$ to the boundary. We define

$$\hat{\rho}_{p,\gamma}^A(\alpha, \beta) = \begin{cases} e^{-\gamma(\alpha|\beta)_p^A} & \text{if } \alpha \neq \beta \\ 0 & \text{otherwise} \end{cases}$$

for all $\alpha, \beta \in \widehat{G}$. By continuity, the inequalities

$$\frac{1}{T}\hat{\rho}_{p,\gamma}^A(\alpha, \beta) \leq \widehat{d}(\alpha, \beta) \leq T\hat{\rho}_{p,\gamma}^A(\alpha, \beta) \tag{8}$$

hold for all $\alpha, \beta \in \widehat{G}$ [2, Section III.H.3].

It is widely known that uniform continuity of a mapping $\varphi : G \to G'$ of hyperbolic groups determines the existence of a continuous extension $\Phi : \widehat{G} \to \widehat{G'}$:

Lemma 3.1 *Let $\varphi : G \to G'$ be a mapping of hyperbolic groups and let d and d' be visual metrics on G and G' respectively. Then the following conditions are equivalent:*

(i) φ is uniformly continuous with respect to d and d';

(ii) φ admits a continuous extension $\Phi : (\widehat{G}, \widehat{d}) \to (\widehat{G'}, \widehat{d'})$.

Indeed, by a general topology result [3, Section XIV.6], every uniformly continuous mapping $\varphi : G \to G'$ admits a continuous extension to the completions.

On the other hand, the completion $(\widehat{G}, \widehat{d})$ is compact. Since every continuous mapping with compact domain is uniformly continuous, it follows that φ, being a restriction of a uniformly continuous extension, is itself uniformly continuous.

We note also that the continuous extension is uniquely defined through

$$[(g_n)_n]\Phi = [(g_n\varphi)_n],$$

for every Gromov sequence $(g_n)_n$ on G.

A group is *virtually free* if it has a free subgroup of finite index. Finitely generated virtually free groups constitute an important subclass of hyperbolic groups. We should mention that the second author developed in [17] a model for the boundary of such a group which allows a huge simplification with respect to the general case.

Lemma 3.2 *Let G be a hyperbolic group and let $d \in V^A(p, \gamma, T)$, $d' \in V^{A'}(p', \gamma', T')$ be visual metrics on G. Let φ be a mapping from (G, d) to (G, d') (respectively from $(\widehat{G}, \widehat{d})$ to $(\widehat{G}, \widehat{d'})$) be a mapping and let $P > 0$ and $Q \in \mathbb{R}$ be constants such that*

$$P(g\varphi|h\varphi)_{p'}^{A'} + Q \geq (g|h)_p^A \tag{9}$$

holds for all $g, h \in G$ (respectively $\widehat{G}$). Then φ satisfies a Hölder condition of exponent $\frac{\gamma'}{\gamma P}$.

Proof. Consider first the case $\varphi : (G, d) \to (G, d')$. Let $g, h \in G$. We may assume that $g\varphi \neq h\varphi$, hence

$$\begin{aligned} d'(g\varphi, h\varphi) &\leq T'\rho^{A'}_{p',\gamma'}(g\varphi, h\varphi) = T'e^{-\gamma'(g\varphi|h\varphi)^{A'}_{p'}} \leq T'e^{-\frac{\gamma'}{P}((g|h)^A_p - Q)} \\ &= T'e^{\frac{\gamma' Q}{P}}e^{-\frac{\gamma'}{P}(g|h)^A_p} = T'e^{\frac{\gamma' Q}{P}}(e^{-\gamma(g|h)^A_p})^{\frac{\gamma'}{\gamma P}} = T'e^{\frac{\gamma' Q}{P}}(\rho^A_{p,\gamma}(g,h))^{\frac{\gamma'}{\gamma P}} \\ &\leq T'e^{\frac{\gamma' Q}{P}}(Td(g,h))^{\frac{\gamma'}{\gamma P}} = T'e^{\frac{\gamma' Q}{P}}T^{\frac{\gamma'}{\gamma P}}(d(g,h))^{\frac{\gamma'}{\gamma P}} \end{aligned}$$

and we are done.

Replacing (7) by (8), we may use the same argument to prove the case $\varphi : (\widehat{G}, \widehat{d}) \to (\widehat{G}, \widehat{d'})$. $\square$

An endomorphism φ of a group G is *trivial* if $G\varphi = 1$. We show next that in the case of nontrivial endomorphisms, Hölder conditions with respect to visual metrics are equivalent to Lispschitz conditions involving the Gromov product.

Proposition 3.3 *Let G be a hyperbolic group and let $d = \sigma^A_{p,\gamma}$, $d' = \sigma^{A'}_{p',\gamma'}$ be visual metrics on G. Let $\varphi : (G, d) \to (G, d')$ be a nontrivial homomorphism and let $r > 0$. Then the following conditions are equivalent:*

(i) φ satisfies a Hölder condition of exponent r;

(ii) there exists a constant $Q \in \mathbb{R}$ such that

$$\frac{\gamma'}{r\gamma}(g\varphi|h\varphi)^{A'}_{p'} + Q \geq (g|h)^A_p \tag{10}$$

holds for all $g, h \in G$.

Proof. (i) $\Rightarrow$ (ii). There exists a constant $K > 0$ such that

$$d'(g\varphi, h\varphi) \leq K(d(g,h))^r$$

for all $g, h \in G$.

Assume first that $g\varphi \neq h\varphi$. Then $g \neq h$ and

$$\begin{aligned} e^{-\gamma'(g\varphi|h\varphi)^{A'}_{p'}} &= \rho^{A'}_{p',\gamma'}(g\varphi, h\varphi) \leq T'd'(g\varphi, h\varphi) \leq T'K(d(g,h))^r \leq T'K(T\rho^A_{p,\gamma}(g,h))^r \\ &\leq T'KT^r(\rho^A_{p,\gamma}(g,h))^r = T'KT^r e^{-r\gamma(g|h)^A_p}, \end{aligned}$$

hence

$$-\gamma'(g\varphi|h\varphi)^{A'}_{p'} \leq \ln(T'KT^r) - r\gamma(g|h)^A_p$$

and so

$$\frac{\gamma'}{r\gamma}(g\varphi|h\varphi)^{A'}_{p'} + \frac{\ln(T'KT^r)}{r\gamma} \geq (g|h)^A_p \tag{11}$$

holds whenever $g\varphi \neq h\varphi$.

Now, since φ is nontrivial, there exists some $a \in A$ such that $a\varphi \neq 1$. We may assume that $d_A(1, a\varphi)$ is minimal. We show that (10) holds for

$$Q = 1 + \frac{\ln(T'KT^r)}{r\gamma} + \frac{\gamma'}{r\gamma}d_{A'}(1, a\varphi).$$

In view of (11), we may assume that $g\varphi = h\varphi$. On the one hand, using (11), we have

$$\begin{aligned}(g\varphi|h\varphi)^{A'}_{p'} &= \tfrac{1}{2}(d_{A'}(p',g\varphi) + d_{A'}(p',h\varphi) - d_{A'}(g\varphi,h\varphi))\\ &\geq \tfrac{1}{2}(d_{A'}(p',g\varphi) + d_{A'}(p',(ha)\varphi) - d_{A'}(g\varphi,(ha)\varphi) - 2d_{A'}(h\varphi,(ha)\varphi))\\ &= (g\varphi|(ha)\varphi)^{A'}_{p'} - d_{A'}(1,a\varphi)\\ &\geq \tfrac{r\gamma}{\gamma'}(g|ha)^A_p - \tfrac{\ln(T'KT^r)}{\gamma'} - d_{A'}(1,a\varphi).\end{aligned}$$

On the other hand, $a\varphi \neq 1$ implies $a \neq 1$ and so

$$\begin{aligned}(g|ha)^A_p &= \tfrac{1}{2}(d_A(p,g) + d_A(p,ha) - d_A(g,ha))\\ &\geq \tfrac{1}{2}(d_A(p,g) + d_A(p,h) - d_A(g,h) - 2d_A(h,ha))\\ &= (g|h)^A_p - 1,\end{aligned}$$

hence

$$\begin{aligned}(g\varphi|h\varphi)^{A'}_{p'} &\geq \tfrac{r\gamma}{\gamma'}(g|ha)^A_p - \tfrac{\ln(T'KT^r)}{\gamma'} - d_{A'}(1,a\varphi) \geq \tfrac{r\gamma}{\gamma'}((g|h)^A_p - 1) - \tfrac{\ln(T'KT^r)}{\gamma'} - d_{A'}(1,a\varphi)\\ &= \tfrac{r\gamma}{\gamma'}((g|h)^A_p - Q)\end{aligned}$$

and so (10) holds as required.

(ii) ⇒ (i). By Lemma 3.2. □

The next technical lemma illustrates an easy way of producing quasigeodesics. The proof is straightforward routine and can be omitted. If (X,d) is a geodesic metric space and

$$x_0 \longrightarrow x_1 \longrightarrow \ldots \longrightarrow x_n \tag{12}$$

is a path in X such that each $x_{i-1} \longrightarrow x_i$ is a geodesic, then (12) induces a canonical mapping $\xi : [0,s] \to X$ such that $s = d(x_0,x_1) + \ldots + d(x_{n-1},x_n)$, $0\xi = x_0$ and $s\xi = x_n$.

Lemma 3.4 *Let (X,d) be a geodesic metric space and let $\xi : [0,s] \to X$ be the canonical mapping induced by*

$$x_0 \longrightarrow x_1 \longrightarrow \ldots \longrightarrow x_n,$$

where each $x_{i-1} \longrightarrow x_i$ is a geodesic. Let $P, L > 0$ and $Q \geq 0$ be such that

$$1 \leq d(x_{i-1},x_i) \leq L \tag{13}$$

and

$$Pd(x_i,x_j) + Q \geq |i-j| \tag{14}$$

for all $i,j \in \{1,\ldots,n\}$. Then ξ is a (λ,K)-quasigeodesic for

$$\lambda = \max\{1, LP\}, \quad K = \max\{2L, \frac{Q+1}{P} + L\}.$$

Two metrics d and d' on a set X are *Hölder equivalent* if the identity mappings $(X,d) \to (X,d')$ and $(X,d') \to (X,d)$ satisfy both a Hölder condition.

The following proposition is the finite version of the well-known analogue result on the equivalence of the visual metrics on the boundary (see [10, Theorem 2.18]).

Proposition 3.5 *All visual metrics on a given hyperbolic group are Hölder equivalent.*

Proof. Let G be a hyperbolic group and let d, d' be visual metrics on G. Let A, A' be finite generating sets of G and assume that $\overline{\Gamma}_A(G)$ (respectively $\overline{\Gamma}_{A'}(G)$) is δ-hyperbolic (respectively δ'-hyperbolic). Let $p, p' \in G$. In view of Proposition 3.3, it suffices to show that there exist constants $P > 0$ and $Q \geq 0$ such that

$$P(g|h)^{A'}_{p'} + Q \geq (g|h)^A_p \tag{15}$$

holds for all $g, h \in G$.

Let $N = N_{A,A'}$ be as in (3) and let $R = R(\delta, N^2, 2N)$ be the constant introduced in Section 2. Let $[g,h]_A$ and $[g,h]_{A'}$ be geodesics in $\overline{\Gamma}_A(G)$ and $\overline{\Gamma}_{A'}(G)$, respectively. We claim that

$$d_A(p, [g,h]_A) \leq N d_{A'}(p', [g,h]_{A'}) + N + d_A(p,p') + R. \tag{16}$$

Assume that $[g,h]_{A'}$ is the path

$$g = g_0 \xrightarrow{a'_1} g_1 \xrightarrow{a'_2} \ldots \xrightarrow{a'_n} g_n = h$$

with $a'_1, \ldots, a'_n \in \widetilde{A'}$. Consider geodesics $g_{i-1} \xrightarrow{u_i} g_i$ in $\overline{\Gamma}_A(G)$ and let $\xi : [0,s] \to (\overline{\Gamma}_A(G), d_A)$ be the canonical mapping induced by the path

$$g = g_0 \xrightarrow{u_1} g_1 \xrightarrow{u_2} \ldots \xrightarrow{u_n} g_n = h.$$

Then $1 \leq d_A(g_{i-1}, g_i) \leq N$ and in view of (4)

$$N d_A(g_i, g_j) \geq d_{A'}(g_i, g_j) = |i - j|$$

holds for all $i, j \in \{1, \ldots, n\}$. By Lemma 3.4, ξ is a $(N^2, 2N)$-quasigeodesic. Note that $0\xi = g$, $s\xi = h$ and $[g,h]_{A'} \cap G \subseteq \mathrm{Im}(\xi)$.

Now

$$\mathrm{Haus}([g,h]_A, \mathrm{Im}(\xi)) \leq R(\delta, N^2, 2N) = R,$$

hence

$$d_A(p, [g,h]_A) \leq d_A(p, \mathrm{Im}(\xi)) + R \leq d_A(p,p') + d_A(p', \mathrm{Im}(\xi)) + R. \tag{17}$$

On the other hand, we have

$$d_{A'}(p', [g,h]_{A'}) \geq d_{A'}(p', [g,h]_{A'} \cap G) - 1$$

and

$$d_A(p', \mathrm{Im}(\xi)) \leq d_A(p', \mathrm{Im}(\xi) \cap G) \leq d_A(p', [g,h]_{A'} \cap G)$$

follows from $[g,h]_{A'} \cap G \subseteq \mathrm{Im}(\xi)$. In view of (4), we get

$$d_{A'}(p', [g,h]_{A'}) \geq d_{A'}(p', [g,h]_{A'} \cap G) - 1 \geq \frac{1}{N} d_A(p', [g,h]_{A'} \cap G) - 1 \geq \frac{1}{N} d_A(p', \mathrm{Im}(\xi)) - 1.$$

Together with (17), this yields (16).

It follows easily from the hyperbolicity conditions (see also [18]) that

$$(g|h)^A_p \leq d_A(p, [g,h]_A) \leq (g|h)^A_p + 2\delta. \tag{18}$$

Together with (16), this yields

$$(g|h)^A_p \leq N d_{A'}(p', [g,h]_{A'}) + N + d_A(p,p') + R.$$

Applying (18) to $d_{A'}(p', [g,h]_{A'})$, we obtain

$$(g|h)^A_p \leq N(g|h)^{A'}_{p'} + 2N\delta' + N + d_A(p,p') + R,$$

hence (15) holds for $P = N$ and $Q = (2\delta' + 1)N + d_A(p,p') + R$. □

4 Endomorphisms of hyperbolic groups

An endomorphism φ of G is *virtually injective* if its kernel is finite. This is a necessary condition for uniform continuity:

Lemma 4.1 *Let G be a hyperbolic group endowed with a visual metric d. Let φ be a uniformly continuous nontrivial endomorphism of G. Then φ is virtually injective.*

Proof. Assume that $d \in V^A(p,\gamma,T)$. Fix $g \in G \setminus \mathrm{Ker}(\varphi)$. Let $\varepsilon = d(1, g\varphi) > 0$ and let $\delta > 0$ be such that

$$\forall x, y \in G \; (d(x,y) < \delta \Rightarrow d(x\varphi, y\varphi) < \varepsilon).$$

For every $h \in \mathrm{Ker}(\varphi)$, we have $d(h\varphi, (hg)\varphi) = d(1, g\varphi) = \varepsilon$, hence $d(h, hg) \geq \delta$. By (7), we get

$$e^{-\gamma(h|hg)_p} = \rho^A_{p,\gamma}(h, hg) \geq \frac{1}{T} d(h, hg) \geq \frac{\delta}{T}$$

and so

$$(h|hg)_p \leq -\frac{\ln \frac{\delta}{T}}{\gamma}.$$

It follows that

$$\begin{aligned} d_A(p,h) &\leq \tfrac{1}{2}(d_A(p,h) + d_A(p,hg) - d_A(h,hg) + 2d_A(h,hg)) = (h|hg)_p + d_A(h,hg) \\ &= (h|hg)_p + d_A(1,g) \leq -\tfrac{\ln \frac{\delta}{T}}{\gamma} + d_A(1,g). \end{aligned}$$

Since A is finite, then $\Gamma_A(G)$ is locally finite, i.e. every ball is finite. Therefore $\mathrm{Ker}(\varphi)$ is finite and φ is virtually injective. $\square$

We need also the following result, whose proof is a straightforward consequence of continuity.

Proposition 4.2 *Let φ be a nontrivial endomorphism of a hyperbolic group G with continuous extension $\Phi : \widehat{G} \to \widehat{G}$. Let d be a visual metric on G. Then the following conditions are equivalent:*

(i) φ satisfies a Hölder condition of exponent r with respect to d;

(ii) Φ satisfies a Hölder condition of exponent r with respect to $\widehat{d}$.

In the main result of the paper, we characterize the uniformly continuous endomorphisms which satisfy a Hölder condition:

Theorem 4.3 *Let φ be a nontrivial endomorphism of a hyperbolic group G and let $d \in V^A(p,\gamma,T)$ be a visual metric on G. Then the following conditions are equivalent:*

(i) φ satisfies a Hölder condition with respect to d;

(ii) φ admits an extension to $\widehat{G}$ satisfying a Hölder condition with respect to $\widehat{d}$;

(iii) there exist constants $P > 0$ and $Q \in \mathbb{R}$ such that

$$P(g\varphi|h\varphi)^A_p + Q \geq (g|h)^A_p \tag{19}$$

for all $g, h \in G$;

(iv) φ is a quasi-isometric embedding of (G, d_A) into itself;

(v) φ is virtually injective and $G\varphi$ is a quasiconvex subgroup of G.

Proof. (i) $\Leftrightarrow$ (ii). By Lemma 3.1 and Proposition 4.2.

(i) $\Leftrightarrow$ (iii). By Proposition 3.3.

We complete the proof by establishing the implications (i) $\Rightarrow$ (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii).

(i) $\Rightarrow$ (v). By Lemma 4.1, φ is virtually injective. In view of Proposition 3.5, we may assume that $p = 1$. Since (i) implies (iii), there exist constants $P > 0$ and $Q \in \mathbb{R}$ such that

$$P(g\varphi|g\varphi)_1^A + Q \geq (g|g)_1^A$$

for every $g \in G$, which is equivalent to

$$Pd_A(1, g\varphi) + Q \geq d_A(1, g).$$

Since $d_A(g, h) = d_A(1, g^{-1}h)$ and $d_A(g\varphi, h\varphi) = d_A(1, (g^{-1}h)\varphi)$, we immediately get

$$Pd_A(g\varphi, h\varphi) + Q \geq d_A(g, h) \tag{20}$$

for all $g, h \in G$.

Let

$$M_\varphi = \max\{d_A(1, a\varphi) \mid a \in A\}.$$

We show now that $G\varphi$ is quasiconvex.

Let $g, h \in G$ and let $g' \xrightarrow{w} h'$ have minimal length among all the paths in $\overline{\Gamma}_A(G)$ such that $g'\varphi = g\varphi$ and $h'\varphi = h\varphi$. In particular, $g' \xrightarrow{w} h'$ is a geodesic. Assume that $w = a_1 \ldots a_n$ with $a_i \in \widetilde{A}$. For $i = 0, \ldots, n$, write $w_i = a_1 \ldots a_i$ and let $(g'w_{i-1})\varphi \longrightarrow (g'w_i)\varphi$ be a geodesic. Let $\xi : [0, s] \to \overline{\Gamma}_A(G)$ be the canonical mapping induced by the path

$$g\varphi = g'\varphi = (g'w_0)\varphi \longrightarrow (g'w_1)\varphi \longrightarrow \ldots \longrightarrow (g'w_n)\varphi = h'\varphi = h\varphi.$$

Suppose that $a_i\varphi = 1$ for some i. Let $w' = w_{i-1}a_{i+1} \ldots a_n$. Since $w'\varphi = w\varphi$, we have $(g'w')\varphi = (g'w)\varphi = h'\varphi = h\varphi$, contradicting the minimality of w. Hence $a_i\varphi \neq 1$ for every i. Since $|a_i\varphi| \leq M_\varphi$, we get

$$1 \leq d_A((g'w_{i-1})\varphi, (g'w_i)\varphi) \leq M_\varphi.$$

Assume that $0 \leq i \leq j \leq n$. By (20), we have

$$Pd_A((g'w_i)\varphi, (g'w_j)\varphi) + Q \geq d_A(g'w_i, g'w_j) = d_A(1, a_{i+1} \ldots a_j).$$

Since $a_{i+1} \ldots a_j$ is a factor of a geodesic, it is itself a geodesic and so $d_A(1, a_{i+1} \ldots a_j) = j - i$. Thus

$$Pd_A((g'w_i)\varphi, (g'w_j)\varphi) + Q \geq |j - i|$$

and it follows from Lemma 3.4 that ξ is a (λ, K)-quasigeodesic for

$$\lambda = \max\{1, M_\varphi P\}, \quad K = \max\{2M_\varphi, \frac{Q+1}{P} + M_\varphi\}.$$

Let $R = R(\delta, \lambda, K)$. Let $[g\varphi, h\varphi]$ be a geodesic in $\overline{\Gamma}_A(G)$ and let $x \in [g\varphi, h\varphi]$. Then

$$\mathrm{Haus}([g\varphi, h\varphi], \mathrm{Im}(\xi)) \leq R$$

and every point in $\mathrm{Im}(\xi)$ is at distance at most M_φ from an element of $G\varphi$, hence

$$d_A(x, G\varphi) \leq R + M_\varphi$$

and so $G\varphi$ is $(R + M_\varphi)$-quasiconvex.

(v) $\Rightarrow$ (iv). Let $K = \mathrm{Ker}(\varphi) \trianglelefteq G$. Let $\pi : G \to G/K$ be the canonical projection and let $\iota : G\varphi \to G$ be inclusion. Then there exists an isomorphism $\overline{\varphi} : G/K \to G\varphi$ such that the diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G \\ \pi\downarrow & & \uparrow\iota \\ G/K & \xrightarrow[\overline{\varphi}]{} & G\varphi \end{array}$$

commutes. Since the composition of quasi-isometric embeddings is still a quasi-isometric embedding, it suffices to show that each one of the homomorphisms $\pi, \overline{\varphi}, \iota$ is a quasi-isometric embedding when we consider a geodesic metric in each of the groups (it does not matter which since the identity $(H, d_A) \to (H, d_B)$ is a quasi-isometry whenever $H = \langle A \rangle = \langle B \rangle$ by (4)).

It is well known that a quotient by a finite normal subgroup constitutes a quasi-isometry (it is a simple exercise to deduce it from the definitions).

Now $\overline{\varphi}$ is an isomorphism and $A\pi\overline{\varphi} = A\varphi$, hence

$$d_{A\varphi}(g\pi\overline{\varphi}, h\pi\overline{\varphi}) = d_{A\pi}(g\pi, h\pi)$$

for all $g, h \in G$ and so $\overline{\varphi} : (G/K, d_{A\pi}) \to (G\varphi, d_{A\varphi})$ is actually an isometry.

Finally, it is well known that the inclusion of a quasiconvex subgroup constitutes a quasi-isometric embedding [2, Section III.Γ.3]. Therefore all three homomorphisms $\pi, \overline{\varphi}, \iota$ are quasi-isometric embeddings and so is their composition φ.

(iv) $\Rightarrow$ (iii). Clearly, φ can be extended to a quasi-isometric embedding $\overline{\varphi}$ of $(\overline{\Gamma}_A(G), d_A)$ into itself. Assume that $\overline{\Gamma}_A(G)$ is δ-hyperbolic. Let $\lambda \geq 1$ and $K \geq 0$ be constants such that

$$\frac{1}{\lambda} d_A(x, y) - K \leq d_A(x\overline{\varphi}, y\overline{\varphi}) \leq \lambda d_A(x, y) + K \tag{21}$$

holds for all $x, y \in \overline{\Gamma}_A(G)$. Write $R = R(\delta, \lambda, K)$. We prove that

$$\lambda (g\varphi|h\varphi)^A_p + \delta + \lambda(\lambda\delta + \frac{3K}{2} + 3R + d_A(p, p\varphi)) \geq (g|h)^A_p \tag{22}$$

holds for all $g, h \in G$.

Let $g, h \in G$. Consider a geodesic triangle $[[p, g, h]]$ with geodesics $[p, g]$, $[g, h]$ and $[p, h]$. Let

$$X = \{x \in [g, h] : d_A(x, [p, g]) \leq \delta\}, \quad Y = \{y \in [g, h] : d_A(y, [p, h]) \leq \delta\}.$$

It is immediate that X and Y are both closed and nonempty. Since $X \cup Y = [g, h]$ is obviously connected, it follows that $X \cap Y \neq \emptyset$. Let $x \in X \cap Y$ and take $g' \in [p, g]$ and $h' \in [p, h]$ such that $d_A(x, g'), d_A(x, h') \leq \delta$.

If $\xi : [0,s] \to [p,g]$ is our geodesic, let $\xi' = \xi\overline{\varphi}$. For all $i, j \in [0,s]$, (21) yields

$$d_A(i\xi', j\xi') = d_A(i\xi\overline{\varphi}, j\xi\overline{\varphi}) \leq d_A(i\xi, j\xi) + K = \lambda|i-j| + K.$$

Similarly,

$$d_A(i\xi', j\xi') \geq \frac{1}{\lambda}|i-j| - K$$

and so ξ' is a (λ, K)-quasigeodesic from $[0,s]$ to $\overline{\Gamma}_A(G)$ such that $0\xi' = p\varphi$, $s\xi' = g\varphi$ and $g'\varphi \in \mathrm{Im}(\xi')$. Fix a geodesic $[p\varphi, g\varphi]$. Then $\mathrm{Haus}([p\varphi, g\varphi], \mathrm{Im}(\xi'))]) \leq R$, hence there exists some $g'' \in [p\varphi, g\varphi]$ such that $d_A(g'', g'\varphi) \leq R$. Similarly, fix geodesics $[p\varphi, h\varphi]$ and $[g\varphi, h\varphi]$. Then there exist some $h'' \in [p\varphi, h\varphi]$ and $x' \in [g\varphi, h\varphi]$ such that $d_A(h'', h'\varphi), d_A(x', x\varphi) \leq R$.

We claim that

$$d_A(g'', g\varphi) - d_A(g\varphi, x') \geq -\lambda\delta - K - 2R. \tag{23}$$

Indeed, in view of (21), we have

$$\begin{aligned} d_A(g'', g\varphi) - d_A(g\varphi, x') &\geq -d_A(g'', x') \geq -d_A(g'', g'\varphi) - d_A(g'\varphi, x\varphi) - d_A(x\varphi, x') \\ &\geq -\lambda d_A(g', x) - K - 2R \geq -\lambda\delta - K - 2R. \end{aligned}$$

Similarly,

$$d_A(h'', h\varphi) - d_A(h\varphi, x') \geq -\lambda\delta - K - 2R. \tag{24}$$

Now (21), (23) and (24) combined yield

$$\begin{aligned} (g\varphi|h\varphi)^A_{p\varphi} &= \tfrac{1}{2}(d_A(p\varphi, g\varphi) + d_A(p\varphi, h\varphi) - d_A(g\varphi, h\varphi)) \\ &= \tfrac{1}{2}(d_A(p\varphi, g'') + d_A(g'', g\varphi) + d_A(p\varphi, h'') + d_A(h'', h\varphi) - d_A(g\varphi, x') - d_A(x', h\varphi)) \\ &= \tfrac{1}{2}(d_A(p\varphi, g'') + d_A(p\varphi, h'') + d_A(g'', g\varphi) - d_A(g\varphi, x') + d_A(h'', h\varphi) - d_A(x', h\varphi)) \\ &\geq \tfrac{1}{2}(d_A(p\varphi, g'\varphi) + d_A(p\varphi, h'\varphi) + d_A(g'', g\varphi) - d_A(g\varphi, x') + d_A(h'', h\varphi) - d_A(x', h\varphi)) - R \\ &\geq \tfrac{1}{2}(d_A(p\varphi, g'\varphi) + d_A(p\varphi, h'\varphi)) - \lambda\delta - K - 3R \\ &\geq \tfrac{1}{2\lambda}(d_A(p, g') + d_A(p, h')) - \lambda\delta - \tfrac{3K}{2} - 3R. \end{aligned}$$

It follows that

$$\begin{aligned} (g\varphi|h\varphi)^A_p &= \tfrac{1}{2}(d_A(p, g\varphi) + d_A(p, h\varphi) - d_A(g\varphi, h\varphi)) \\ &\geq \tfrac{1}{2}(d_A(p\varphi, g\varphi) + d_A(p\varphi, h\varphi) - 2d_A(p, p\varphi) - d_A(g\varphi, h\varphi)) \\ &= (g\varphi|h\varphi)^A_{p\varphi} - d_A(p, p\varphi) \\ &\geq \tfrac{1}{2\lambda}(d_A(p, g') + d_A(p, h')) - \lambda\delta - \tfrac{3K}{2} - 3R - d_A(p, p\varphi). \end{aligned}$$

On the other hand,

$$\begin{aligned} (g|h)^A_p &= \tfrac{1}{2}(d_A(p, g) + d_A(p, h) - d_A(g, h)) \\ &= \tfrac{1}{2}(d_A(p, g') + d_A(g', g) + d_A(p, h') + d_A(h', h) - d_A(g, x) - d_A(x, h)) \\ &= \tfrac{1}{2}(d_A(p, g') + d_A(p, h') + d_A(g', g) - d_A(g, x) + d_A(h', h) - d_A(x, h)) \\ &\leq \tfrac{1}{2}(d_A(p, g') + d_A(p, h') + d_A(g', x) + d_A(h', x)) \\ &\leq \tfrac{1}{2}(d_A(p, g') + d_A(p, h')) + \delta \end{aligned}$$

and combining the previous ineqialities we get

$$\begin{aligned} (g\varphi|h\varphi)^A_p &\geq \tfrac{1}{2\lambda}(d_A(p, g'\varphi) + d_A(p, h'\varphi)) - \lambda\delta - \tfrac{3K}{2} - 3R - d_A(p, p\varphi) \\ &\geq \tfrac{1}{\lambda}(g|h)^A_p - \tfrac{\delta}{\lambda} - \lambda\delta - \tfrac{3K}{2} - 3R - d_A(p, p\varphi). \end{aligned}$$

Therefore (22) holds and we are done. □

5 Simplifications

Under which circumstances does every uniformly continuous endomorphism satisfy a Hölder condition? We have the following remark:

Lemma 5.1 *Let G be a hyperbolic group and let $d \in V^A(p,\gamma,T)$ be a visual metric on G. Then the following conditions are equivalent:*

(i) every uniformly continuous endomorphism of G satisfies a Hölder condition with respect to d;

(ii) $G\varphi$ is a quasiconvex subgroup of G for every endomorphism of G uniformly continuous with respect to d.

Proof. (i) $\Rightarrow$ (ii). Both conditions hold for the trivial endomorphism. For nontrivial endomorphisms we use Theorem 4.3.

(ii) $\Rightarrow$ (i). Let φ be a nontrivial endomorphism of G, uniformly continuous with respect to d. By Lemma 4.1, φ is virtually injective. Now we apply Theorem 4.3. □

We recall that a group G is said to be *co-hopfian* if every monomorphism of G is an automorphism.

We can improve Theorem 4.3 for virtually free groups and torsion-free co-hopfian hyperbolic groups. Examples in the later class have been provided by Rips and Sela [14] and Sela [15], namely:

- non-elementary torsion-free hyperbolic groups which admit no nontrivial cyclic splittings [14];
- non-elementary torsion-free freely indecomposable hyperbolic groups [15].

However, many torsion-free hyperbolic groups fail to be co-hopfian, such as infinite cyclic groups. For more interesting examples, see [12].

Corollary 5.2 *Let φ be a nontrivial endomorphism of a finitely generated virtually free or torsion-free co-hopfian hyperbolic group G and let $d \in V^A(p,\gamma,T)$ be a visual metric on G. Then the following conditions are also equivalent to the conditions of Theorem 4.3:*

(i) φ is uniformly continuous with respect to d;

(ii) φ admits a continuous extension to the completion $(\widehat{G},\widehat{d})$;

(iii) φ is virtually injective.

Proof. Assume first that G is torsion-free co-hopfian hyperbolic. Since G is torsion-free, every virtually injective endomorphism φ is a monomorphism. Since G is co-hopfian, φ is actually an isomorphism and so $G\varphi = G$ is trivially a quasi-convex subgroup of G. Thus condition (iii) is equivalent to the conditions in Theorem 4.3, which trivially imply condition (i).

Now (i) $\Leftrightarrow$ (ii) follows from Lemma 3.1 and (i) $\Rightarrow$ (iii) follows from Lemma 4.1.

Assume now that G is virtually free. By [1, Corollary 4.2], every finitely generated subgroup of a finitely generated virtually free group is quasiconvex, hence condition (i) is equivalent to the conditions in Theorem 4.3.

The equivalence (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) has been proved by the second author in [17], but can also be deduced directly from Lemmas 3.1 and 4.1. □

If φ is not an endomorphism, then Corollary 5.2 does not hold, even for an infinite cyclic group. For $x \in \mathbb{R}$, we denote by $\lfloor x \rfloor$ the greatest integer $n \leq x$.

Example 5.3 *Let d be the prefix metric on $F_{\{a\}}$ and let $\varphi : F_{\{a\}} \to F_{\{a\}}$ be defined by*

$$a^n\varphi = \begin{cases} a^{\lfloor \sqrt{n} \rfloor} & \text{if } n \geq 0 \\ a^n & \text{otherwise} \end{cases}$$

Then φ is uniformly continuous but satisfies no Hölder condition with respect to d.

Indeed, it is a simple exercise to show that

$$\forall \varepsilon > 0 \; \forall m, n \in \mathbb{Z} \; (d(a^m, a^n) < \min\{\frac{1}{2}, \frac{1}{2}\varepsilon^{2-\log_2 \varepsilon}\} \Rightarrow d(a^m\varphi, a^n\varphi) < \varepsilon),$$

hence φ is uniformly continuous with respect to d.

However, for all $r, K > 0$, we have

$$rm - \lfloor \sqrt{m} \rfloor > \log_2 K$$

for m large enough. It is easy to check that, for such m and $n > (\sqrt{m}+1)^2$, we have

$$d(a^m\varphi, a^n\varphi) > K(d(a^m, a^n))^r,$$

thus φ satisfies no Hölder condition with respect to d.

In the case of torsion-free hyperbolic groups, it would be enough of course to prevent the existence of monomorphisms with non quasiconvex image. The following example shows that such a situation cannot always be avoided.

Example 5.4 *There exists a torsion-free hyperbolic group G having a non quasiconvex subgroup isomorphic to G.*

Indeed, let a, b, t be distinct letters and write $A = \{a, b\}$ and $B = \{a, b, t\}$. We fix words $u = abab^2 \dots ab^{20}$ and $v = baba^2 \dots ba^{20}$. Let H be the group defined by the presentation

$$\langle B \mid t^{-1}atu, t^{-1}btv \rangle. \tag{25}$$

Let R denote the set of all cyclic conjugates of the two (cyclically reduced) relators and their inverses. A *piece* of (25) is a maximal common prefix of two distinct elements of R. It is easy to see that the longest pieces of (25) are $b^{18}ab^{19}$, $a^{18}ab^{19}$ and their inverses and have therefore length 38. On the oher hand, the length of each relator is $3 + 20 + \frac{20(20+1)}{2} = 233$. Since $38 < \frac{1}{6}233$, the presentation (25) satisfies the small cancellation condition $C'(\frac{1}{6})$. Now it follows from a theorem of Gromov [7] that H is hyperbolic.

Let F denote the subgroup of H generated by a, b. The subgroup K of F_A generated by u and v cannot have rank 1 since $uv \neq vu$ in F_A. Since F_A is hopfian, it follows that K is free on $\{u, v\}$. Hence H is an HNN extension of F_A and so there is a canonical isomorphism $F_A \to F$. Thus F is a free subgroup of H with basis A. Moreover, since the finite order elements of the HNN extension H must be conjugates of the finite order elements of F (see [13]), then H is torsion-free.

Since F is a normal subgroup of H, we have $t^{-n}at^n \in F$ for every $n \geq 0$. Consider the geodesic metrics d_A and d_B on F and H. We have

$$d_B(1, t^{-n}at^n) \leq 2n + 1 \tag{26}$$

for every $n \geq 0$. We prove that

$$d_A(1, t^{-n}at^n) = 230^n \tag{27}$$

by induction on n. The case $n = 0$ being trivial, assume that $n \geq 1$ and $d_A(1, t^{-(n-1)}at^{n-1}) = 230^{n-1}$. It follows that there exist $m = 230^{n-1}$ letters $c_1, \ldots, c_m \in \widetilde{A}$ such that $t^{-(n-1)}at^{n-1} = c_1 \ldots c_m$ in reduced form. Hence

$$t^{-n}at^n = (t^{-1}c_1t)\ldots(t^{-1}c_mt).$$

We have $t^{-1}c_it \in \{u, v, u^{-1}, v^{-1}\}$ for $i = 1, \ldots, m$. Moreover, since $u = a \ldots b$ and $v = b \ldots a$, and $c_1 \ldots c_m$ is reduced, there is no reduction between the reduced forms over $\widetilde{A}$ of two consecutive $t^{-1}c_it$. Hence the length of the reduced form of $t^{-n}at^n$ over $\widetilde{A}$ is $230m = 230^n$. Since F is free on A, we get (27).

Now it follows from (26) and (27) that the embedding $(F, d_A) \to (H, d_B)$ is not a quasi-isometric embedding and so F is not an *undistorted* subgroup of H. By a theorem of Short [16] (see also [2, Lemma Γ.3.5]), F is a non quasiconvex subgroup of H.

Consider now the free product $G = H * F$. Since it is well known that hyperbolic groups are closed under free product, G is hyperbolic. Moreover, being a free product of torsion-free groups, it is torsion-free as well. Let $K = \langle H \cup aHa^{-1} \rangle \leq G$ (where a comes from the second factor in $H * F$). It is easy to check that

$$K \cong H * H. \tag{28}$$

Indeed, we define a homomorphism $\varphi : H * H \to K$ by sending an element h from the first factor H into $h \in K$, and an element h from the second factor H into $aha^{-1} \in K$. This is clearly surjective, and injectivity follows from the free product normal form.

We consider now the sequence of embeddings

$$G = H * F \xrightarrow{\theta} H * H \xrightarrow{\varphi} K < H * F = G \tag{29}$$

where θ acts as the identity $H \to H$ with respect to the first factors and as the inclusion $F \to H$ for the second ones. Using indices 1 and 2 to distinguish generators from different free factors in the free products, we fix now the finite generating sets C and D for the G and $H * H$, respectively:

$$C = \{a_1, b_1, t_1, a_2, b_2\}, \quad D = \{a_1, b_1, t_1, a_2, b_2, t_2\}.$$

Let d_C and d_D denote the corresponding geodesic metrics on G and $H * H$, respectively.

For each $n \geq 0$, let w_n denote the (unique) reduced word over $\widetilde{\{a_2, b_2\}}$ representing the element $t_2^{-n}a_2t_2^n \in F$. It follows from (27) and the free product normal form that

$$d_C(w_n) = 230^n. \tag{30}$$

Now by (26) we have $d_D(w_n\theta) \leq 2n + 1$. We claim that

$$d_C(w_n\theta\varphi) \leq 6n + 3. \tag{31}$$

Indeed, each generator a_2, b_2, t_2 of $H * H$ is sent by φ into $a_2a_1a_2^{-1}, a_2b_1a_2^{-1}, a_2t_1a_2^{-1}$, respectively, and so $d_D(w_n\theta) \leq 2n + 1$ yields $d_C(w_n\theta\varphi) \leq 3(2n + 1) = 6n + 3$. Thus (31) holds. Together with (30), this implies that (29) is not a quasi-isometric embedding. By the aforementioned theorem of Short, G has a non quasiconvex subgroup isomorphic to itself.

If the embedding in such an example can be taken to be uniformly continuous with respect to some visual metric, we shall have proved that quasiconvexity cannot be removed from condition (v) in Theorem 4.3(v). But we have no answer yet.

6 Open problems

The main open problem left by this work relates to the possibility of removing quasiconvexity from condition (v) of Theorem 4.3.

Problem 6.1 *Does every uniformly continuous endomorphism of a hyperbolic group (with respect to a visual metric) satisfy a Hölder condition? If not, would it satisfy some other type of condition?*

It would be interesting to discuss replacing Hölder conditions by LIpschitz conditions:

Problem 6.2 *When does an automorphism of a hyperbolic (virtually free) group satisfy a Lipschitz condition?*

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