On using customer order time windows to optimize online retail inventory management

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Master's dissertation

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To my family

Resumo

O cenário de retalho online representa novos desafios, mas também uma oportunidade para as empresas aproveitarem as informações adicionais fornecidas e melhorarem as suas políticas de inventário. A diferença entre o momento em que uma encomenda é recebida e o momento em que o cliente deseja que ela seja entregue fornece um intervalo de tempo que permite flexibilidade adicional para o retalhista.

Este trabalho é baseado em teses anteriores, onde a política de inventário (s, Q) tem em conta explicitamente o intervalo de tempo entre o momento de encomenda e o momento de entrega. Neste trabalho, classificámos a procura em diferentes tipos de acordo com a política de inventário com revisão contínua e com revisão periódica, e encontrámos as expressões que caracterizam cada tipo de procura.

Apresentámos uma política de inventário (s, Q) que usa explicitamente a janela de tempo para minimizar os custos de encomenda, posse e de *stockout*. Considerando a flexibilidade proporcionada pelo intervalo de tempo entre o momento de encomenda e o momento de entrega, encontrámos expressões para os parâmetros ótimos da política. Considerámos que as chegadas das encomendas assim como o intervalo de tempo entre o momento de encomenda e o momento de entrega são estocásticos e explorámos diferentes distribuições de probabilidade. A política ótima é obtida considerando os cenários de uma ou múltiplas encomendas em trânsito.

Os testes realizados mostram que, quando comparada com a política ótima (s, Q) do retalho tradicional, a nossa política oferece poupanças que, em média, variam entre 11.91% e 28.50% ao usar nível de serviço α e entre 11.91% e 28.36% ao usar nível de serviço β.

Apresentámos também uma política de inventário (R, s, S) que usa explicitamente o intervalo de tempo e que tenta obter um determinado nível de serviço β e um determinado tempo entre reabastecimentos considerando cenários de uma ou múltiplas encomendas em trânsito. Considerámos que as chegadas das encomendas assim como o intervalo de tempo entre o momento de encomenda e o momento de entrega são estocásticos e explorámos diferentes distribuições de probabilidade.

Os testes realizados mostram que, quando comparada com a política (R, s, S) do retalho tradicional, a nossa política oferece reduções no nível médio do inventário físico que, em média, variam entre 38.31% e 68.32%.

Abstract

The online retail setting represents new challenges, but also an opportunity for companies to take advantage of the additional available information and improve their inventory policies. The difference between the time an order is received and the time at which the customer wants it delivered provides an order window that allows additional flexibility for the retailer.

This work is based in previous thesis, where the (s, Q) inventory policy explicitly accounts for the order window. In this work, we classify the demand in different types according to the inventory policy with continuous review and periodic review, and we find the expressions that characterize each type of demand.

We present an (s, Q) inventory policy that explicitly accounts for the ordering window when minimizing ordering, holding and stockout costs. Considering the flexibility provided by the ordering window we find expressions for the optimal parameters of the policy. We consider that the order arrival as well as the ordering window are stochastic and we explore different probability distributions. The optimal policy is obtained considering both single and multiple on-order scenarios.

Our experiments show that, when compared to the traditional retail (s, Q) optimal policy, our policy provides savings, that on average, range between 11.91% and 28.50% with α service level metrics and between 11.91% and 28.36% with β service level metrics.

We also present an (R, s, S) inventory policy that explicitly accounts for the ordering window and that tries to achieve a desired fill rate and time between replenishments considering both single and multiple on-order scenarios. We consider that the order arrival as well as the ordering window are stochastic and we explore different probability distributions.

Our experiments show that, when compared to the traditional retail (R, s, S) policy, our policy provides reductions in the average on-hand inventory that, on average, range between 38.31% and 68.32%.

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Contents

Acronyms

ADI	Advance demand information
AUSPRC	Target allowed (average) units short per replenishment cycle
B2B	Business to business
B2C	Business to consumer
CD	Customer delivery time
CO	Customer ordering time
CV	Coefficient of variance
EOQ	Economic order quantity
ESPRC	Expected shortage per replenishment cycle
ETRC	Expected total relevant costs
HC	Holding cost
IP	Inventory position
L	Replenishment lead time
MOO	Multiple on-order
OC	Ordering cost
OH	On-hand
OO	On-order
OW	Order window
Q	Order quantity
R	Review interval
s	Reorder point
SS	Safety Stock
SC	Stockout cost
SOO	Single on-order
TC	Total cost
UT	Unit time

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1 Introduction

E-commerce is the sale or purchase of goods or services through electronic means, such as the Internet. According to Gunasekaran et al. (2002), e-commerce is the process of conducting business electronically among various entities in order to satisfy an organizational or individual objective. It includes all aspects of trading, including commercial market creation, ordering, supply chain management and the transfer of money. Many e-commerce initiatives have risen in a short period of time including online retailing (e-tailing) that is the focus of this work.

With the increase of internet penetration in all regions of the globe, the e-commerce has risen considerably over the last years. From 2014 to 2015 the global e-commerce turnover growth was 19.9% with the Asian-Pacific region presenting the best result (Foundation 2016b).

The European e-commerce turnover managed to increase 13.3% from 2014 to 2015, a number much higher than the 1.0% growth of general retail in Europe, so nearly all growth in retail comes from e-commerce (Foundation 2016a).

In 2016, 57% of European Internet users shop online, but only 16% of Small and Medium Enterprises sell online and only 7.5% of those sell online across borders. In Portugal, 70% of the population uses the Internet and 35% shops online. Although European e-commerce turnover has been growing steadily over the years and it is expected that it will continue to grow, there are still many opportunities for improvement (Foundation 2016a).

Globally, retailers in every sector are improving the ability of meeting customers' expectations with improved supply chains and inventory management policies. To meet customers' new expectations, online retail offers clear advantages in terms of choice, ease of search, shopping cross-category and even buying cross-border, however, most customers prefer to have both a physical and an online experience so they expect an omni-channel approach (Foundation 2016a). This brings additional complexity for retailers as they need to manage their operations holistically.

This work takes on B2C online retailing that, at present, seems still to be of lower volume than B2B, but this may change in the future (Gunasekaran et al. 2002). The motivation for this work arises from the grocery sector where it is common for customers not choosing to receive the products as soon as possible, which is rather different from other sectors in which the presence of the customer is not critical upon the delivery of goods. It's known that the online retail in the grocery sector is small in comparison with other sectors, as it's shown in the survey "Total Retail 2016" made by PwC, where 72% of the sample still preferred to make grocery purchases in store, while just 20% preferred to do it online (PwC 2016). According to the study "Online Shopping Customer Experience" made by UPS, specially for valuable or important purchases, an important shipping service for customers is the possibility to request a particular delivery window (UPS 2012). The option of choosing the delivery date is also appearing in other sectors, and a good example is the Dell's online Intelligent Fulfillment initiative that allows the customer to choose a specific date for delivery (Gunasekaran et al. 2002). Grocery stores operate on very thin margins, so a good policy of inventory management is essential to be successful in the online grocery business.

In an omni-channel approach some retailers use dedicated warehouses (darkstores) completely dedicated to fulfill online demand for products with high turnover and low perishability, and support stores to fulfill other ones. The expanded operational capacity, the improved picking and delivery productivity, and the increased customer service levels leveraged by the higher stock availability are the direct advantages of this approach (Espinós 2015).

This work considers a scenario where a customer places an order online and schedules a date for the delivery. The difference between the time the order is received and the time of delivery provides an order window that gives additional flexibility for the retailer and we try to explore an inventory policy that uses this feature to minimize the retailer's expected costs due to ordering, holding and stockout costs.

This policy implies that the company only commits to a customer if there is product available in the network to satisfy his order, which means that even if the retailer does not have enough on-hand stock he can commit to a customer order if he has in-transit orders that will arrive before the date of delivery. Stockout will then represent cases where the retailer cannot commit to a customer order. Notice that, in our setting, retailers commit uppon arrival of orders and they will be able to satisfy the customer request when the order is due, which is different from the usual concept of advance demand information (ADI) where orders are accepted but some may need to be backordered (or cancelled) when the order is due. ADI research motivation comes, specially, from the B2B setting, by motivating the members of the supply chain to share information among them. In this setting there exists a distinction on when the order is announced and when the order is due. Our motivation arises from the B2C, specifically from the grocery sector, where there is no such distinction. The retailer with ADI normally has information about announced orders and an estimated due-date but the customer may change his order and the due-date may change (see, for example, the work of Gayon et al. (2009) or the work of Benbitour and Sahin (2015)), but in our case we have firm orders from customers with a fixed due-date because a customer chooses the delivery date when he is available to receive the goods.

The division of the demand according to its urgency is then essential to take advantage of the flexibility provided by the order window, but it also implies that the delivery date should be linked to product availability either by choosing the date previously, or with warnings that show when the product is available. Some grocery retailers already have delivery time windows for different zip codes with dynamic pricing policies (Asdemir et al. 2009). Here the same may be applied but linked to products availability.

To the best of our knowledge, the previous work done by Espinós (2015) and Soler (2017) were the first time that such characteristic was incorporated in the e-commerce inventory management and we pretend to extend this line of research. This study is based in previous work done by Espinós (2015) and Soler (2017) where an (s, Q) inventory policy explicitly accounts for the ordering window and was explored in single on-order (SOO) and multiple on-order (MOO) scenarios. The policy considers that the customer demand as well as the customer order window are stochastic and the replenishments to the darkstore have a short and deterministic lead time.

This study has three main objectives. Firstly, a characterization of types of demand that are related to the online retail considering a customer order window for different distributions of the order window is provided. Secondly, a revision and refinement of the policy for the (s, Q) continuous system is performed. Finally, a model for periodic review (R, s, S) policy is developed and assessed.

The remainder of the thesis is organized as follows: Chapter 2 presents a literature review on inventory management in online retail, presents a suggested optimal (s, Q) inventory management policy for the traditional retail and an approach to the traditional retail periodic review, reorder point, order-up-to level (R, s, S) policy.

Chapter 3 reviews the previous work done by Espinós (2015) and Soler (2017).

Chapter 4 presents the approach followed to develop the model for continuous review, which includes the division and classification of the demand types during the lead time and their characterization. We also explain the simulations done to validate the expressions obtained. The expressions for the optimal parameters of the policy are presented. Finally, numerical applications of the policy are presented by analyzing computational experiments in which the advantages of this policy in comparison to a traditional (s, Q) policy are assessed, both with α service level and β service level.

Chapter 5 presents the approach followed to develop the model for periodic review, which includes a different classification of the demand types and their characterization. We also explain the simulations done to validate the expressions obtained. Then we present a numerical study to assess the advantages of this policy in comparison to a traditional (R, s, S) policy.

Finally, Chapter 6 summarizes the results/findings and contributions of this study, the limitations that we found and proposes several future research directions.

2 Literature review

The literature review will be divided in three main sections. First, we review three research streams: inventory management problems and suggested policies in single-echelon systems, demand classification in inventory management and advance demand information in the online retail context. Secondly, an approach for an optimal continuous review (s, Q) inventory policy for traditional retail is presented and explained. Finally, a traditional retail (R, s, S) periodic review, reorder point, order-up-to level policy is presented and explained.

2.1 Inventory management in online retail

The growing of e-commerce has led to substantial several modifications in different fields. Modifications concerning supply chain management procedures represent one of the focal points of interest (Keskinocak and Tayur 2001). The first research stream related to our work is inventory management problems and suggested policies in single-echelon systems. A study done by Hovelaque et al. (2007) aims at examining whether inventory location and ordering policies can affect supply chain efficiency when both online and physical channels coexist. Their perspective is that the optimization of costs involves not only transport optimization, but also, on a larger scale, inventory policy and the management of product flows throughout the entire supply chain. They propose a model to study and compare the efficiency of three main organizational models identified by them and that are currently implemented: “store-picking”, “warehouse-picking” and “drop-shipping”. Their model for order policies is a traditional newsboy-based approach which takes the main features of the three types of organization into consideration. Hovelaque et al. (2007) present these organizational models in the following way:

- “Store-picking” consists in satisfying an online order off the shelf in the closest shop to the customer’s home. All the decisions about inventory management are made by the retailer because he needs to place his own orders to the suppliers according to the expected demands from both online and traditional customers, so the supplier only receives one aggregated order. The customer places the order online and the information is sent to the store closest to his home, where the products are picked from the shelves and packaged to be delivered to the customer.
- “Warehouse-picking” dedicates a warehouse reserved for online orders. The orders are prepared in these warehouses and shipped out to the customers. The need for a dedicated warehouse can lead to higher fixed costs and, because orders are centralized, the delivery times may be longer. In this organizational model, the supplier receives two types of orders: from traditional channel and from online channel.
- “Drop-shipping” is a model where the supplier receives the online orders from customers and manages, on its own, the stock and the orders delivery. The retailer receives the online order and sends it directly to the supplier who decides on the quantities to be manufactured. With this model, the responsibility for any over-stocking falls entirely on the supplier.

They conclude that a drop-shipping organizational model is always better for the retailer and, conversely, always less advantageous for the supplier because the entire risk burden of shortages and over-stocking falls on the supplier solely.

Online retailing is growing fast and to remain competitive retailers are searching for ways to reduce the fulfillment costs. A robust inventory management policy that seeks to minimize the costs is critical. According to Xu et al. (2017), the literature on inventory policy of online retailing mainly considers two kinds of online retailers, pure-play online retailers and retailers operating in a multi-channel or dual-channel environment. For what concerns studies about the pure-play online retailers, they refer the studies of Chen et al. (2005) were they analyze the strategy of maintaining an online retailer's own inventory, the study of Allgor et al. (2004) extending the use of a traditional two-stage serial inventory system in a setting of online retailing and the studies of Xu (2005) and Acimovic (2012) where they try to minimize the transportation costs or lower outbound shipping costs by designing an inventory allocation policy in the first study or by designing a replenishment policy in the second study.

In this work we consider a pure-play online retailer and an organizational model similar to the "Warehouse picking" as we consider a darkstore completely dedicated to fulfill online demand, so although the darkstore may arise from a retailer that already has physical stores, we do not consider them in this study. We classify our problem as a single-echelon inventory control problem because we focus on determining the appropriate level of inventory for an individual unit (the darkstore) within the supply chain network. Ekanayake et al. (2016) present a review of the recent literature and a comparison of single-echelon and multi-echelon systems. The results of their research suggest that single-echelon optimisation produces optimal average inventory and fill rate values for individual nodes, but non-optimal values for the entire supply chain network as a whole, and that multi-echelon systems provide the optimal inventory levels for the entire supply chain network altough some units compromise on their individual performance levels in the best interest of the entire network.

In the online retail, there is a difference between when an order is requested and when that order is depleted from inventory to be delivered. This means that the retailer knows the demand before having to fulfill it, and this allows the retailer to adapt its actions according to the known future demand. This order window offers a lot of opportunities of research.

The second research stream related to this work is demand classification in inventory management. Being able to take advantage of the flexibility provided by the order window is essential and, to do so, we divide the demand during the lead time in different classes allowing the policy to take advantage of this scenario. Some literature has similar approaches by classifying the demand in multiple classes. A good categorization of the existing literature about demand classification is given by Arslan et al. (2007). However, most of the existing literature is very different from this work as they classify the demand according to shipping costs, priority levels or channels of distribution. We classify the demand from online customers according to the time they placed an order and the time they want it to be delivered. Demand is classified only when an order is received and according to its order window.

Acimovic (2012) states this time window can provide a significant benefit to the online retailer and presents 3 possibilities to take advantage of it:

1) Calculate optimal future strategies: The retailer has time to make calculations that cannot be made in real time and that can help making better choices regarding inventory management.
2) Wait for inventory it knows is in-transit to arrive: If inventory will arrive soon, the online retailer can delay the shipment until then.
3) Move items between fulfillment centers: Use the time window to move orders to others fulfillment centers and ship them to customers from there by taking advantage of different transport costs.

This work studies the benefit associated with this time window based in the second possibility, as the policy we present considers inventory that is in-transit to commit to customers' orders.

The other stream of research that is related to this work is research about advance demand information (ADI). There are several papers that investigate the value of advance demand information and its interactions with inventories. Research in this stream can be classified based on the way the underlying supply system is modelled. Supply systems with endogenously determined lead times (production/inventory systems) behave differently than supply systems with exogenous lead times (pure inventory systems) (Karaesmen et al. 2004). According to this classification, this work can be classified as a pure inventory system research.

In the literature about ADI there is a similar definition to our definition of order window, that is called "demand lead time". A portfolio of customers with different demand lead times results in what is called ADI. This definition was first introduced by Hariharan and Zipkin (1995) in a study where they consider an inventory scenario where customers provide advance warning of their demands and they concluded that the effect of using ADI is equivalent to reducing the supply lead time and it can reduce safety stock levels and costs significantly when used effectively.

ADI was widely studied since then and it can be classified in "Perfect ADI" and "Imperfect ADI". Under "Perfect ADI", the supplier receives reliable information about customer demand and consequently the orders due-date or quantity do not change over time. Under "Imperfect ADI", the order information is uncertain and can change before the due-date of the order. Probably this is the most common scenario, as a customer initially gives information about a possible order but then may want to update the order. DeCroix and Mookerjee (1997) study the question of when it is optimal to acquire costly advance information about stochastic demand. They analyze a periodic review system where the supplier has the option to purchase ADI. They present the optimal information purchase policy and the value of dynamically purchasing ADI. Gallego and Özer (2001) explore the value of ADI by studying optimal replenishment policies for a single-stage periodic review *(s,S)* inventory system with ADI with and without fixed costs. They consider the case of "Perfect ADI" and their results show that ADI can provide important cost reductions. The same authors extend the model to the case of multi-stage systems (Gallego and Özer 2003). Tan et al. (2007) study the effect of "Imperfect ADI" in a multi-period scenario, and they show that the optimal ordering policy is a function of the amount of ADI obtained. They find that "Imperfect ADI" becomes most beneficial for lower levels of imperfectness of ADI and higher variability in demand. They also study the impact "Imperfect ADI" has in the ordering and rationing decisions, because it allows to make better decisions on when to start rejecting lower class demand (Tan et al. 2009). Benbitour and Sahin (2015) consider four types of ADI in a single-stage system, where the customer order quantity is a random variable that follows a normal distribution, to evaluate the impact of the "Imperfect ADI" on the performance of the system. They concluded the same as Tan et al. (2007): imperfectness of demand information reduces the benefits of ADI. They also found that imperfect due-dates deteriorate the system's performance more than imperfect demand quantities. They believe this result is due to the phenomenon of "Cross demands". A recent study about "Imperfect ADI" was done by Huang and Van Mieghem (2014) where they analyze how the recent clickstream tracking technology can be used to improve operational decisions and inventory management by estimating the converted demands in the future.

From our point of view, the literature that is closer to our work are the papers from Hariharan and Zipkin (1995) and Gallego and Özer (2001). The first one by considering that customers provide advanced warnings of their demands and using that information to obtain an optimal policy, and the second because their approach is similar to the approach in this work as they divide demand in two parts: the observed part, referring to the known demands (observed before the current period and that will prevail in a future period) and the unobserved part (unknown demands that have not been observed yet).

This work is different from the work done by Hariharan and Zipkin (1995) because they don't consider ordering costs in their policy which simplifies it, as they would order whenever they want without costs. The approach to the problem is quite different as their policy considers the demand in a period equal to the supply lead time reduced by the demand lead time, while we divide the demand in different types in order to establish the new policy. It is also different from the work done by Gallego and Özer (2001) because they aggregate the known demand from previous periods, while we divide the predicted demand of one product during the lead time in different types of demand, and they commit to all orders but may need to backorder some of them. We believe the novelty of our approach is that, in our policy, the retailer only commits to a customer if there is product available to satisfy his order (on-hand or in-transit) when the order is received. Thus, instead of allowing backorders, we have lost sales, which means that we will not commit to a customer order that arrives after another one but is to be delivered before the first one, if we already were not committed to the first customer. See Figure 1 for a better understanding. In this example, we will not commit to customer Y order if we already were not committed to customer X order.

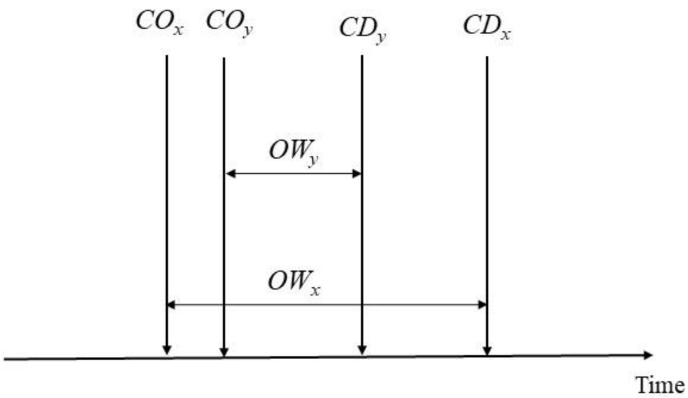

Figure 1 - Commitment uppon arrival (CO - Customer order; CD – Delivery date, OW – Order window)

Resuming, our concept of committing uppon arrival of an order is what distinguishes our work from the previous literature about ADI where orders are accepted but some may need to be backordered (or cancelled) when the order is due. As concluded by Benbitour and Sahin (2015), the imperfect due-dates deteriorate the system's performance due to the phenomenon of "Cross demands" that are not allowed in our policy as we explained above.

2.2 Traditional retail (s, Q) policy

The policy we present in this section was proposed by Silver et al. (1998) to obtain the optimal (s, Q) values in the traditional retail context. The policy they present is interesting for faster-moving items, α service level metrics and normally distributed demand during the lead time. In a continuous review system (s, Q) a replenishment action can be taken immediately after any demand transaction, however only after a replenishment lead time L, the order is available for satisfying customer demands. Therefore, an order should be placed when the available inventory is adequate to avoid a stockout over the replenishment lead time L. The inventory is continuously monitored and when it reaches a certain quantity s, called the reorder point, a replenishment of size Q is triggered. The policy they presented was built on the following assumptions:

1) The demand is stationary.
2) All demand transactions are unit size. A replenishment of size Q is triggered when the inventory position reaches exactly s.
3) Crossing of orders are not permitted, the lead time is assumed constant.
4) The average level of backorders is negligibly small when compared to the average level of on-hand stock because the unit shortage costs are assumed to be very high.
5) Forecast errors have a normal distribution with average zero and standard deviation σ_L over a lead time L.
6) In the iterative procedures, the value of Q is assumed to have been predetermined.
7) The cost of the control system does not depend on the specific value of s selected.

The notation used was:

A	Ordering cost, in €/replenishment
B_1	Stockout cost, in €/stockout
D	Demand per year, in units/year
k	Safety factor
L	Replenishment lead time in years
$P_{u\geq}(k)$	Probability that a unit normal (mean 0, standard derivation 1) variable takes on a value of k or higher
Q	Order quantity, in units
r	Inventory carrying charge, in €/€/year
s	Reorder point, in units
SS	Safety stock, in units
v	Unit variable cost, in €/unit
$\hat{x}_L$	Forecast demand over a replenishment lead time, in units
σ_L	Standard deviation of errors of forecast over a replenishment lead time, in units

In this policy, the reorder point is calculated according to equation (2.1) rounded to the nearest integer:

$$s = \hat{x}_L + k\sigma_L \tag{2.1}$$

The expected total relevant costs (ETRC) of the policy is the sum of ordering costs (OC), holding costs (HC) and stockout costs (SC) and is expressed in equation (2.2):

$$ETRC(s,Q) = OC + HC + SC = A\frac{D}{Q} + \left[\frac{Q}{2} + \int_0^s (s - x_0) f_x(x_0)dx_0\right] vr + B_1 \frac{D}{Q}\int_s^\infty f_x(x_0)dx_0 \tag{2.2}$$

The expression for the total cost approximated for the case where the lead time demand follows a normal distribution is:

$$ETRC\ (k,Q) = A\frac{D}{Q} + \left[\frac{Q}{2} + k\sigma_L\right] vr + B_1 \frac{D}{Q} p_{u\geq(k)} \tag{2.3}$$

To find the values of k and Q that minimize the total cost, the partial derivates are set equal to zero and the results are:

$$\frac{\delta ETRC\ (k,Q)}{\delta k} = 0 \rightarrow k = \sqrt{2\,ln\left(\frac{DB_1}{\sqrt{2\pi}Qvr\sigma_L}\right)} \tag{2.4}$$

$$\frac{\delta ETRC\ (k,Q)}{\delta Q} = 0 \rightarrow Q = EOQ\sqrt{1+\frac{B_1}{A}\mathrm{P}_{u\geq}(k)} \tag{2.5}$$

Where:

$$EOQ = \sqrt{\frac{2AD}{vr}} \tag{2.6}$$

The calculation of the parameters k and Q depend on each other so, for the simultaneous determination of k and Q, an iterative procedure is suggested by Silver et al. (1998). They suggest starting with $Q = EOQ$ (this value is the quantity that minimizes the sum OC+HC), then using this value in equation (2.4) to find the corresponding k and finally use k in equation (2.5) to find a new Q and so forth. This procedure is repeated until the values of k and Q do not change significantly and because of the convex nature of the functions involved, convergence to the optimal pair (k and Q) is ensured.

2.3 Traditional retail (R, s, S) policy

In a (R, S) system an order of variable size is placed every review interval (R) to raise the inventory level to the order-up-to level (S). According to Silver et al. (1998), a situation/policy where a periodic review system (R, S) is used is equivalent to the (s, Q) system if the following transformations are made:

Table 1 - Transformations for equivalency between (s, Q) and (R, S) policies

(s, Q)	(R, S)
s	S
Q	DR
L	$R + L$

When using this type of systems two additional assumptions were made by Silver et al. (1998):

1) A replenishment order is placed every review, hence there is a negligible chance of no demand between reviews.
2) The value of R is assumed to be predetermined.

A more complex system is the periodic review, reorder point, order-up-to level policy (R, s, S) where the inventory is examined every R units of time and, if the inventory position is at or below the reorder point s, then a replenishment, that raises it to the order-up-to-level S, is placed. The work done by Silver et al. (2009) allows us to think of this systems in terms of a continuous review system with "effective" lead time $L + \tau$ (see Figure 2), where the random variable τ is the time between the instant when the inventory position drops to the reorder point and the next review.

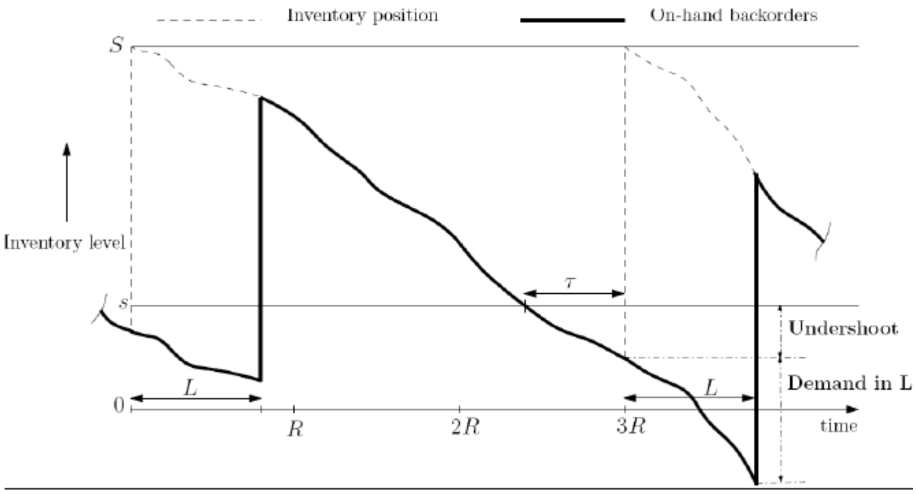

Figure 2 - Periodic review, reorder point, order-up-to level system (Silver et al. 2009)

The assumptions made by Silver et al. (2009) in their work are:

1) Every R units of time the inventory position is reviewed. R is prespecified and not controllable.
2) The replenishment lead time L, from when a replenishment is triggered until it is available in stock, is constant.
3) Demands in disjoint intervals of time are independent, stationary, and normally distributed variables.
4) In a stockout situation, there is complete backordering of any demand.
5) The service measure used is the fill rate, the fraction of demand to be satisfied from stock.
6) Instead of explicitly incorporating setup and carrying costs, a target average time between consecutive replenishments is specified.

For convenience, they use $R = 1$, which means that the review interval is redefined as unit time (UT). Some additional notation was used (time variables are defined in units of R):

P_2	Desired value of the fill rate
n	Desired average number of review intervals between consecutive replenishments (integer)
S	Order-up-to level, in units
μ	Average demand in a unit time interval, in units
σ	Standard deviation of demand in a unit time interval, in units
CV	$\frac{\sigma}{\mu}$ is the coefficient of variation of demand in a unit time interval
τ	Random variable representing the time from when the inventory position hits s until the next review instant
$f_\tau(\tau_0)$	Probability density function of τ
X	Total demand in $L + \tau$, in units
σ_X	Standard deviation of X, in units

Silver et al. (2009) present an approach in which they model this system as a continuous review, reorder point system. In their approach the "effective" lead time $L + \tau$ is a random variable with mean $\mathrm{E}[L + \tau]$ and variance $\mathrm{Var}[\tau]$.

They obtain the density function of τ, equation (2.7). Because it is not possible to analytically develop expressions for the moments of τ, they present expressions to obtain $\mathrm{E}[\tau]$ and $\mathrm{Var}[\tau]$ for values of CV between 0.1 and 0.5 and values of *n* between 2 and 6 (Table 2 and Table 3).

$$f_\tau(\tau_0) = \frac{m}{CV\sqrt{2\pi}} \sum_{i=1}^{\infty} \frac{1}{\sqrt{(i-\tau_0)^3}} \exp\left[-\frac{(m+\tau_0-i)^2}{2(CV)^2(i-\tau_0)}\right] \quad 0 < \tau_0 < 1 \tag{2.7}$$

Where:

$$m = \frac{S-s}{\mu} \tag{2.8}$$

Table 2 - Fractional polynomial approximations of $\mathrm{E}[\tau]$ as a function of CV for selected values of n (Silver et al. 2009)

$\boldsymbol{n}$	Approximation (note $\mathbf{C \equiv CV}$)
2	$0.53608 + 0.44271\left(\frac{1}{C} - 3.333\right) + 1.7634\left(\frac{1}{\sqrt{C}} - 1.826\right) + 1.0508\left(\frac{ln(C)}{\sqrt{C}} + 2.198\right)$
3	$0.51211 + 1.8625(C - 0.3) - 1.143(C^2 - 0.09) + 3.1367\,(C^2\,ln(C) + 0.1084)$
4	$0.50325 + 2.1455(\sqrt{C} - 0.5477) - 0.65943(\sqrt{C}\,ln(C) + 0.6594) + 0.50973(\sqrt{C}\,ln(C)^2 - 0.794)$
5	$0.50079 - 0.13438\left(\frac{1}{C} - 3.333\right) - 0.39946\left(\frac{1}{\sqrt{C}} - 1.826\right) - 0.28188\left(\frac{ln(C)}{\sqrt{C}} + 2.198\right)$
6	$0.50004 - 0.00237\left(\frac{1}{C^2}\ 11.11\right) - 0.03307\left(\frac{1}{C} - 3.333\right) - 0.02296\left(\frac{ln(C)}{C} + 4.013\right)$

Table 3 - Fractional polynomial approximations of $\mathrm{Var}[\tau]$ as a function of CV for selected values of n (Silver et al. 2009)

$\boldsymbol{n}$	Approximation (note $\mathbf{C \equiv CV}$)
2	$0.07401 + 0.5338(C - 0.3) - 0.58217(C^2 - 0.09)$
3	$0.08283 + 0.36794\,(\sqrt{C} - 0.5477) - 0.35809(\sqrt{C}\,ln(C) + 0.6594)$
4	$0.08387 - 0.12888\left(\frac{1}{\sqrt{C}} - 1.826\right) - 0.10939(ln(C) + 1.204)$
5	$0.08371 + 0.01876\,\left(\frac{1}{C} - 3.333\right) + 0.00887\left(\frac{ln(C)}{C} + 4.013\right)$
6	$0.08352 - 0.00078\left(\frac{1}{C^2} - 11.11\right) + 0.00503\left(\frac{1}{C} - 3.333\right)$

The procedure they presented for calculating the reorder point and the order-up-to level is:

1) Obtain E[τ] and Var[τ] from Table 2 and Table 3
2) Calculate the mean and variance of X

$$E[X] = (E[\tau] + L)\mu \tag{2.9}$$

$$Var[X] = (E[\tau] + L)\sigma^2 + \mu^2 Var[\tau] \tag{2.10}$$

3) Calculate the target allowed (average) units short per replenishment cycle (AUSPRC)

$$AUSPRC = (1 - P_2)E[Q] \tag{2.11}$$

4) Choose k to satisfy

$$G_u(k) = \frac{(1 - P_2)E[Q]}{\sigma_X} = \frac{(1 - P_2)n\mu}{\sigma_X} \tag{2.12}$$

5) Calculate the reorder point

$$s = E[X] + \; k\sigma_X \tag{2.13}$$

6) Calculate the order-up-to level

$$S = s + n\mu - E[\tau]\mu \tag{2.14}$$

They use this procedure for given values of CV, μ, n, L and P_2. We will adapt this approach to the online retail scenario.

3 Review of the previous work

This chapter presents a brief review of the work done by Espinós (2015) and Soler (2017) that precedes the work done in this thesis. First, the division of demand proposed by Espinós (2015) is presented and reviewed. Then the policy is presented for the SOO scenario and for the MOO scenario, and the findings made by these authors are explained.

3.1 Demand context in online retail for a continuous (s, Q) review policy

In online retail the demand during the lead time is more complex than in traditional retail so, in the previous work (Espinós 2015), the demand during the lead time was divided in three types taking in account four moments:

- The time at which the customer places the order (*CO*);
- The time at which the customer wants the delivery (*CD*);
- The time at which the retailer orders a replenishment (*O*);
- The time at which the order arrives (*A*).

Figure 3 helps to understand how to classify the demand. The time between the replenishment and the order arrival is represented by L and it's the supplier's lead time, during which customers' orders arrive. The order window is defined as the time between the customer order (*CO*) and the delivery time (*CD*), that is, $OW = CD - CO$.

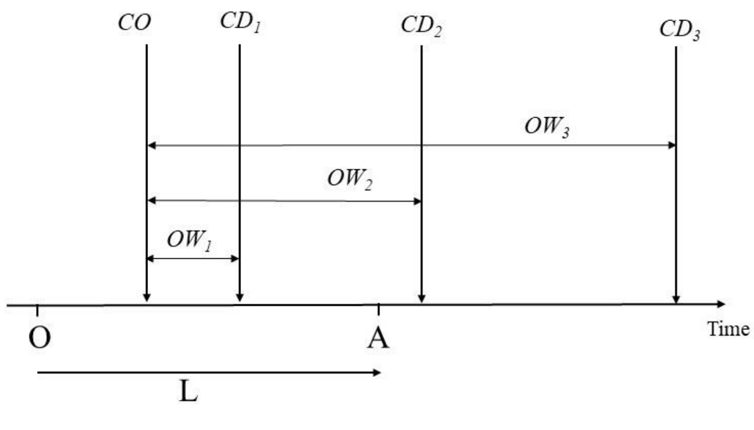

Figure 3 - Demand classification in a (s, Q) policy. Adapted from (Espinós 2015)

For simplicity, O will be set equal to 0 (the count of time starts when a replenishment is made) and A equal to L. When a customer places an order during the lead time $(0 < CO < L)$ and establishes the time when the order must be received, the demand originated can be classified as follows:

- $\boldsymbol{d_1}$ - Demand of type 1: $CO + OW < L$ and $OW < L$

The delivery takes place before the end of the lead time, which means that the retailer can only fulfill the order with the stock on-hand (similar to the traditional retail scenario).

- $\boldsymbol{d_2}$ - Demand of type 2: $CO + OW \geq L$ and $OW < L$

The delivery takes place after the end of the lead time, which means that the retailer can wait for the stock in-transit to fulfill it. Demand of type 2 gives the retailer the opportunity to satisfy orders with in-transit inventory and use the on-hand inventory to satisfy the demand of type 1.

- $\boldsymbol{d_3}$ - Demand of type 3: $OW \geq L$

The order window is big enough such that a new replenishment can be ordered and received by the retailer before the delivery time set by the customer. This means that if demand of type 3 triggers a new replenishment by dropping the inventory position below (or equal to) the reorder level, then the order would be received before the delivery time. Demand of type 3 will never originate a stockout because we assume infinite inventory at the supplier. This demand is similar to the case where the supply lead time is lower than the demand lead time in the study of Hariharan and Zipkin (1995) and in which they say this case would represent perfect customer service. Another approach to deal with this type of demand is given by Karaesmen et al. (2002). They propose to delay the replenishment, so in our case the replenishment would be delayed by $OW - L$ units of time (we wait that d_3 becomes d_2).

3.2 Continuous $(\boldsymbol{s}, \boldsymbol{Q})$ policy for online retail

This section presents the (s, Q) policy for the online retail based on the demand context explained in 3.1 and in the methodology used in the traditional retail policy presented in 2.2.

To obtain an adapted policy for the online retail scenario, Espinós (2015) ignored demand of type 3 because, like explained before, we would have unlimited inventory to fulfill these orders and it allowed to simplify the study. The study of the probability of stockout was focused on demand of type 1 and demand of type 2 by looking at the (s, Q) system as a two-bin system. When the inventory position drops to s and a replenishment is ordered, the first bin contains a stock quantity of s, which is the stock on-hand, and the second bin contains the quantity of Q, which is the stock on-order. Demand of type 1 will be fulfilled from the first bin, while demand of type 2 will be fulfilled from the second bin and from the first bin if necessary.

A stockout will occur if demand of type 1 is not fulfilled because the first bin is out of stock or if demand of type 2 is greater than Q, and part of that demand would need to be fulfilled from the on-hand stock (the first bin).

With this approach Espinós (2015) defined the probability of stockout (P_s) as:

$$\begin{aligned} P_s &= P(d_1 > s \ \cap d_2 \leq Q) + P(d_1 + d_2 > s + Q \ \cap d_2 > Q) \\ &= P(d_1 + max(0; d_2 - Q) > s) \end{aligned} \tag{3.1}$$

Where s can take two values depending on how the stockout could occur, if due only to d_1 (case A) or to both d_1 and d_2 (case B), which lead to equation (3.2).

$$s = max(\mu_1 + k\sigma_1; \mu_1 + \mu_2 + k\sigma_L - Q) = max(s_A; s_B) \tag{3.2}$$

In order to be conservative, s should be high enough such that in both cases (A) and (B), a certain service level is assured, so the higher value of s is chosen.

The expression for the ETRC was also modified into equation (3.3) and is also dependent on each case.

$$ETRC\ (k,Q) = A\frac{D}{Q} + \left[\frac{Q}{2} + \max(k\sigma_1; \mu_2 - Q + k\sigma_L)\right]vr + B_1\frac{D}{Q}P_s \qquad (3.3)$$

With the expression for the total cost defined, the optimal pair (s, Q) that minimizes the costs can be figured out with the approach explained in 2.2 through iterations until the values of k and Q do not change significantly.

The following work done by Soler (2017) expanded the policy, originally proposed for SOO scenario, to the MOO scenario and made some modifications to the policy in order to make it suitable for the MOO scenario. The modifications were related to the expressions for the reorder point s, the probability of stockout and the ETRC.

The modification to the probability of stockout and the reorder point was the introduction of a new variable OO (equation (3.4) - the quantity on-order/in-transit) that replaced Q in equations (3.1) and (3.2) resulting in equations (3.5) and (3.6).

$$OO = max(Q; \mu_L) \qquad (3.4)$$

$$P_s = P(d_1 + max(0; d_2 - OO) > s) \qquad (3.5)$$

$$s = max(\mu_1 + k\sigma_1;\ \mu_1 + \mu_2 + k\sigma_L - OO) \qquad (3.6)$$

Soler (2017) stated that case B is unlikely to happen because it would imply that $\sigma_2 \gg \sigma_1$ so that σ_L could be very large compared to σ_1 to make $s_B > s_A$. This would only happen if the demand's coefficient of variance is greater than one. The new expression proposed for the reorder point was:

$$s = \mu_1 + k\sigma_1 \qquad (3.7)$$

The expression for the ETRC was also modified into equation (3.8).

$$ETRC\ (k,Q) = A\frac{D}{Q} + \left[\frac{Q}{2} + k\sigma_1\right]vr + B_1\frac{D}{Q}P_s \qquad (3.8)$$

In this work, we will extend this line of research by making the characterization of each type of demand and by refining the policy, in which respects to the reorder point and stockout probability, because, as Soler (2017) stated, these were aspects that were subject to improvement in future work. We will also extend the policy to deal with β service level.

4 Model development for continuous review

This chapter will provide the characterization of the demand in online retail and a revised (s, Q) policy for online retail based in the work done by Espinós (2015) and in the traditional retail approach presented in 2.2. The demand during the lead time was divided in different types in previous work (Espinós 2015) in a continuous review policy setting. However, no information is given about the characterization of such demand types. We aim to fill this gap. Numerical applications are also presented in this chapter.

4.1 Demand characterization

This section presents the approach followed to characterize the demand and the simulations done to validate the results obtained.

4.1.1 Main assumptions and notation

For the characterization of demand during the lead time we made the following assumptions:

1) We consider that the time between orders is exponentially distributed and demand follows a Poisson distribution with constant rate. This means that demand arrivals constitute a Poisson process.
2) The lead time (L) is constant.
3) All demand transactions are of unit size.
4) For the order window (*OW*) we consider three cases: the order window follows a uniform distribution, a normal distribution, or an exponential distribution.
5) The customer order time (*CO*) has a uniform distribution between 0 and L.
6) The different types of demand are independent, *OW* and *CO* are also independent.

Assumption 6) is reasonable because it's common that customers have independent demands and we classify/segment them according to the time between the placement of theirs orders and the time when they want the orders delivered, so the different demand types will remain independent. Gallego and Özer (2001) make a similar assumption in their work when they divide the demand, and they also explain that it would be difficult to keep a manageable state space otherwise. Concerning the *OW* and the *CO*, their independence is natural as different customers ordering in the same moment have different needs for the time when they want the order delivered.

For the characterization of demand during the lead time we use the following notation:

λ_{CO}	Average demand per unit time
L	Lead time
μ_1	Average of demand of type 1 during the lead time
μ_2	Average of demand of type 2 during the lead time
μ_3	Average of demand of type 3 during the lead time
μ_L	Average of total demand during the lead time
σ_1	Standard deviation of demand of type 1 during the lead time
σ_2	Standard deviation of demand of type 2 during the lead time
σ_3	Standard deviation of demand of type 3 during the lead time
σ_L	Standard deviation of total demand during the lead time

4.1.2 Analytical approach

The total demand during the lead time is easy to obtain because the demand follows a Poisson distribution and lead time is constant. If X is the total demand during the lead time, the expected value is $E[X] = E[D]E[L]$ and the variance is $Var[X] = Var[D]E[L] + E[D]^2Var[L]$ where D is the average demand per unit time and L is the lead time. According to our notation:

$$\mu_L = \lambda_{CO}L \tag{4.1}$$

$$\sigma_L = \sqrt{(\lambda_{CO}L)} = \sqrt{\mu_L} \tag{4.2}$$

$$d_L \sim Poisson(\mu_L) \tag{4.3}$$

To determine the parameters for the three different types of demand, we followed the approach known as "thinning" a Poisson Process. This happens when we are interested in counting various special types of the events being counted. The events being counted correspond to the total demand and the different types of demand are the special types. According to Ross (1996) if $N_i(t)$ represents the number of events of type *i* that occur by time *t*, then $N_i(t)$ are Poisson independent random variables with mean given by $\lambda \times t \times P_i(t)$, where $P_i(t)$ is the probability that an event occurring at time *t* is of type *i*. (Here we consider a constant rate for the demand, however if the demand were not constant then we simply had to substitute λ for $\lambda(t)$).

Sigman (2007) calls this property by Partitioning Theorems for Poisson processes and random variables:

- Partitioning a Poisson random variable: If $X \sim Poisson(\alpha)$ and if each object of X is, independently, type 1 with probability p or type 2 with probability $q = 1 - p$, then $X_1 \sim Poisson(p\alpha)$, $X_2 \sim Poisson(q\alpha)$ and they are independent.
- Partitioning a Poisson process (PP): If $\psi \sim PP(\lambda)$ and if each arrival of ψ is, independently, type 1 with probability p or type 2 with probability $q = 1 - p$ then, the two resulting processes are themselves Poisson and independent: $\psi_1 \sim PP(p\lambda), \psi_2 \sim PP(q\lambda)$.

The above generalizes to $k \geq 2$ types (type *i* with probability P_i) yielding independent Poisson processes with rates $\lambda_i = P_i \times \lambda$, $i \in [1, 2, ..., k]$ (Sigman 2007).

When we divided the demand in three types (section 3.1), we set bounds on the values that the order window (*OW*) and the customer order (*CO*) can take for each type of demand. We use those bounds to calculate the probability of an order being of each one of the demand types. See Figure 4 for a better understanding. According to the division done in section 3.1, an order is demand of type 1 if $CO + OW < L$ and $OW < L$, which according to Figure 4, is the region below the line $CO = L - OW$. The same method is applied to demand of type 2 and type 3. This representation makes it easier to identify the limits of integration for the expressions that allowed us to obtain the probabilities.

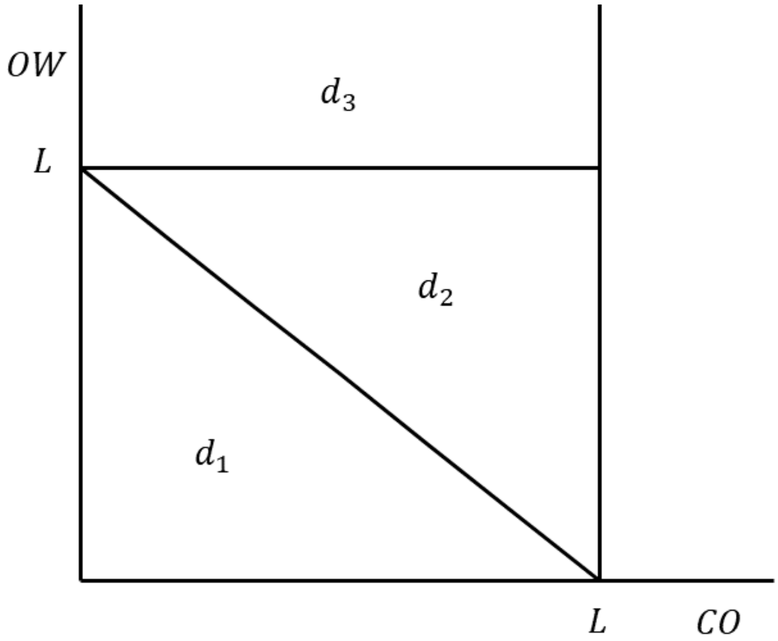

Figure 4 - 2D representation of the division of demand in a (s, Q) policy

Following the previous explanation, we obtain the expressions:

$$P(d_1) = P(CO + OW < L \cap OW < L) \tag{4.4}$$

$$P(d_2) = P(CO + OW \geq L \cap OW < L) \tag{4.5}$$

$$P(d_3) = P(OW \geq L) \tag{4.6}$$

Once we obtain the probabilities, we know how the total demand is distributed among the three types of demand, so the average of each type of demand can be calculated as $\mu_{\mathrm{x}} = P(d_x)\mu_{\mathrm{L}}$.

The expressions for d_1 and d_2 led to the calculation of a double integral (equations (4.7) and (4.8)). The expression for d_3 led to the calculation of a simple integral (equation (4.9)).

$$P(d_1) = \int_0^L \int_0^{L-OW} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \tag{4.7}$$

$$P(d_2) = \int_0^L \int_{L-OW}^{L} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \tag{4.8}$$

$$P(d_3) = \int_L^\infty f_{OW}(OW) d_{OW} \tag{4.9}$$

Where:

$$f_{CO}(CO) = \begin{cases} \frac{1}{L} & for\ CO \in [0; L] \\ 0 & otherwise \end{cases} \tag{4.10}$$

The expressions for the probabilities change accordingly to the distribution assumed for the OW. Through equations (4.12) to (4.32) the results are presented for uniform, exponential and normal distributions. We will present the formulas for the average of each type of demand. The standard deviation can be obtained by equation (4.11).

$$\sigma_x = \sqrt{\mu_x}. \tag{4.11}$$

a) OW ~Uniform[a;b]

$$f_{OW}(OW) = \begin{cases} \frac{1}{b-a} & for\ OW \in [a;b] \\ 0 & otherwise \end{cases} \tag{4.12}$$

In this case we need to divide the results in two scenarios, because the limits for the integral of the OW are limited by the values of a and b.

- $b > L$ and $a < L$ (If $a > L$, everything is d_3)

$$P(d_1) = \int_a^L \int_0^{L-OW} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \tag{4.13}$$

$$P(d_2) = \int_a^L \int_{L-OW}^{L} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \tag{4.14}$$

$$P(d_3) = \int_L^b f_{OW}(OW) d_{OW} \tag{4.15}$$

$$\mu_1 = \left[\frac{(a - L)^2}{2L(b - a)}\right] \lambda_{co} L \tag{4.16}$$

$$\mu_2 = \left[\frac{L^2 - a^2}{2L(b - a)}\right] \lambda_{co} L \tag{4.17}$$

$$\mu_3 = \left[\frac{b - L}{b - a}\right] \lambda_{co} L \tag{4.18}$$

- $b \leq L$

$$P(d_1) = \int_a^b \int_0^{L-OW} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \tag{4.19}$$

$$P(d_2) = \int_a^b \int_{L-OW}^{L} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \tag{4.20}$$

$$P(d_3) = 0 \tag{4.21}$$

$$\mu_1 = \left[1 - \frac{a + b}{2L}\right] \lambda_{CO} L \tag{4.22}$$

$$\mu_2 = \left[\frac{a + b}{2L}\right] \lambda_{CO} L \tag{4.23}$$

$$\mu_3 = 0 \tag{4.24}$$

b) OW ~Exponential(λ_{OW})

$$f_{OW}(OW) = \begin{cases} \lambda_{OW} e^{-\lambda_{OW} OW} & for\ OW \geq 0 \\ 0 & for\ OW < 0 \end{cases} \tag{4.25}$$

Using expressions (4.7) through (4.9), the results obtained are:

$$\mu_1 = \left[\frac{e^{-\lambda_{OW} L} - 1}{\lambda_{OW} L} + 1\right] \lambda_{CO} L \tag{4.26}$$

$$\mu_2 = \left[\frac{1 - e^{-\lambda_{OW} L}(\lambda_{OW} L + 1)}{\lambda_{OW} L}\right] \lambda_{CO} L \tag{4.27}$$

$$\mu_3 = \left[e^{-\lambda_{OW} L}\right] \lambda_{CO} L \tag{4.28}$$

c) OW ~Normal(μ_{OW} , σ_{OW})

$$f_{OW}(OW) = \frac{1}{\sqrt{2\pi(\sigma_{ow})^2}} e^{-\frac{(OW - \mu_{ow})^2}{2(\sigma_{ow})^2}} \tag{4.29}$$

Note that the OW must take positive values, so values used for μ_{OW} and σ_{OW} should be chosen wisely.

Using expressions (4.7) through (4.9), the results obtained are:

$$\mu_1 = \left(\left[\sqrt{\left(\frac{\pi}{2}\right)} (L - \mu_{OW}) \left(erf\left(\frac{L - \mu_{OW}}{\sqrt{2}\sigma_{OW}}\right) + erf\left(\frac{\mu_{OW}}{\sqrt{2}\sigma_{OW}}\right)\right) + \sigma_{OW} \left(e^{-\frac{(L - \mu_{OW})^2}{2\,\sigma_{OW}^2}} - e^{-\frac{\mu_{OW}^2}{2\,\sigma_{OW}^2}}\right)\right] \frac{1}{L\sqrt{2\pi}}\right) \lambda_{CO} L \tag{4.30}$$

$$\mu_2 = \left(\left[\sqrt{\left(\frac{\pi}{2}\right)} (\mu_{OW}) \left(erf\left(\frac{L - \mu_{OW}}{\sqrt{2}\sigma_{OW}}\right) + erf\left(\frac{\mu_{OW}}{\sqrt{2}\sigma_{OW}}\right)\right) + \sigma_{OW} \left(-e^{-\frac{(L - \mu_{OW})^2}{2\,\sigma_{OW}^2}} + e^{-\frac{\mu_{OW}^2}{2\,\sigma_{OW}^2}}\right)\right] \frac{1}{L\sqrt{2\pi}}\right) \lambda_{CO} L \tag{4.31}$$

$$\mu_3 = \frac{1}{2}\left(erfc\left(\frac{L - \mu_{OW}}{\sqrt{2}\sigma_{OW}}\right)\right) \lambda_{CO} L \tag{4.32}$$

The expressions obtained here are the expressions that allow to obtain the values of the parameters of the types of demand during the lead time. Notice, however, that the probability of an order being of a specific type of demand changes with time, which is obvious by looking at Figure 4. Because the parameters of interest for the policy are the average and the standard deviation during the lead time we presented a method to obtain them directly, however another approach is presented in Annex A where the time is taken into consideration and a more detailed analysis is presented.

According to Silver et al. (1998), if the demand during the lead time is higher than 10 units and the ratio $\frac{\sigma_x}{\mu_x}$ is lower than 0.5, then the normal distribution is probably a good approximation for the demand. This means that if the average of demand is higher than 10 units the Poisson distributed demand with average μ_x can be approximated by a normal distribution with average μ_x and standard deviation $\sqrt{\mu_x}$, thus in the policy presented in this work we assume that demand during the lead time follows a normal distribution (as in the previous work done by Espinós (2015) and also in the work of Soler (2017)). This assumption is relatively usual in the literature and it is also referred by Hariharan and Zipkin (1995) in their study.

4.2 Validation of results through simulation

In order to validate the results obtained in the analytical approach presented in 4.1.2, we built a simulator that returns the percentage of each type of demand, its average and standard deviation and the coefficient of correlation between each type of demand. The inputs in this simulator are the OW distribution parameters, the average demand per unit time (λ_{CO}) and the lead time (L).

4.2.1 Simulation design

We simulate 2000 orders with the time between orders following an exponential distribution with mean $\frac{1}{\lambda_{CO}}$ and then the simulator counts the total number of orders and the number of orders that are from each type of demand during the lead time. Since we assume that transactions are unit size, the demand in units is equal to the number of orders.

The simulator uses 10000 iterations and was performed for 100 different combinations of inputs for each one of the 3 possibilities of the OW distribution.

Table 4 - Possible input values considered for the lead time and for demand per unit time

Demand and lead time		
$\boldsymbol{L}$	$1, 2, 3, 4, 5, 6, 7, 10, 20$	UT
$\boldsymbol{\lambda_{CO}}$	$\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1, \frac{5}{3}, \frac{10}{3}, \frac{20}{3}, 10, 25, 50, 100, 250, 500, 1000$	$\frac{Units}{UT}$

Table 5 - Possible input values considered for the order window parameters with uniform distribution

OW	Uniform	
$\boldsymbol{a}$	$0, \frac{1}{5}L,\ \frac{1}{4}L,\ \frac{1}{3}L, \frac{2}{5}L, \frac{1}{2}L, \frac{3}{5}L, \frac{2}{3}L, \frac{3}{4}L, \frac{4}{5}L, L$	UT
$\boldsymbol{b}$	$\frac{1}{4}L,\ \frac{1}{3}L, \frac{1}{2}L, \frac{2}{3}L, \frac{3}{4}L, L, \frac{5}{4}L, \frac{4}{3}L, \frac{3}{2}L, \frac{7}{4}L, 2L$	UT

Table 6 - Possible input values considered for the order window parameters with exponential distribution

OW	Exponential	
$\boldsymbol{\lambda_{OW}}$	$\frac{1}{20}, \frac{1}{10}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{1}{2}, 1, \frac{4}{3}, \frac{3}{2}, 2, \frac{5}{2}, \frac{\ln(2)}{L}, \frac{\ln(4)}{L}, \frac{\ln\left(\frac{4}{3}\right)}{L}$	$\frac{1}{UT}$

Table 7 - Possible input values considered for the order window with normal distribution

OW	Normal	
μ_{OW}	$\frac{1}{5}L,\ \frac{1}{4}L,\ \frac{1}{3}L, \frac{1}{2}L, \frac{2}{3}L, \frac{3}{4}L, \frac{4}{5}L, L, \frac{5}{4}L, \frac{4}{3}L, \frac{3}{2}L, 2L$	UT
σ_{OW}	$\frac{1}{50}\mu_{OW}, \frac{1}{20}\mu_{OW}, \frac{1}{10}\mu_{OW}, \frac{1}{5}\mu_{OW}, \frac{3}{10}\mu_{OW}, \frac{4}{10}\mu_{OW}, \frac{5}{10}\mu_{OW}$	UT

The values for the 100 combinations used in the simulations were chosen randomly from the values in Table 4, Table 5, Table 6 and Table 7 without allowing repeated combinations to ensure that all possible scenarios were tested.

4.2.2 Results

This sub-section is dedicated to illustrating the validation of the expressions obtained in the analytical approach by comparison with the simulated results. Table 8 provides the mean absolute percentage error (MAPE) between the simulated results and the results obtained using the expressions in 4.1.2.

Table 8 - Validation results for continuous review

Error (%)								
OW	μ_1	μ_2	μ_3	μ_L	σ_1	σ_2	σ_3	σ_L
Uniform	0.27%	0.18%	0.10%	0.13%	0.53%	0.49%	0.26%	0.42%
Exponential	0.22%	0.30%	1.88%	0.15%	0.45%	0.51%	1.21%	0.50%
Normal	3.63%	0.86%	2.54%	0.10%	1.96%	0.88%	1.27%	0.40%

It has been possible to verify the analytical approach with the simulation done. The average percentage of error between the experimental and the analytical values is very small, except for the values for the demand of type 1 when the *OW* has a normal distribution and the values for the demand of type 3 when the *OW* has a normal distribution or an exponential distribution. Concerning the errors for the demand of type 3, they are due to very low values of demand where they have decimal values (both the demand obtained in the simulation and by the expressions have values of order 10^{-2} or lower, when the probability of being demand of type 3 is close to zero and the total demand has low values). Concerning the error for the demand of type 1 when the *OW* has a normal distribution, it is due to the simulations were the ratio $\frac{\sigma_{ow}}{\mu_{ow}}$ for the *OW* distribution parameters is high enough to generate negative values for the *OW*. The maximum ratio used was 0.5, and because the *OW* follows a normal distribution there is the probability of 2.275% to generate negative values for the *OW*.

With the simulation done we also verified the independence between the types of demand, as the coefficients of correlation between them are very small and negligible, which verified assumption 6) as a reasonable one. The histograms obtained by using the results of the iterations also allowed to verify that for values of the average demand approximately higher than 10 for each one of the types of demand, they can be approximated by a normal distribution.

4.3 Revised continuous (s, Q) policy for online retail

This section introduces the findings and modifications made to the policy presented in the previous work done by Espinós (2015) and Soler (2017) and the respective validations. Then the revised policy is presented.

4.3.1 Modifications to the policy presented in the previous work

In the online retail scenario, the difference between the time an order is received and the time when it will be delivered provides an order window that the retailer can use to make better decisions concerning the inventory management. Our inventory policy tries to explore this feature.

In previous work, it was assumed that demand of type 2 could have an impact on the reorder point and in the probability of stockout. In this thesis, we prove that demand of type 2 has no such impact and we present the following preposition.

PREPOSITION 1. The probability of stockout (P_s) and the reorder level s are defined by:

$$P_s = P(d_1 > s) \tag{4.33}$$

$$s = \mu_1 + k\sigma_1 \tag{4.34}$$

PROOF:

In order to validate this reorder point consider the following scenario, represented in Figure 5. An order of size Q (Q_1) is triggered at moment $t = 0$ when the inventory position reaches s and it will arrive after L. After the replenishment the inventory position is $s + Q$. A new order (Q_2) would be placed when the inventory position drops from $s + Q$ to s and it would arrive after L. That is between two replenishments the demand is Q, because we are reviewing the inventory continuously. The proof is built around this scenario where we have two cycles. We study the probability of stockout when there are two cycles (P_s'), to see if demand other than demand of type 1 has impact on the probability of stockout. Then we compare it with the probability of stockout of a single cycle and we extend the conclusion to the situations where there are more cycles.

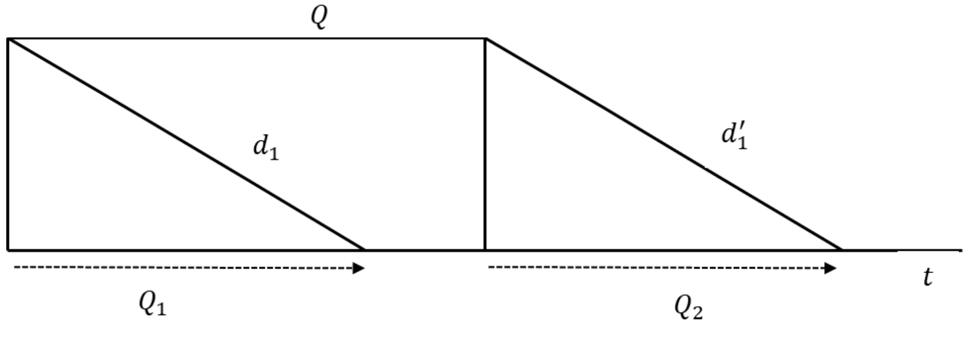

Figure 5 - Scenario representation

The probability of stockout in the scenario with two cycles is given by equation (4.35). The numerator represents the situation where we have a stockout in the first cycle due to d_1, or a stockout in the second cycle due to d_1' and the demand that happened between the replenishments. The denominator is equal to the number of cycles.

$$\begin{aligned} P_s' &= \frac{P(d_1 > s \cup d_1' + Q > s + Q)}{2} = \frac{P(d_1 > s \cup d_1' > s)}{2} \\ &= \frac{P_1 + P_1 - P(d_1 > s \cap d_1' > s)}{2} \end{aligned} \tag{4.35}$$

Then we compare this probability with the probability of stockout when there is a single cycle:

$$P_s' \ ?\ P(d_1 > s\ )\ (=)\ P_1 - \frac{P(d_1 > s \cap d_1' > s\ )}{2} ?\ P_1\ (=)$$

$$(=) - \frac{P(d_1 > s \cap d_1' > s\ )}{2}\ ?\ 0$$

Where ? is a mathematical operator that we are trying to find.

Because a probability is always greater than or equal to zero, then ? is ≤ and $P_s' \ \leq\ P(d_1 > s\ )$. Therefore, when we have two cycles the probability of stockout is lower or equal than in a single cycle. In order to be conservative, we define the probability of stockout by the equality and we extend this proof to scenarios where there are more cycles. From the probability of stockout, it follows that the reorder point is $s = \mu_1 + k\sigma_1$.

This proof suits the SOO scenario represented in Figure 5 where there is no overlap between cycles. Thus, cycles are independent and $\frac{P(d_1 > s \cap d_1' > s\ )}{2}$ would be $\frac{(P_1)^2}{2}$. This proof also suits the MOO scenario, in which there is overlap between cycles. Consequently, there is dependency between cycles but the probability $\frac{P(d_1 > s \cap d_1' > s\ )}{2}$ is still greater than or equal to zero.

4.3.2 Total cost in the continuous (s, Q) policy with α service level metrics

The new expressions for the reorder point and probability of stockout imply that the total cost formula has to be adapted. In the next expressions we use $P_{u_1 \geq}(k)$ instead of P_s.

The expression for the ETRC with the new modifications is :

$$ETRC\ (k, Q) = A\frac{D}{Q} + \left[\frac{Q}{2} + k\sigma_1\right] vr + B_1\frac{D}{Q}P_{u_1 \geq}(k) \tag{4.36}$$

For each one of the different OW studied in this thesis the expressions for the total costs are presented next:

a) OW ~Uniform[$a;b$]

- $b > L$ and $a < L$

$$ETRC\ (k, Q) = A\frac{D}{Q} + \left[\frac{Q}{2} + k\sqrt{\left[\frac{(a\ -\ L)^2}{2\,L(b\ -\ a)}\right]\lambda_{co}L}\right] vr + B_1\frac{D}{Q}P_{u_1 \geq}(k) \tag{4.37}$$

- $b < L$

$$ETRC\ (k, Q) = A\frac{D}{Q} + \left[\frac{Q}{2} + k\sqrt{\left[1 - \frac{a+b}{2L}\right]\lambda_{CO}L}\right] vr + B_1\frac{D}{Q}P_{u_1 \geq}(k) \tag{4.38}$$

b) OW ~Exponential(λ_{OW})

$$ETRC\ (k, Q) = A\frac{D}{Q} + \left[\frac{Q}{2} + k\sqrt{\left[\frac{e^{-\lambda_{OW}L} - 1}{\lambda_{OW}L} + 1\right]\lambda_{CO}L}\right] vr + B_1\frac{D}{Q}P_{u_1 \geq}(k) \tag{4.39}$$

c) $OW \sim \text{Normal}(\mu_{OW}, \sigma_{OW})$

$$ETRC\ (k, Q) = A\frac{D}{Q} + Z + B_1\frac{D}{Q}P_{u_1\geq}(k) \tag{4.40}$$

$$Z = \left[\frac{Q}{2} + k\sqrt{\left(\left[\sqrt{\left(\frac{\pi}{2}\right)}(L - \mu_{OW})\left(erf\left(\frac{L - \mu_{OW}}{\sqrt{2}\sigma_{OW}}\right) + erf\left(\frac{\mu_{OW}}{\sqrt{2}\sigma_{OW}}\right)\right) + \sigma\left(e^{-\frac{(L-\mu_{OW})^2}{2\sigma_{OW}^2}} - e^{-\frac{\mu_{OW}^2}{2\sigma_{OW}^2}}\right)\right]\frac{1}{L\sqrt{2\pi}}\right)\lambda_{CO}L}\right]vr$$

With the expressions for the total cost defined, we can find the optimal pair (s, Q) that minimizes the costs with the approach explained in 2.2. First obtain the partial derivatives (equations (4.41) and (4.42)) and then through iterations until the values of k and Q do not change significantly. In these expressions $P_{u_1\geq}(k)$ can be easily obtained in an Excel spreadsheet, however, if needed, an alternative is $\frac{1}{2}erfc\left(\frac{k}{\sqrt{2}}\right)$. Details about the derivations and the Excel functions are presented in Annex B.

$$\frac{\delta ETRC\ (k, Q)}{\delta k} = 0 \rightarrow k = \sqrt{2\,ln\left(\frac{DB_1}{\sqrt{2\pi}Qvr\sigma_1}\right)} \tag{4.41}$$

$$\frac{\delta ETRC\ (k, Q)}{\delta Q} = 0 \rightarrow Q = EOQ\sqrt{1 + \frac{B_1}{A}P_{u_1\geq}(k)} \tag{4.42}$$

Instead of using this procedure, if the retailer is interested in a certain α service level then k can be obtained from equation (4.43) and then use a predetermined Q to calculate the ETRC or calculate the Q that minimizes ETRC and the respective total cost.

$$\alpha = 1 - P_s = 1 - P_{u_1\geq}(k) = 1 - \frac{1}{2}erfc\left(\frac{k}{\sqrt{2}}\right) \tag{4.43}$$

When using a predetermined Q, the EOQ is a common practice (Silver et al. 1998). It is also the value used to start the iterative procedure.

4.3.3 Total cost in the continuous (s, Q) policy with β service level metrics

As suggested by Espinós (2015) and by Soler (2017) a possible extension for this work is to use a different service level measure. Here we present the derivation of the total cost with β service level for normal distributed demand.

To define a policy with β service level and normal distributed demand, additional notation and definitions are needed:

B_2	Fractional charge per unit short
$G_u(k)$	$\int_k^\infty (u_0 - k)\frac{1}{\sqrt{2\pi}}e^{\left(-\frac{u_0^2}{2}\right)}du_0$, is the loss function used to calculate the expected shortage per replenishment cycle (ESPRC)
ESPRC	Equal to $\sigma\ G_u(k)$

According to Silver et al. (1998), in a continuous policy for the traditional retail, the total cost with β service level and normal distributed demand is given by:

$$ETRC\ (k, Q) = A\frac{D}{Q} + \left[\frac{Q}{2} + k\sigma_L\right]vr + \frac{B_2 v\sigma_L G_u(k)D}{Q} \tag{4.44}$$

To adapt equation (4.44) for the online retail, we need to replace the standard deviation of the total demand by the standard deviation of demand of type 1 and change the variable in the $G_u(k)$ function.

$$ETRC\ (k,Q) = A\frac{D}{Q} + \left[\frac{Q}{2} + k\sigma_1\right] vr + \frac{B_2 v\sigma_1 G_{u_1}(k) D}{Q} \tag{4.45}$$

The expressions for each one of the different OW studied in this thesis can be obtained by replacing σ_1 for the corresponding value obtained in 4.1.2.

With the expressions for the total cost defined, we use the same approach used in the α service level to find the optimal pair (s, Q) that minimizes the costs with the iterative procedure. First obtain the partial derivatives (equations (4.46) and (4.47)) and then through iterations until the values of k and Q do not change significantly. In these expressions $G_{u_1}(k)$ can be easily obtained in an Excel spreadsheet, however, if needed, an alternative is $\left(\frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} - \frac{1}{2}k\ erfc\left(\frac{k}{\sqrt{2}}\right)\right)$.

Details about the derivations and the Excel functions are presented in Annex B.

$$\frac{\delta ETRC\ (k,Q)}{\delta k} = 0 \rightarrow P_{u_1 \geq}(k) = \frac{rQ}{DB_2} \tag{4.46}$$

$$\frac{\delta ETRC\ (k,Q)}{\delta Q} = 0 \rightarrow Q = EOQ\sqrt{1 + \frac{B_2\sigma_1 v}{A} G_{u_1}(k)} \tag{4.47}$$

Instead of using this procedure, if the retailer is interested in a certain β service level then k can be obtained from equation (4.48) and then use a predetermined Q to calculate the ETRC or calculate the Q that minimizes ETRC and the respective total cost.

$$G_{u_1}(k) = \frac{Q}{\sigma_1}\left(\frac{1-\beta}{\beta}\right) (=) \left(\frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} - \frac{1}{2}k\ erfc\left(\frac{k}{\sqrt{2}}\right)\right) = \frac{Q}{\sigma_1}\left(\frac{1-\beta}{\beta}\right) \tag{4.48}$$

When using a predetermined Q and the β service level, the recommendation of Silver et al. (1998) adapted for the online retail scenario is:

$$Q = \frac{1}{\beta}\sqrt{\frac{2AD}{rv} + {\sigma_1}^2} \tag{4.49}$$

If σ_1 is large relative to Q, a more accurate formula should be used instead of (4.48). The formula adapted from Silver et al. (1998) to the online retail scenario is:

$$G_{u_1}(k) - G_{u_1}\left(k + \frac{Q}{\sigma_1}\right) = \frac{Q}{\sigma_1}\left(\frac{1-\beta}{\beta}\right) (=) \tag{4.50}$$

$$(=) \left(\frac{e^{-\frac{k^2}{2}}}{\sqrt{2\pi}} - \frac{1}{2}k\ erfc\left(\frac{k}{\sqrt{2}}\right)\right) - \left(\frac{e^{-\frac{\left(k+\frac{Q}{\sigma_1}\right)^2}{2}}}{\sqrt{2\pi}} - \frac{1}{2}\left(k + \frac{Q}{\sigma_1}\right) erfc\left(\frac{k + \frac{Q}{\sigma_1}}{\sqrt{2}}\right)\right)$$
$$= \frac{Q}{\sigma_1}\left(\frac{1-\beta}{\beta}\right)$$

4.4 Numerical study

In this section, a numerical study is presented so that conclusions can be taken from the results obtained. The objective of this section is to assess how the online policy compares to the traditional policy and how it behaves with different parameters. We evaluate the optimal values of the traditional retail policy into the online policy and compare the results to assess the benefits for the retailer of using the adapted policy.

4.4.1 Continuous (s, Q) policy with α service level metrics

For each combination of parameters, different variations have been tested, each of these possible combinations will be a scenario.

We have generated different coefficients of variance (CV):

$$\text{CV} = \frac{\sigma_L}{\mu_L} = \frac{\sqrt{\mu_L}}{\mu_L} \ (assuming\ Poisson\ demand) \tag{4.51}$$

Because we are assuming that demand follows a Poisson distribution and then we approximate to the normal distribution, the maximum CV used was 0.2 because higher values for the CV would require low values of demand and the approximation to the normal distribution may not be appropriate in those scenarios. Thus, we generated CV's ranging from 0.05 to 0.2.

We also generate different proportions of demand of type 1 over the total demand during the lead time. To generate the different proportions a uniform order window was used:

- OW~$[0, 0.5L]$ – 75% d_1
- OW~$[0, L]$ – 50% d_1
- OW~$[0.5L, L]$ – 25% d_1

The values for $d_1(\mu_1, \sigma_1)$ to be used in the total cost function, equation (4.36), and in the iterative procedure, equations (4.41) and (4.42), are obtained through the formulas in 4.1.2. To compare the online policy (OR) to the traditional policy (TR) when using a specified cost per stockout occasion (B_1) the following input data was used:

A	30 €/replenishment
B_1	60 €/stockout
D	3000 units/year
r	0.24 €/€/year
v	12 €/unit

To obtain the pretended CV's using this data, we needed to use different values for the lead time. In Table 9 we present examples of the values needed to simulate the intended CV's.

Table 9 - Values of the lead time to obtain the intended demand and respective CV

CV	0.05	0.1	0.2
μ_L	400	100	25
L	≈ 49 days	≈ 13 days	≈ 4 days

Firstly, we compare the cost reduction from using the online policy with optimal pairs (s, Q) obtained from the iterative procedure. Secondly, we compare the cost reduction from using the new policy when the retailer is limited by the supplier to a certain order quantity. Then, we study the impact of different order windows in the cost reduction obtained using the optimal pair (s, Q) correspondent to each order window used. Finally, we assess how the cost reduction from using the new policy varies with the values of the fixed costs of the policy.

Figure 6 illustrates the cost reduction that results from using the new policy with α service level for the online retail. For each scenario, the optimal values of the pair (s, Q) were obtained through the iterative procedure, for both the traditional and the online policy, and the costs associated with those pairs were compared. Numerical results are presented in Annex C.

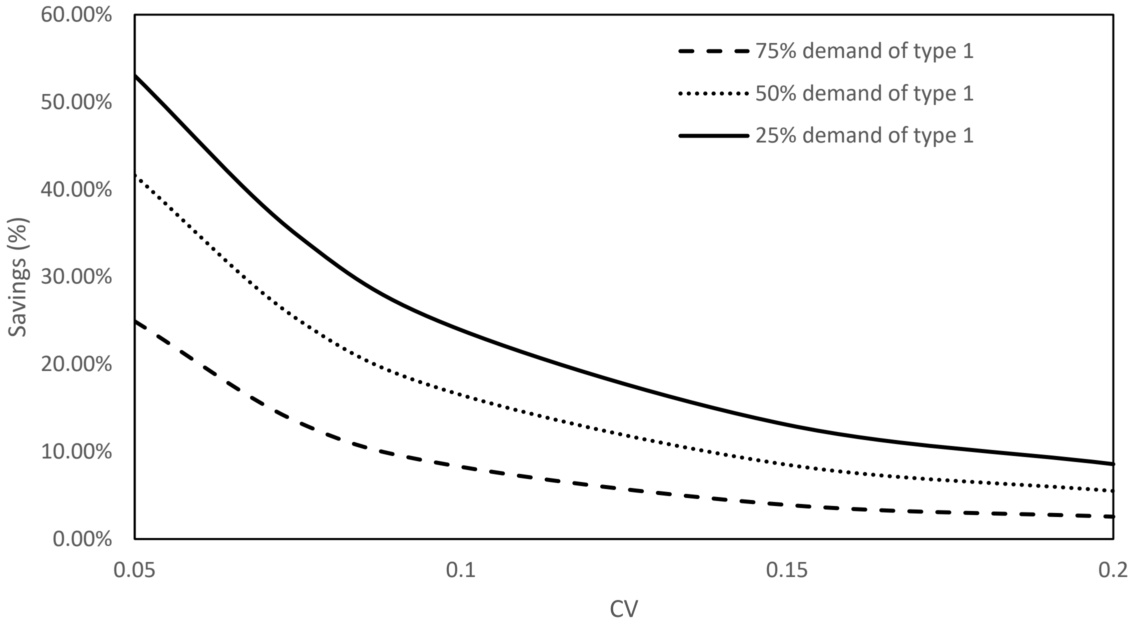

Figure 6 - Policies comparison with α service level with optimal pairs (s, Q) for different percentages of d_1

The results obtained show that, the lower the percentage of demand of type 1 is, the higher the savings from using the new policy proposed for the online retail are. When the percentage of demand of type 1 is 75% of the total demand, most of the demand during the lead time is of type 1, which is the most similar scenario to the traditional retail and that is why the savings are lower.

Concerning to the results in respect to CV, the percentage of savings lowers for higher CV's. For higher CV's the average and the standard deviation of the total demand are closer to each other. Due to this fact, the average of demand of type 1 and its standard deviation are closer to the average and standard deviation of the total demand and that explains why the savings are lower.

Notice that the difference between the averages and the standard deviations of demand of type 1 and the total demand is lower for higher CV's and for higher percentages of demand of type 1, which explains the lower savings and the lower values of the total cost.

We also tested the policies in a situation where the retailer cannot choose the order quantity because the supplier imposes it. Figure 7 shows the percentage of savings for different possible orders quantity Q imposed by the supplier. Demand of type 1 is 25% or 75% of the total demand and CV varies between 0.05 and 0.2. In general, for the same order quantity, lower CV's and lower percentages of demand of type 1 provide higher percentages of savings due to greater differences between the averages and standard deviations of demand of type 1 and the total demand. For the same percentage of demand of type 1, first the savings increase with the order quantity but above a certain quantity, the percentage of savings starts to decrease. This is because the difference between the stockout costs (lower in the traditional policy due to the higher safety factor) increases with Q and the difference between the holding costs decreases. Numerical results for three specific order quantities are available in Annex C.

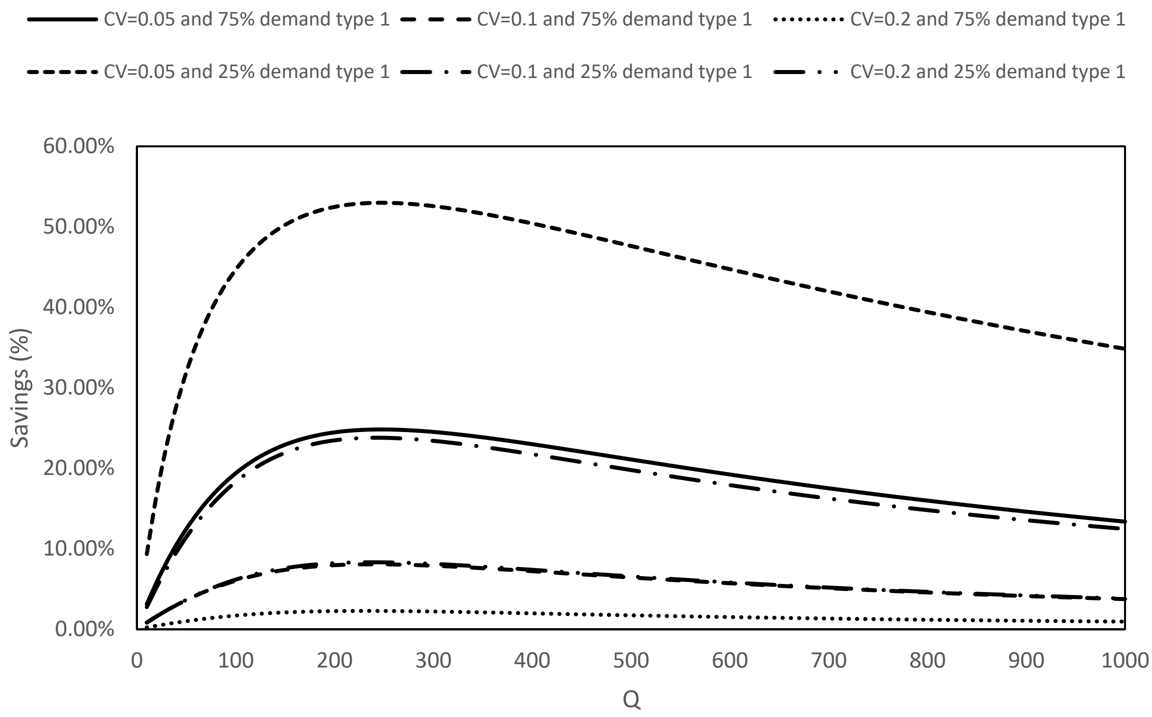

Figure 7 - Policies comparison with α service level for different values of the order quantity, different CV's and for different percentages of d_1

The contribution of different order windows is also studied. We tested different scenarios were the CV varies between 0.05 and 0.2. The policies are compared when the order window has the same mean but different parameters or distributions. The values tested for the mean of the order window were: $0.25L$, $0.5L$ and $0.75L$. Some of the results obtained in the experiments are presented in Figure 8 through Figure 10. Numerical results are presented in Annex C.

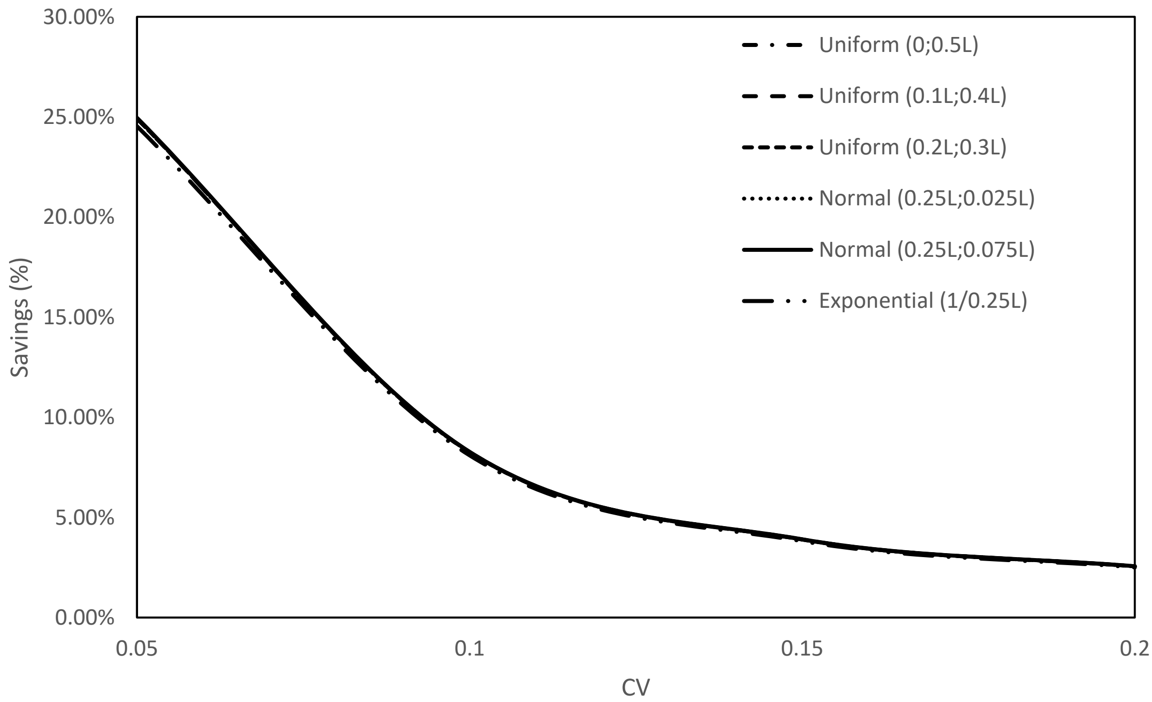

Figure 8 - Policies comparison with α service level for different order windows with mean $0.25L$

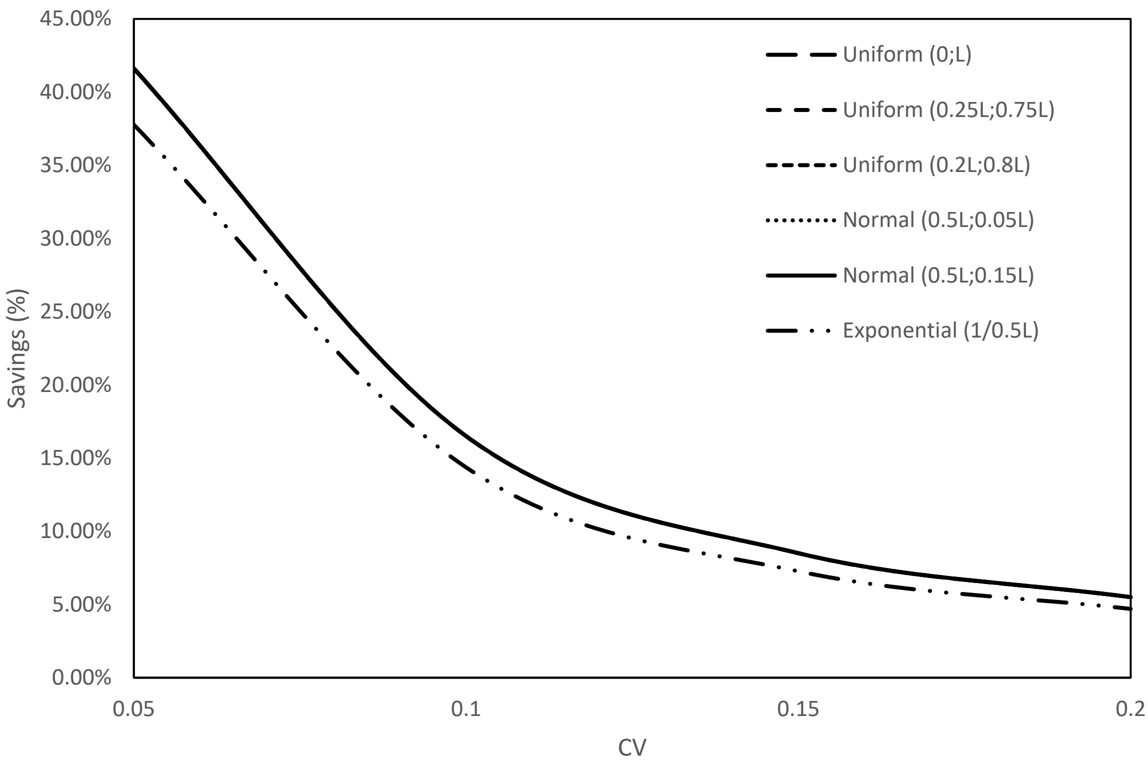

Figure 9 - Policies comparison with α service level for different order windows with mean $0.5L$

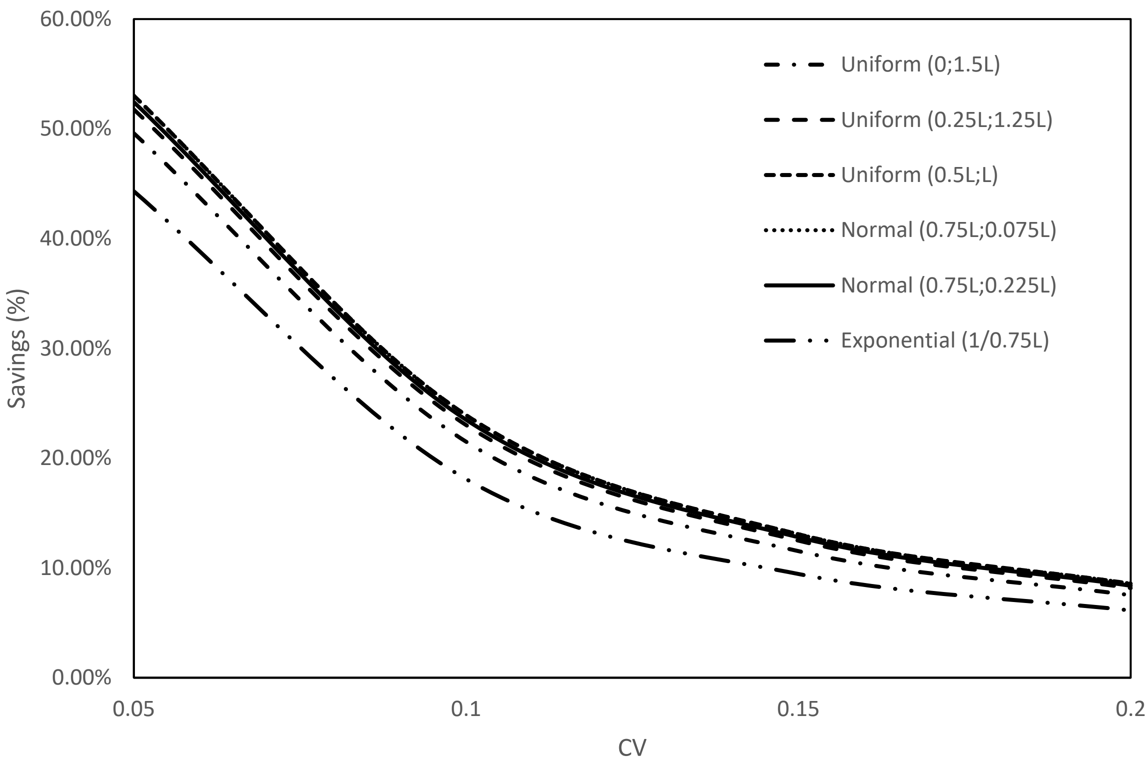

Figure 10 - Policies comparison with α service level for different order windows with mean $0.75L$

For order windows with the same mean, Figure 8 through Figure 10 show that an exponential order window provides lower percentages of savings than normal or uniform distributed order windows. This is because in an exponential distribution the mean is higher than the median. Even order windows with the same distribution, uniform or normal, can present different percentages of savings due to differences in the parameters. For normal distributed order windows, lower ratios $CV_{\mathrm{OW}} = \frac{\sigma_{OW}}{\mu_{OW}}$ provide higher percentages of savings when the mean value is higher, because the probability of appearance of demand of type 2 and type 3 is higher. For the same reason, with uniform distributed order windows, higher values of the mean and lower differences between the lower and upper bounds (lower variances) provide higher percentages of savings. Concerning the CV's of the demand, higher values provide lower percentages of savings.

Finally, we assessed how the cost reduction from using the new policy varies with the values of the fixed costs of the policy. Experiments were made with different values of the costs A (ordering cost), $H = vr$ (holding cost), and B_1 (stockout cost) and different percentages of demand of type 1 for the scenario with CV = 0.1. We assess how varying the values of the costs affects the percentage of savings obtained by implementing the policy adapted for the online retail for the optimal values (s, Q). In all the experiments, the percentage of demand of type 1 varies between 1% and 100% however, for percentages below 10%, the results need to be analyzed taking into account that the approximation to the normal distribution may not be a good approximation.

The experiments realized allowed to take some conclusions. By analyzing Figure 11, lower values of A combined with lower values of the percentage of demand of type 1 provide higher percentages of savings. Lower values of A correspond to lower values of the EOQ, lower values of the optimal Q and higher safety factors k which is lower in the online policy. Savings are due to the fact that the difference between the holding costs is higher than the difference between the stockout costs, and that is why the savings are higher for lower values of A.

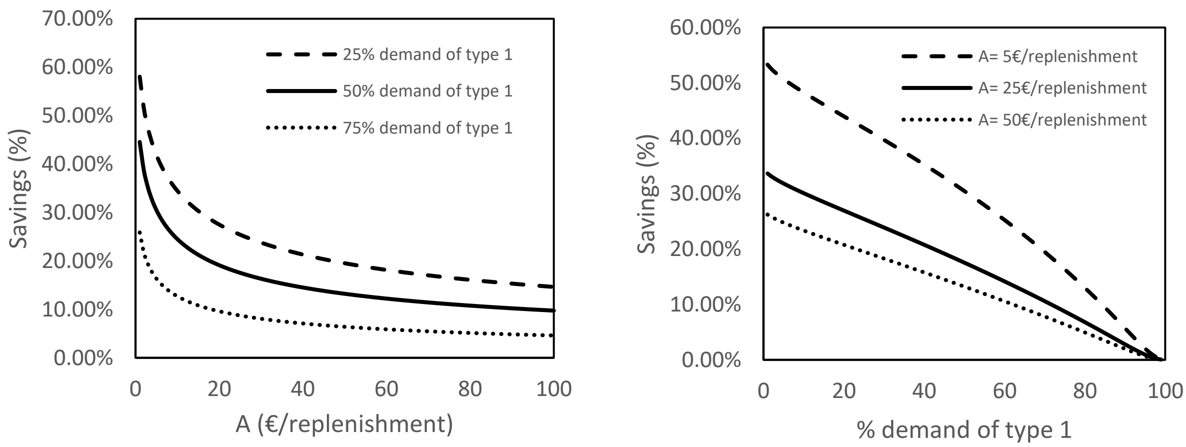

Figure 11 - Policies comparison with α service level for different percentages of d_1 and different values of A

From Figure 12, higher values of B_1 combined with lower values of the percentage of demand of type 1 provide higher percentage of savings. For the same percentage of demand of type 1, because the optimal safety factor k increases and the optimal order quantity Q decreases for higher values of B_1, the savings increase slightly with B_1. Increasing differences between the holding costs and decreasing differences between the stockout costs are the reason for the higher percentages of savings.

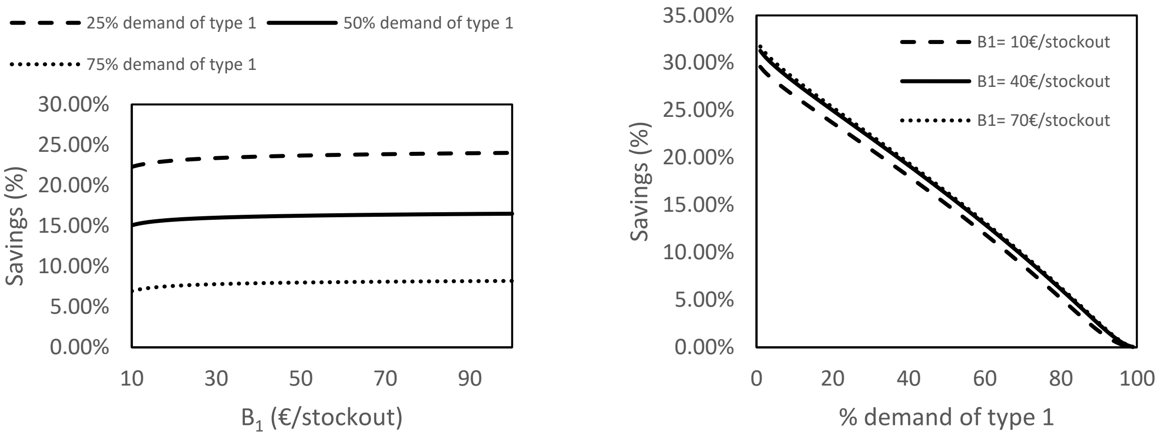

Figure 12 - Policies comparison with α service level for different percentages of d_1 and different values of B_1

Analyzing Figure 13, higher values of H combined with lower values of the percentage of demand of type 1 provide higher percentages of savings. Savings increase for higher values of H and for lower values of the percentage of demand of type 1 because the increasing differences between the holding costs of both policies are higher than the increasing differences in the stockout costs. Higher safety factors related to the traditional policy originate lower stockout costs but much higher holding costs. This fact leads to higher savings for the online policy and, although the total cost also increases with higher values of H, the increase in the difference is enough to increase the percentage of savings.

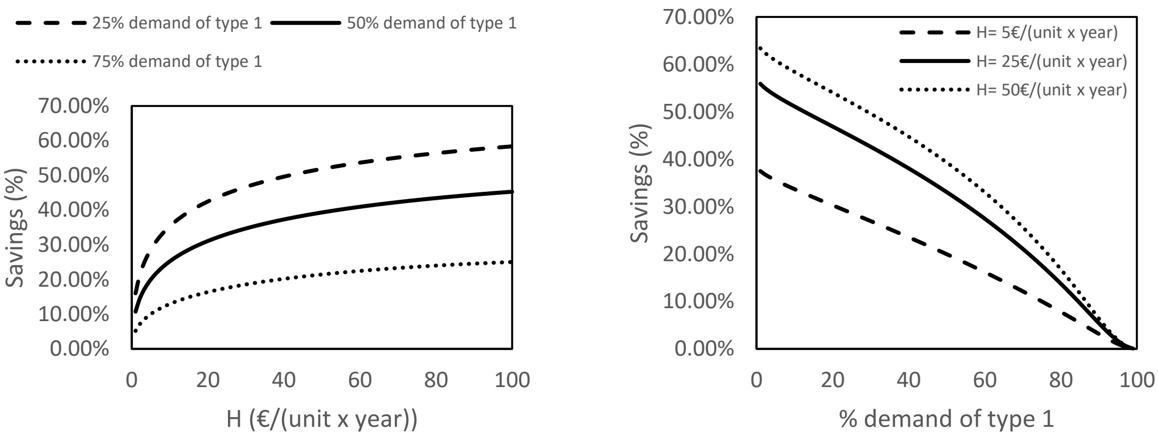

Figure 13 - Policies comparison with α service level for different percentages of d_1 and different values of H

4.4.2 Continuous (s, Q) policy with β service level metrics

For the policy with β service level we repeat the methodology used for the policy with α service level and the same tests were made, with the objective to compare it to the traditional policy with β service level.

We have generated different coefficients of variance according to equation (4.51). The same procedure explained in 4.4.1 was used to obtain the coefficients of variance, and the percentages of demand of type 1.

The values for d_1 (μ_1, σ_1) to be used in the total cost function, equation (4.45), and in the iterative procedure, equations (4.46) and (4.47), are obtained though the formulas in 4.1.2. To compare the online policy (OR) to the traditional policy (TR) when using a specified fractional charge per unit short (B_2) the following input data was used:

A	30 €/replenishment
B_2	0.25
D	3000 units/year
r	0.24 €/€/year
v	12 €/unit

Figure 14 illustrates the cost reduction that provides the use of the new policy with β service level for the online retail. For each scenario, the optimal values of the pair (s, Q) were obtained through the iterative procedure, for both the traditional and the online policy, and the costs associated with those pairs were compared. Numerical results are presented in Annex C.

The conclusions we can take from the results are similar to the α service level setting: lower percentages of demand of type 1 are related to higher savings from using the new policy proposed for the online retail.

Concerning the results in respect to CV, the percentage of savings lowers for higher CV's. This was also observed in the α service level scenario and the explanation is the same.

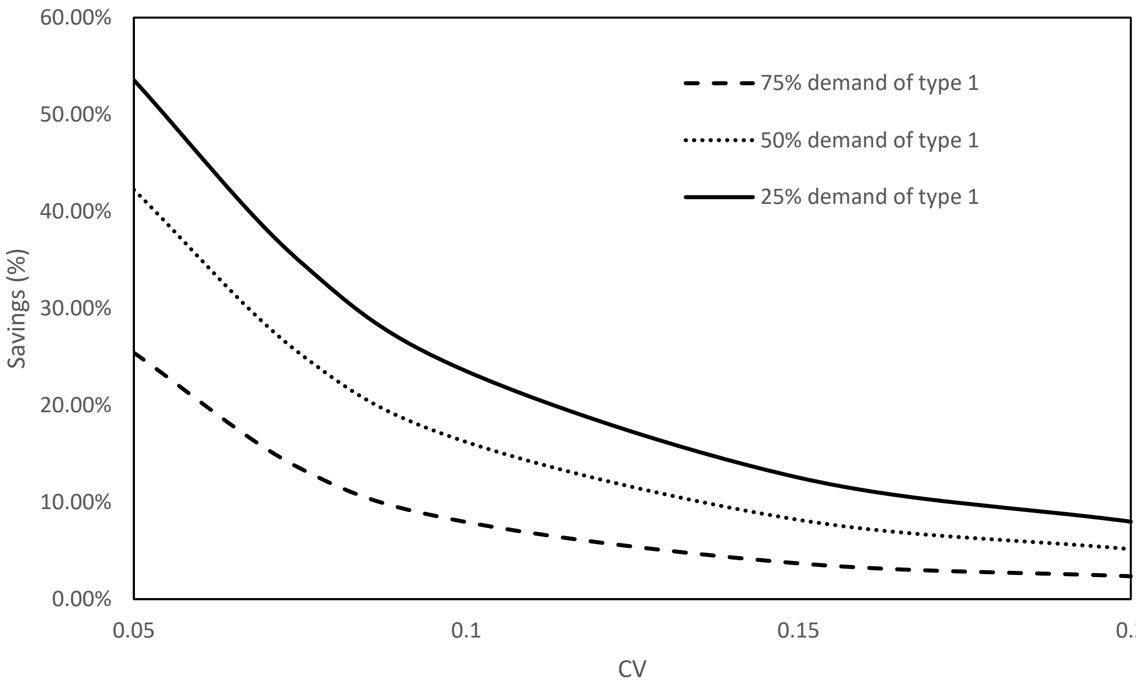

Figure 14 - Policies comparison with β service level with optimal pairs (s, Q) for different percentages of d_1

In a situation where the retailer cannot choose the order quantity, Figure 15 shows the percentage of savings for different possible orders quantity Q imposed by the supplier. Demand of type 1 is 25% or 75% of the total demand and CV varies between 0.05 and 0.2.

The conclusions are similar to the α service level policy. For the same percentage of demand of type 1, first the savings increase with the order quantity but above a certain quantity, the percentage of savings starts to decrease. For the same order quantity, lower CV's and lower percentages of demand of type 1 provide higher percentages of savings. Numerical results for three specific order quantities are available in Annex C.

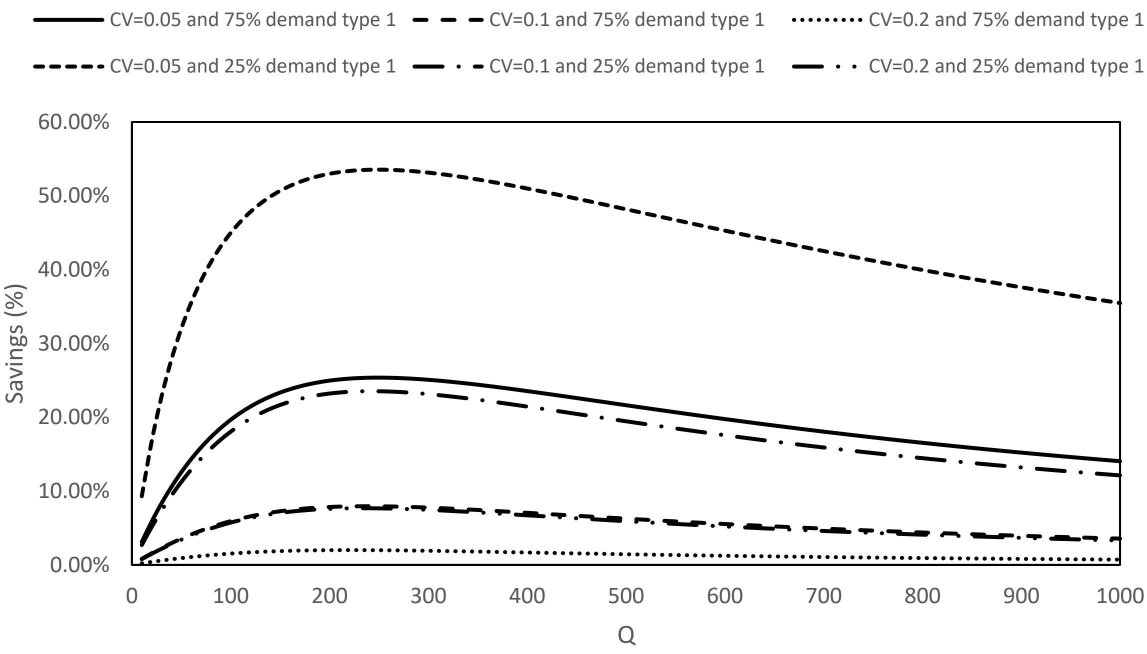

Figure 15 - Policies comparison with β service level for different values of the order quantity, different CV's and for different percentages of d_1

The contribution of different order windows was also studied with the same procedure explained in 4.4.1 for the α service level policy. The values used for the CV's and for the mean of the order windows are the same. Some of the results obtained in the experiments are present in Figure 16 through Figure 18. Numerical results are presented in Annex C.

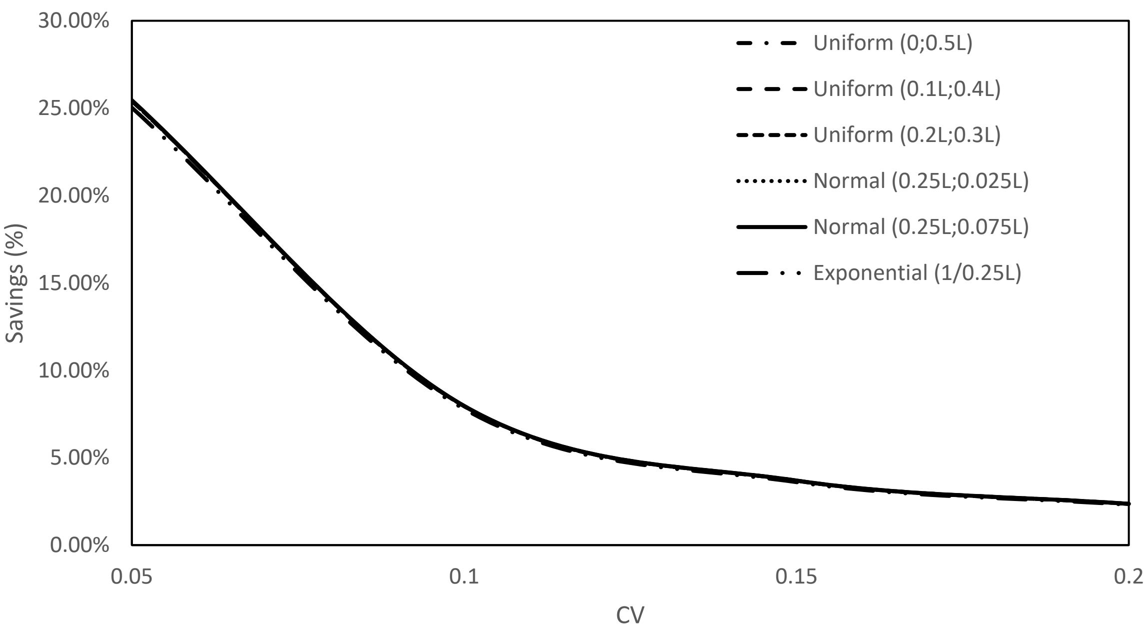

Figure 16 - Policies comparison with β service level for different order windows with mean $0.25L$

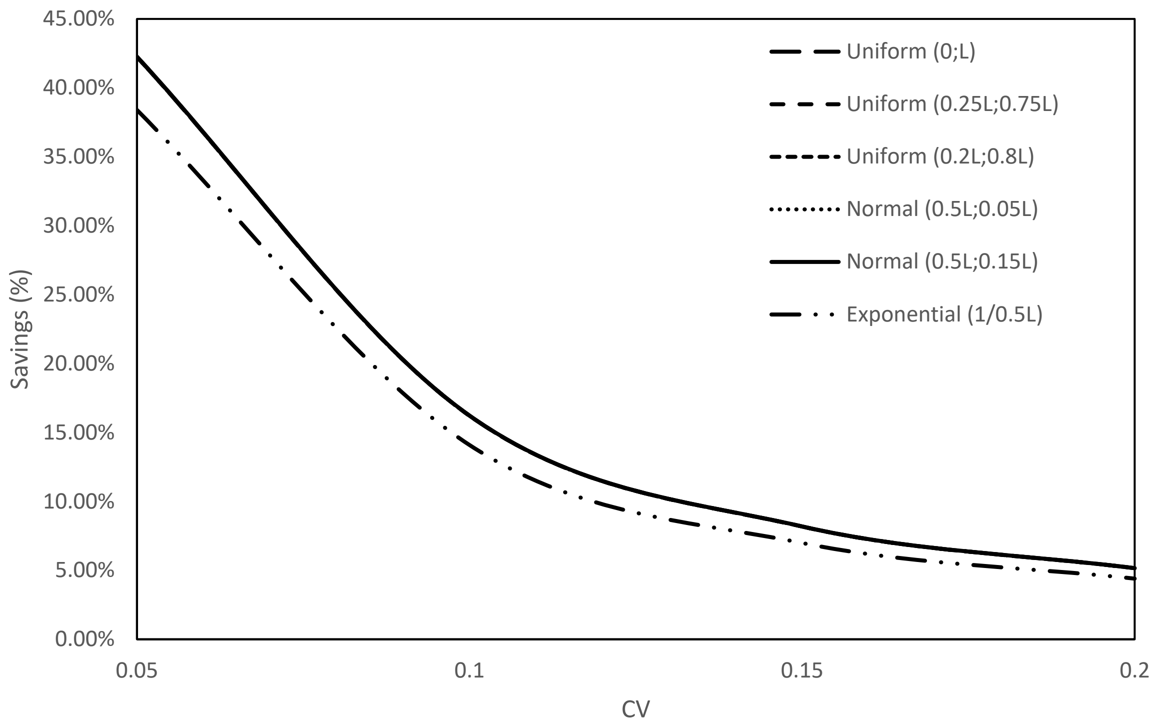

Figure 17 - Policies comparison with β service level for different order windows with mean $0.5L$

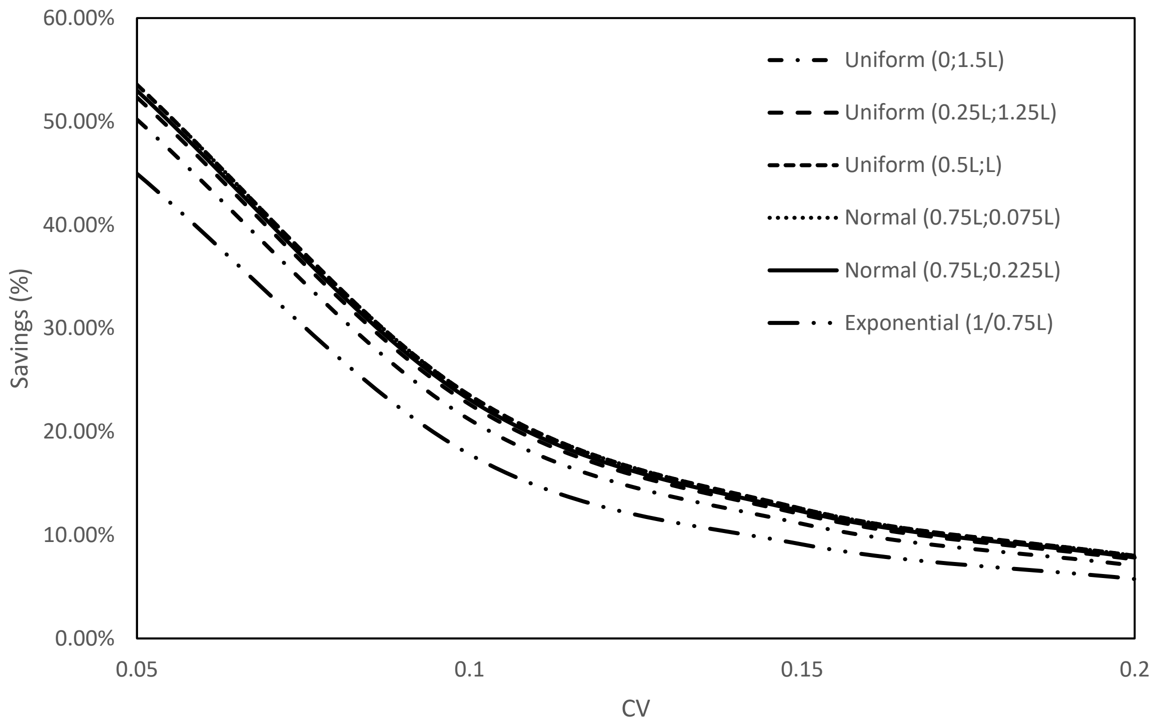

Figure 18 - Policies comparison with β service level for different order windows with mean $0.75L$

The conclusions we can take from analyzing Figure 16 through Figure 18 are similar to the ones made for the α service level policy. An exponential order window provides lower percentages of savings than normal or uniform distributed order windows and even order windows with the same distribution, uniform or normal, can present different percentages of savings due to differences in the parameters. Concerning the CV's of the demand, higher values provide lower percentages of savings.

To assess how the variation of the fixed costs affects the cost reduction from using the new policy for optimal values (s, Q), we realized experiments with different values of the costs A (ordering cost), v (variable unit cost), r (inventory carrying charge), B_2 (fractional charge per unit short) and different percentages of demand of type 1 for the scenario with CV = 0.1. The costs v and r were studied separately because, with β service level, the cost v affects the stockout cost while in the α service level scenario that does not happen. In all the experiments, the percentage of demand of type 1 varies between 1% and 100% however, for percentages below 10%, the results need to be analyzed taking into account that the approximation to the normal distribution may not be a good approximation.

The experiments realized allowed to take some conclusions. Analyzing Figure 19 lower values of A with lower values of the percentage of demand of type 1 provide higher percentage of savings. Lower values of A originate lower values of the optimal order quantity Q and higher values of the safety factor k, which implies a lower $G_u(k)$. Consequently, there are higher differences in the holding costs and lower differences in the stockout costs for low values of A. Therefore, higher percentages of savings are obtained.

From Figure 20, higher values of B_2 with lower values of the percentage of demand of type 1 provide higher percentages of savings. For the same percentage of demand of type 1, savings are due to increasing differences between holding costs and decreasing differences between stockout costs. These variations are due to higher optimal safety factors k and lower order quantities Q for higher values of B_2, which justifies the slightly increase in savings.

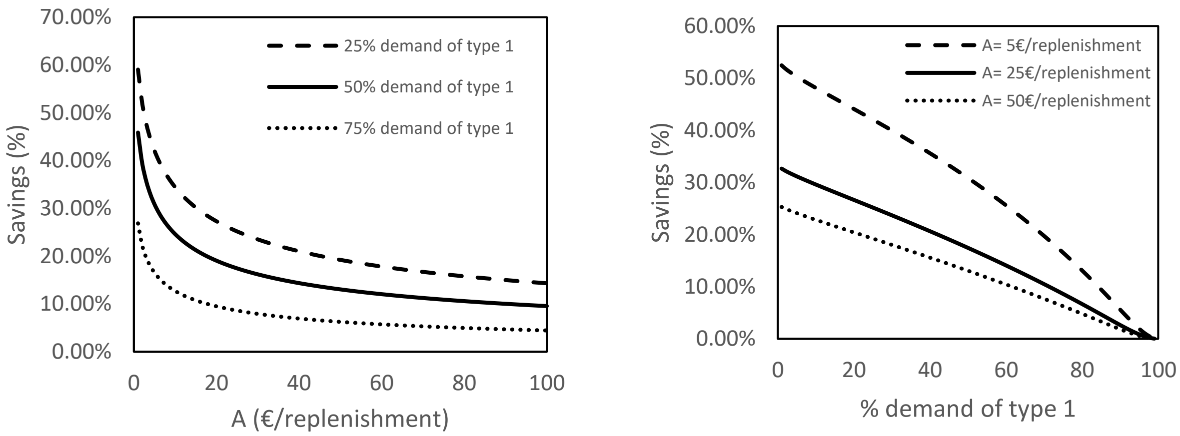

Figure 19 - Policies comparison with β service level for different percentages of d_1 and different values of A

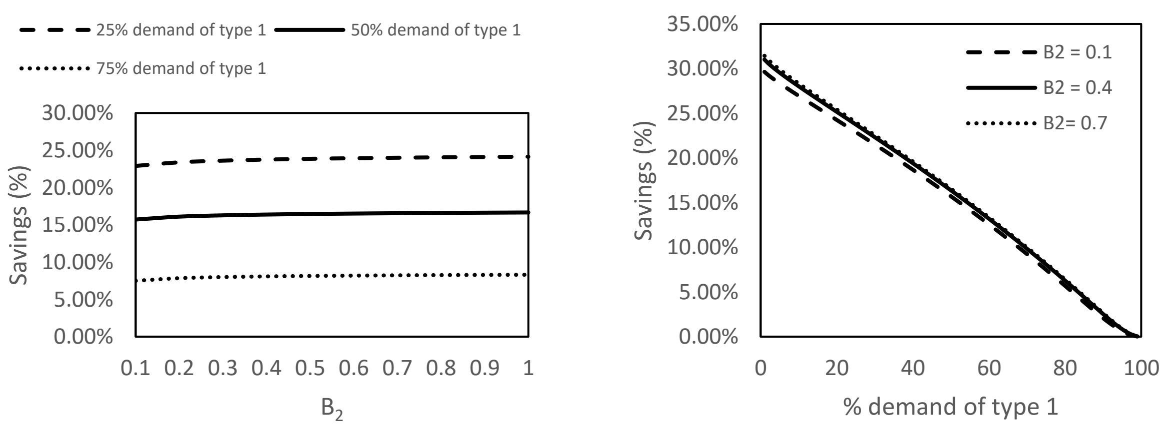

Figure 20 - Policies comparison with β service level for different percentages of d_1 and different values of B_2

Analyzing Figure 21, higher values of r combined with lower values of the percentage of demand of type 1 provide higher percentages of savings. This observation is explained by the fact that, for the same percentage of demand of type 1, the increasing differences between the holding costs are higher than the increasing differences between the stockout costs for increasing values of r. For the same r, lower percentages of demand of type 1 provide higher values of the safety factor k and lower order quantities Q, which result in higher percentages of savings.

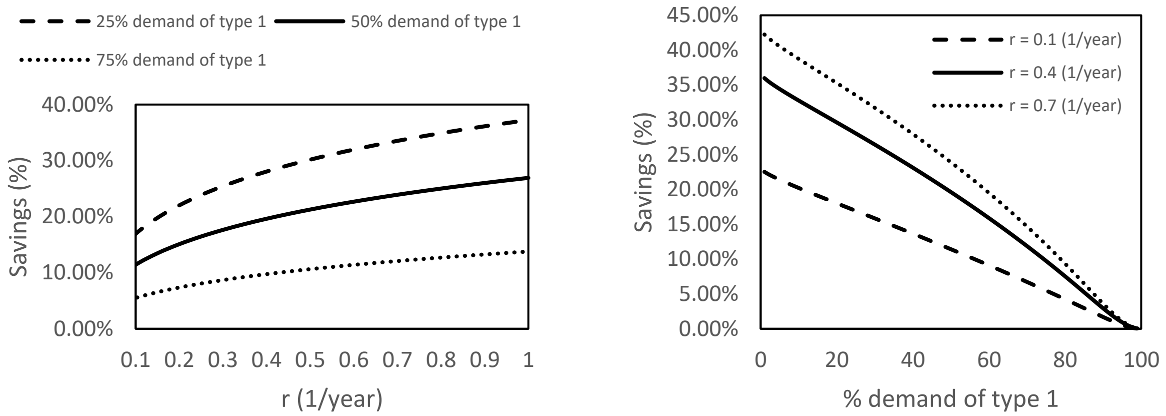

Figure 21 - Policies comparison with β service level for different percentages of d_1 and different values of r

Looking at Figure 22, higher values of v with lower values of the percentage of demand of type 1 provide higher percentages of savings. Higher values of v generate increasing differences between the holding costs that are higher than the increasing differences between the stockout costs, for the same percentage of demand of type 1. Also, for the same v, there are higher savings for lower percentages of demand of type 1 and the explanation is similar to the explanation for the r costs.

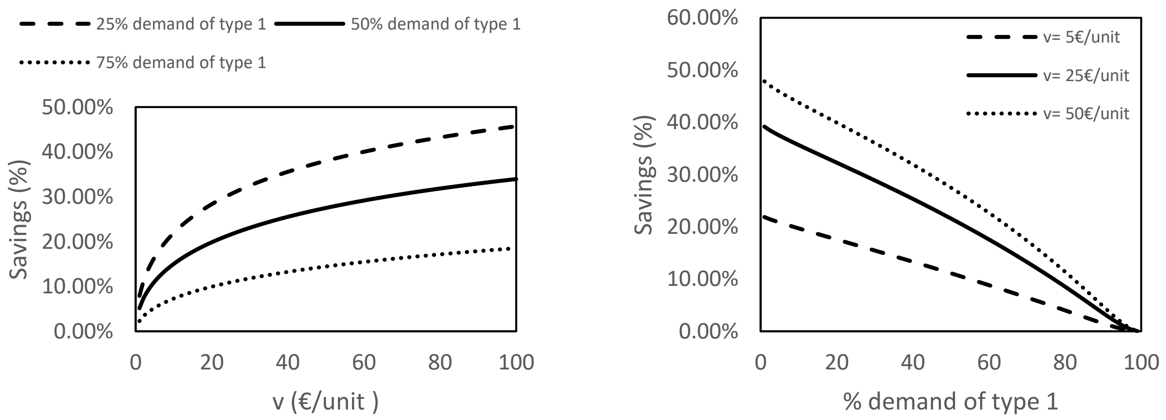

Figure 22 - Policies comparison with β service level for different percentages of d_1 and different values of v

5 Model development for periodic review

This chapter will provide the characterization of the demand in online retail for a (R, s, S) system and an online retail policy based in the traditional retail approach presented in 2.3. The demand during the protection period is divided in different types in a periodic review policy setting. Numerical applications are also presented in this chapter.

5.1 Demand context in online retail for a periodic $(\boldsymbol{R}, \boldsymbol{s}, \boldsymbol{S})$ review policy

In a periodic review, reorder point, order-up-to-level system (R, s, S), according to the work done by Silver et al. (2009), the key time period over which protection is required is of duration $L + \tau$, where τ is a random variable. The demand during the "effective" lead time (protection period) was divided in four types taking in account five moments:

- The time at which the customer places the order (*CO*);
- The time at which the customer wants the delivery (*CD*);
- The time between reviews - review interval (R_1);
- The time at which the order arrives (*A*);
- The time at which the inventory drops to the reorder point (*O*).

Figure 23 helps to understand how to classify the demand. The time between the replenishment and the order arrival is represented by L and it's the supplier's lead time. Inventory is examined every R units of time and τ is the time between the instant when the inventory position drops to the reorder point and the next review. The order window is defined as the time between the customer order (*CO*) and the delivery time (*CD*), that is, $OW = CD - CO$.

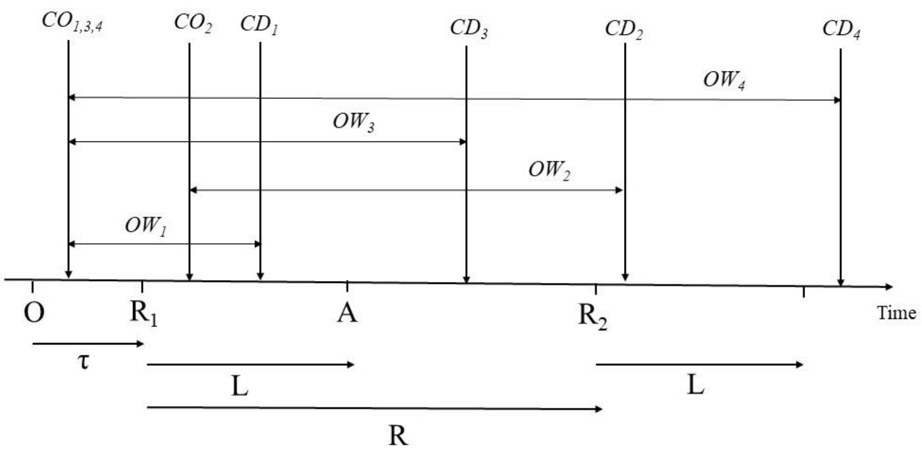

Figure 23 - Demand classification in a (R, s, S) policy

For simplicity, O will be set equal to 0 (the count of time starts when the inventory hits the reorder point) and A equal to $L+\tau$. When a customer places an order during the protection period ($0 < CO < L+\tau$) and establishes the time when the order must be received, the demand originated can be classified as follows:

- $\boldsymbol{d_1}$ - Demand of type 1: $0 < CO < L+\tau$ and $0 < CO + OW < L+\tau$

The delivery takes place before the end of the protection period, which means that the retailer can only fulfill the order with the stock on-hand (similar to the traditional retail scenario).

- $\boldsymbol{d_2}$ - Demand of type 2: $\tau \le CO < L+\tau$ and $L+\tau \le CO+OW < L+\tau+R$

The customer order takes place after the review time (during the lead time) and the delivery takes place after the end of the protection period but before the next replenishment (made in R_2) arrives, which means that the retailer can wait for the stock in-transit (of the replenishment made in R_1) to fulfill it. Demand of type 2 gives the retailer the opportunity to satisfy orders with in-transit inventory and use the on-hand inventory to satisfy the demand of type 1 (similar to demand of type 2 in a (s,Q) policy).

- $\boldsymbol{d_3}$ - Demand of type 3: $0 < CO < \tau$ and $L+\tau \le CO+OW < L+\tau+R$

The customer order takes place before the review time and the delivery takes place after the end of the protection period but before the next replenishment (made in R_2) arrives, which means the order window is big enough such that a new replenishment can be ordered and received by the retailer before the delivery time set by the customer. Demand of type 3 will never originate a stockout because we assume infinite inventory at the supplier (similar to demand of type 3 in a (s,Q) policy).

- $\boldsymbol{d_4}$ - Demand of type 4: $0 < CO < L+\tau$ and $CO+OW \ge L+\tau+R$

This type of demand is similar to the demand of type 3, but it was considered separately to facilitate calculations. The delivery takes place after the next replenishment (made in R_2) arrives. A new replenishment can be ordered and received by the retailer before the delivery time set by the customer. Demand of type 4 will never originate a stockout because there is infinite inventory to fulfill this type of orders.

5.2 Demand characterization

This section presents the approach followed to characterize the demand and the simulations done to validate the results obtained.

5.2.1 Main assumptions and notation

For the characterization of demand during the "effective" lead time we made the following assumptions:

1) We consider that the time between orders is exponentially distributed and demand in a unit time follows a Poisson distribution with constant rate.
2) The review interval (R) is constant.
3) The lead time (L) is constant.
4) The time between the instant when the inventory position drops to the reorder point and the next review (τ) has a uniform distribution between 0 and R. The customer order time (CO) has a uniform distribution between 0 and $L+\tau$.
5) All demand transactions are of unit size.

6) For the order window (OW) we consider two cases: the order window follows a uniform distribution or an exponential distribution.
7) The different types of demand are independent, OW and CO are also independent.

Although the density function of τ proposed by Silver et al. (2009) is given by equation (2.7), we made assumption 4) to facilitate our approach. Silver et al. (2009) state that the original distribution of τ tends to uniformity as both n and CV increase. They also made a similar assumption when they make some adjustments to the estimates of E[τ] and Var[τ] and consider the uniform distribution as the unmodified (original) distribution of τ.

For the characterization of demand during the "effective" lead time we use the following notation:

λ_{CO}	Average demand per unit time
L	Lead time
R	Review interval
τ	Time between the instant when the inventory position drops to the reorder point and the next review
μ_1	Average of demand of type 1 during the "effective" lead time
μ_2	Average of demand of type 2 during the "effective" lead time
μ_3	Average of demand of type 3 during the "effective" lead time
μ_4	Average of demand of type 4 during the "effective" lead time
μ_L	Average of total demand during the "effective" lead time
σ_1	Standard deviation of demand of type 1 during the "effective" lead time
σ_2	Standard deviation of demand of type 2 during the "effective" lead time
σ_3	Standard deviation of demand of type 3 during the "effective" lead time
σ_4	Standard deviation of demand of type 4 during the "effective" lead time
σ_L	Standard deviation of total demand during the "effective" lead time

5.2.2 Analytical approach

The approach followed in this section is similar to the explained in 4.1.2.

The total demand during the "effective" lead time is easily obtainable. If X is the total demand during the "effective" lead time, the expected value is $E[X] = E[D]E[L+\tau]$ and the variance is $Var[X] = Var[D]E[L+\tau] + E[D]^2Var[L+\tau]$ where D is the average demand per unit time and $L+\tau$ is the "effective" lead time. According to our notation:

$$\mu_L = \lambda_{CO}(E[L+\tau]) = \lambda_{CO}\left(L+\frac{R}{2}\right) \tag{5.1}$$

$$\sigma_L = \sqrt{\lambda_{CO}\left(L+\frac{R}{2}\right) + \lambda_{CO}^2\frac{R^2}{12}} \tag{5.2}$$

We use the bounds on the values that the order window (OW) and the customer order (CO) can take for each type of demand (section 5.1), to calculate the probability of an order being of each one of the demand types. See Figure 24 for a better understanding. According to the division done in section 5.1, an order is demand of type 1 if $0 < CO < L+\tau$ and $0 < CO + OW < L+\tau$, which according to Figure 24 is the region below the line $CO = L+\tau-OW$. The same method is applied to demand of type 2, type 3 and type 4. This representation makes it easier to identify the limits of integration for the expressions that allowed us to obtain the probabilities.

Following the previous explanation, we obtain the expressions:

$$P(d_1) = P(0 < CO < L+\tau \cap 0 < CO + OW < L+\tau) \tag{5.3}$$

$$P(d_2) = P(\tau < CO < L+\tau \cap L+\tau \leq CO + OW < L+\tau+R) \tag{5.4}$$

$$P(d_3) = P(0 < CO < \tau \cap L+\tau \leq CO + OW < L+\tau+R) \tag{5.5}$$

$$P(d_4) = P(0 < CO < L+\tau \cap CO + OW \geq L+\tau+R) \tag{5.6}$$

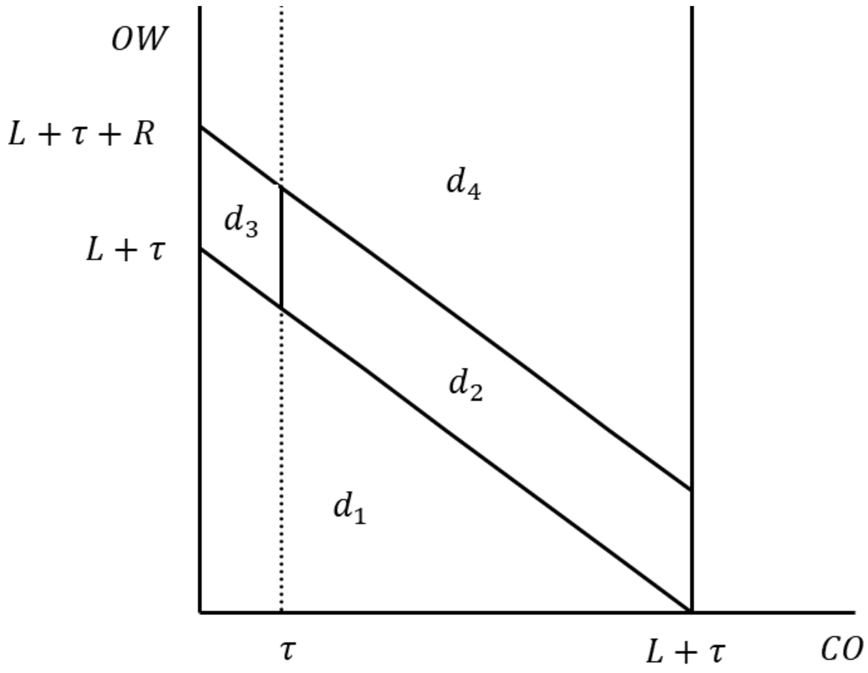

Figure 24 - 2D representation of the division of demand in a (R, s, S) policy

Once we obtain the probabilities, we know how the total demand is distributed among the four types of demand, so the average of each type of demand can be calculated as $\mu_x = P(d_x)\mu_L$.

The previous expressions led to the calculation of triple integrals (equation (5.7) through (5.11)) for each type of the demand. Details about the approach to obtain these expressions are presented in Annex D.

$$P(d_1) = \int_0^R \left[\int_0^{L+\tau} \int_0^{L+\tau-OW} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \right] f_{\tau w}(\tau) d_\tau \tag{5.7}$$

$P(d_2)\ if\ L \geq R$

$$\begin{aligned} P(d_2) = &\int_0^R \left[\int_0^R \int_{L+\tau-OW}^{L+\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \right] f_{\tau w}(\tau) d_\tau + \\ &\int_0^R \left[\int_R^L \int_{L+\tau-OW}^{L+\tau+R-OW} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \right] f_{\tau w}(\tau) d_\tau + \\ &\int_0^R \left[\int_L^{L+R} \int_{\tau}^{L+\tau+R-OW} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \right] f_{\tau w}(\tau) d_\tau \end{aligned} \tag{5.8}$$

$P(d_2)\ if\ L < R$

$$P(d_2) = \int_0^R \left[\int_0^L \int_{L+\tau-OW}^{L+\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \right] f_{\tau w}(\tau) d_\tau + \int_0^R \left[\int_L^R \int_\tau^{L+\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \right] f_{\tau w}(\tau) d_\tau + \int_0^R \left[\int_R^{L+R} \int_\tau^{L+\tau+R-OW} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \right] f_{\tau w}(\tau) d_\tau \tag{5.9}$$

$$P(d_3) = \int_0^R \left[\int_L^{L+\tau} \int_{L+\tau-OW}^{\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \right] f_{\tau w}(\tau) d_\tau + \int_0^R \left[\int_{L+\tau}^{L+R} \int_0^{\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \right] f_{\tau w}(\tau) d_\tau + \int_0^R \left[\int_{L+R}^{L+R+\tau} \int_0^{L+\tau+R-OW} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \right] f_{\tau w}(\tau) d_\tau \tag{5.10}$$

$$P(d_4) = \int_0^R \left[\int_R^{L+\tau+R} \int_{L+\tau+R-OW}^{L+\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \right] f_{\tau w}(\tau) d_\tau + \int_0^R \left[\int_{L+\tau+R}^{\infty} \int_0^{L+\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \right] f_{\tau w}(\tau) d_\tau \tag{5.11}$$

Where:

$$f_{CO}(CO) = \begin{cases} \dfrac{1}{L+\tau} & for\ CO \in [0; L+\tau] \\ 0 & otherwise \end{cases} \tag{5.12}$$

$$f_{\tau w}(\tau) = \begin{cases} \dfrac{L+\tau}{\left(L+\frac{R}{2}\right)R} & for\ \tau \in [0; R] \\ 0 & otherwise \end{cases} \tag{5.13}$$

The expressions for the probabilities change accordingly to the distribution assumed for the OW. Through equations (5.15) to (5.39) the results are presented for uniform and exponential distributions.

a) OW ~Uniform[a;b]

$$f_{OW}(OW) = \begin{cases} \dfrac{1}{b-a} & for\ OW \in [a; b] \\ 0 & otherwise \end{cases} \tag{5.14}$$

In this case we need to divide the results in various scenarios, because the limits for the integral of the OW are limited by the values of a and b. Trying to minimize the expressions we use the auxiliary notation presented in Annex D.

Demand of type 1:

- $b \leq L$

$$P(d_1) = \int_0^R \left[\int_a^b \int_0^{L+\tau-OW} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW}\right] f_{\tau W}(\tau) d_\tau \tag{5.15}$$

$$\mu_1 = \left[-\frac{a+b-2L-R}{2L+R}\right]\left(\lambda_{CO}\left(L+\frac{R}{2}\right)\right) \tag{5.16}$$

- $b > L$

$$P(d_1) = \int_Y^X \left[\int_a^{L+\tau} \int_0^{L+\tau-OW} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW}\right] f_{\tau W}(\tau) d_\tau + \int_X^R \left[\int_a^b \int_0^{L+\tau-OW} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW}\right] f_{\tau W}(\tau) d_\tau \tag{5.17}$$

$$\mu_1 = \left[\frac{3(a-b)(a+b-2L-R-X)(R-X) + (-a+L+X)^3 + (a-L-Y)^3}{3(-a+b)R(2L+R)}\right] \times \left(\lambda_{CO}\left(L+\frac{R}{2}\right)\right) \tag{5.18}$$

Demand of type 2:

- $L \geq R$

$$P(d_2) = \int_0^R \left[\int_O^N \int_{L+\tau-OW}^{L+\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW}\right] f_{\tau W}(\tau) d_\tau + \int_0^R \left[\int_D^J \int_{L+\tau-OW}^{L+\tau+R-OW} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW}\right] f_{\tau W}(\tau) d_\tau + \int_0^R \left[\int_C^H \int_\tau^{L+\tau+R-OW} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW}\right] f_{\tau W}(\tau) d_\tau \tag{5.19}$$

$$\mu_2 = \left[\frac{-C^2 + H^2 - N^2 + O^2 + 2DR - 2JR + 2C(L+R) - 2H(L+R)}{(a-b)(2L+R)}\right] \times \left(\lambda_{CO}\left(L+\frac{R}{2}\right)\right) \tag{5.20}$$

- $L < R$

$$P(d_2) = \int_0^R \left[\int_O^N \int_{L+\tau-OW}^{L+\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW}\right] f_{\tau W}(\tau) d_\tau + \int_0^R \left[\int_E^Q \int_\tau^{L+\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW}\right] f_{\tau W}(\tau) d_\tau + \int_0^R \left[\int_F^I \int_\tau^{L+\tau+R-OW} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW}\right] f_{\tau W}(\tau) d_\tau \tag{5.21}$$

$$\mu_2 = \left[\frac{-B^2 - F^2 + G^2 + \mathrm{I}^2 + 2\mathrm{E}L - 2\mathrm{I}L - 2LQ - 2\mathrm{I}R + 2F(L+R)}{(a-b)(2L+R)}\right] \times \left(\lambda_{CO}\left(L + \frac{R}{2}\right)\right) \tag{5.22}$$

Demand of type 3:

- $b > L$

$$P(d_3) = \int_0^Y \left[\int_T^b \int_{L+\tau-OW}^{\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW}\right] f_{\tau w}(\tau) d_\tau + \int_Y^X \left[\int_T^{L+\tau} \int_{L+\tau-OW}^{\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW}\right] f_{\tau w}(\tau) d_\tau \tag{5.23}$$

$$\mu_3 = \left[\frac{-3RT^2 + 3b^2(R-X) + X^3 + 6bL(-R+X)}{3(a-b)R(2L+R)} + \frac{6LT(R-Y) - 3L^2(X-Y) + 3T^2Y - Y^3}{3(a-b)R(2L+R)}\right]\left(\lambda_{CO}\left(L + \frac{R}{2}\right)\right) \tag{5.24}$$

- $a < L + R$

$$P(d_3) = \int_0^Y \left[\int_a^W \int_0^{\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW}\right] f_{\tau w}(\tau) d_\tau + \int_Y^X \left[\int_{L+\tau}^{W} \int_0^{\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW}\right] f_{\tau w}(\tau) d_\tau \tag{5.25}$$

$$\mu_3 = \left[\frac{-3WX^2 + 2X^3 + 3aY^2 - 2Y^3 + 3L(X^2 - Y^2)}{3(a-b)R(2L+R)}\right]\left(\lambda_{CO}\left(L + \frac{R}{2}\right)\right) \tag{5.26}$$

- $b > L + R$

$$P(d_3) = \int_Z^K \left[\int_M^{L+R+\tau} \int_0^{L+\tau+R-OW} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW}\right] f_{\tau w}(\tau) d_\tau + \int_K^R \left[\int_M^{b} \int_0^{L+\tau+R-OW} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW}\right] f_{\tau w}(\tau) d_\tau \tag{5.27}$$

$$\mu_3 = \left[\frac{(K + L - M + R)^3 - (L - M + R + Z)^3}{3(a-b)R(2L+R)} - \frac{(b-M)(b-K-2L+M-3R)(K-R)}{(a-b)R(2L+R)}\right]\left(\lambda_{CO}\left(L + \frac{R}{2}\right)\right) \tag{5.28}$$

Demand of type 4:

- $a < R$ and $R < b < L + R + \tau$

$$P(d_4) = \int_0^K \left[\int_R^{L+R+\tau} \int_{L+\tau+R-OW}^{L+\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW}\right] f_{\tau w}(\tau) d_\tau + \int_K^R \left[\int_R^{b} \int_{L+\tau+R-OW}^{L+\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW}\right] f_{\tau w}(\tau) d_\tau \tag{5.29}$$

$$\mu_4 = \left[\frac{-L^3 + (K+L)^3}{3(a-b)R(2L+R)} + \frac{(b-R)^2(-K+R)}{2(-a+b)\left(L + \frac{R}{2}\right)R}\right]\left(\lambda_{CO}\left(L + \frac{R}{2}\right)\right) \tag{5.30}$$

- $a \geq R$ and $R < b < L + R + \tau$

$$P(d_4) = \int_P^K \left[\int_a^{L+R+\tau} \int_{L+\tau+R-OW}^{L+\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \right] f_{\tau w}(\tau) d_\tau + \int_K^R \left[\int_a^b \int_{L+\tau+R-OW}^{L+\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \right] f_{\tau w}(\tau) d_\tau \quad (5.31)$$

$$\mu_4 = \left[-\frac{(K-P)(K^2+3L^2+3LP+P^2+K(3L+P)-3(a-R)^2)}{3(a-b)R(2L+R)} - \frac{(a-b)(a+b-2R)(-K+R)}{2(-a+b)\left(L+\frac{R}{2}\right)R} \right] \left(\lambda_{CO} \left(L + \frac{R}{2} \right) \right) \quad (5.32)$$

- $b \geq L + R + \tau$

$$P(d_4) = \int_0^P \left[\int_a^b \int_0^{L+\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \right] f_{\tau w}(\tau) d_\tau + \int_P^K \left[\int_{L+R+\tau}^b \int_0^{L+\tau} f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \right] f_{\tau w}(\tau) d_\tau \quad (5.33)$$

$$\mu_4 = \left[\frac{LP}{(L+\frac{R}{2})R} + \frac{P^2}{2(L+\frac{R}{2})R} + \frac{(K-P)(-3b(K+2L+P)}{3(a-b)R(2L+R)} + \frac{2(K^2+3L^2+3LP+P^2+K(3L+P))+3(K+2L+P)R)}{3(a-b)R(2L+R)} \right] \left(\lambda_{CO} \left(L + \frac{R}{2} \right) \right) \quad (5.34)$$

b) OW ~Exponential(λ_{OW})

$$f_{OW}(OW) = \begin{cases} \lambda_{\text{ow}} e^{-\lambda_{\text{ow}} OW} & for\ OW \geq 0 \\ 0 & for\ OW < 0 \end{cases} \quad (5.35)$$

Using expressions (5.7) through (5.11), the results obtained are:

$$\mu_1 = \left[\frac{2\left(\frac{e^{-\lambda_{OW}(L+R)}\left(-1+e^{\lambda_{OW}R}\right)}{\lambda_{OW}^2} - \frac{R}{\lambda_{OW}} + LR + \frac{R^2}{2} \right)}{R(2L+R)} \right] \left(\lambda_{CO} \left(L + \frac{R}{2} \right) \right) \quad (5.36)$$

$$\mu_2 = \left[\frac{2e^{-\lambda_{OW}(L+R)}\left(-1+e^{\lambda_{OW}L}\right)\left(-1+e^{\lambda_{OW}R}\right)}{\lambda_{OW}(2L+R)} \right] \left(\lambda_{CO} \left(L + \frac{R}{2} \right) \right) \quad (5.37)$$

$$\mu_3 = \left[\frac{2e^{-\lambda_{OW}(L+2R)}\left(-1+e^{\lambda_{OW}R}\right)\left(1+e^{\lambda_{OW}R}(-1+\lambda_{OW}R)\right)}{\lambda_{OW}^2\ R(2L+R)} \right] \times \left(\lambda_{CO} \left(L + \frac{R}{2} \right) \right) \quad (5.38)$$

$$\mu_4 = \left[\frac{2e^{-\lambda_{OW}(L+2R)}\left(1-e^{\lambda_{OW}R}+e^{\lambda_{OW}(L+R)}\lambda_{OW}R\right)}{\lambda_{OW}^2 R(2L+R)} \right] \left(\lambda_{CO} \left(L + \frac{R}{2} \right) \right) \quad (5.39)$$

Concerning the standard deviation of each type of demand, we couldn't get an expression. This should be studied in a future work.

5.3 Validation of results through simulation

In order to validate the results obtained in the analytical approach presented in 5.2.2, we modified the simulator used for continuous review to deal with the periodic review system. The simulator returns the percentage of each type of demand, its average and standard deviation and the coefficient of correlation between each type of demand. The inputs in this simulator are the OW distribution parameters, the average demand per unit time (λ_{CO}), the lead time (L) and the review interval (R).

5.3.1 Simulation design

The simulator explained in 4.2.1 was modified to be used for the new classification of demand in the (R, s, S) policy. The procedure is the same: we simulate 2000 orders with the time between orders following an exponential distribution with mean $\frac{1}{\lambda_{CO}}$ and then the simulator counts the total number of orders and the number of orders that are from each type of demand during the "effective" lead time.

The simulator uses 10000 iterations and was performed for 50 different combinations of inputs for each one of the 2 possibilities of the OW distribution.

Table 10 - Possible input values considered for the review interval

Review interval		
$\boldsymbol{R}$	$1, 2, 3, 4, 5, 6, 7, 8, 9$	UT

The values for the 50 combinations used in the simulations were chosen randomly from the values in Table 4, Table 5, Table 6 and Table 10 without allowing repeated combinations to ensure that all possible scenarios were tested.

5.3.2 Results

Here the expressions obtained in the analytical approach are validated by comparison with the simulated results. Table 11 provides the mean absolute percentage error (MAPE) between the simulated results and the results obtained using the expressions obtained in 5.2.2.

Table 11 – Validation results for periodic review

Error (%)						
OW	μ_1	μ_2	μ_3	μ_4	μ_L	σ_L
Uniform	0.62%	0.14%	0.39%	0.08%	0.20%	0.39%
Exponential	0.29%	0.32%	0.90%	1.37%	0.18%	0.44%

It has been possible to verify the analytical approach with the simulation done. The average percentage of error between the experimental and the analytical values is very small, except for the values for the demand of type 3 and 4 when the OW has an exponential distribution. These errors are due to very low values of demand where they have decimal values (both the demand obtained in the simulation and by the expressions have values of order 10^{-2} or lower when the probability of being demand of type 3 or type 4 is close to zero and the total demand has low values).

With the simulation done we could not verify assumption 7) because the types of demand seem to be dependent, as the coefficients of correlation between demand of type 1, 3 and 4 are not negligible. This dependence may be because the probability of these 3 types of demand are dependent on the value of τ (see the difference in the limits of integration of the probabilities in 5.2.2).

Here we present an approximation for the standard deviation of demand of type 1, in order to use it in a numerical study.

$$\sigma_1 = C\sqrt{\lambda_{CO}\left(L+\frac{R}{2}\right)P(d_1)+\lambda_{CO}^2\,\frac{R^2}{12}\,P(d_1)} \tag{5.40}$$

Where:

- $C = 1.17$ if OW~Uniform
- $C = 1.04$ if OW~Exponential

This approximation was obtained by multiplying the variance of the total demand by the probability of type 1 and a constant. The value for the constant was obtained by searching for the value that minimizes the MAPE between the simulated standard deviations of demand of type 1 and the values obtained with this formula. When the OW follows a uniform distribution we only used the instances where the lower bound of the uniform distribution is 0, since for positive values the ratio of d_1 starts to change considerably. The errors of using this approximation are 3.2% for the uniform OW and 6.7% for the exponential OW. An extension of the expression for positive lower bounds of the uniform distribution would be a good direction for future work.

In the numerical study we assume the demand in the protection period follows a normal distribution, an assumption also made by Silver et al. (2009) in their work. In Annex E we present an assessment of these assumptions.

5.4 Numerical study

In this section, a numerical study is presented so that conclusions can be taken from the results obtained. The objective of this section is to assess how the online policy compares to the traditional policy and how it behaves with different parameters. We evaluate the values of the traditional retail policy into the online policy and compare the results to assess the benefits for the retailer of using the adapted policy.

5.4.1 Methodology

In this policy, instead of explicitly incorporating costs, we specify the average time between consecutive replenishments and a fill rate, so the comparison of policies will not be done by measuring the costs reduction. Instead we will measure the reduction in the average on-hand inventory obtained by using the adapted policy. We also study the impact of different order windows in the reduction in the average on-hand inventory,

Because we assume the demand per unit time follows a Poisson distribution in our analytical approach and because Table 2 and Table 3 can only be used for CV's (for the demand per unit time) between 0.1 and 0.5 , in the experiments done we used CV's that range from CV = 0.1 ($\mu = 100$) and CV = 0.25 ($\mu = 16$).

The experiments done show the reduction in the average OH inventory for different percentages of demand of type 1, for different values of the fill rate, for different values of the average time between consecutive replenishments and for different order windows. In all the experiments L is equal to 4 and R is equal to 1.

The approach presented in 2.3 is used to calculate s and S for the traditional retail. Then we adapted the approach and used it to calculate s and S for the online policy using equations (5.41) through (5.46):

1) Obtain E[τ] and Var[τ] from Table 2 and Table 3
2) Calculate the mean and variance of d_1

$$E[d_1] = \mu_1 = (E[\tau] + L)\mu P(d_1) \tag{5.41}$$

$$Var[d_1] = \sigma_1^2 = min\left[\left((E[\tau] + L)\sigma^2 P(d_1) + \mu^2 Var[\tau]\, P(d_1)\right) C^2;\ \sigma_L^2\right] \tag{5.42}$$

3) Calculate the target allowed (average) units short per replenishment cycle (AUSPRC)

$$AUSPRC = (1 - \beta)E[Q] \tag{5.43}$$

4) Choose k to satisfy

$$G_{u_1}(k) = \frac{(1-\beta)E[Q]}{\sigma_1} = \frac{(1-\beta)n\mu}{\sigma_1} \tag{5.44}$$

5) Calculate the reorder point

$$s = \mu_1 + k\sigma_1 \tag{5.45}$$

6) Calculate the order-up-to level

$$S = s + n\mu - E[\tau]\mu \tag{5.46}$$

5.4.2 Comparison of policies

In this sub-section we present and discuss the results obtained in the experiments done, where the traditional policy was compared with the online policy. We made experiments separately for exponential distributed OW and for uniform distributed OW because the standard deviation of demand of type 1 is calculated differently.

In the experiments done for different values of *n* the fill rate used is 95% and the values for the CV are 0.1 or 0.2. Analyzing Figure 25 and Figure 26, lower values of *n* combined with lower percentages of demand of type 1 provide higher percentages of reductions in the average OH inventory. Lower values of *n* originate lower values of $E[Q]$, and consequently lower values of $G_u(k)$. Lower values of $G_u(k)$ are related to higher values of the safety factor. Because the difference between the safety factor of both policies is higher for lower values of *n*, the percentage of reduction is higher. Numerical results are presented in Annex F.

In the experiments done for different values of β the average time between consecutive replenishments is 4 and the values for the CV are 0.1 or 0.2. Analyzing Figure 27 and Figure 28, lower values of β combined with lower percentages of demand of type 1 provide higher percentages of reductions in the average OH inventory. Lower values of β are related to lower values of safety stock to meet the desired fill rate and because the difference between the safety factor of both policies is higher for lower values of β, the percentage of reduction is higher. Numerical results are presented in Annex F.

The difference between the averages and the standard deviations of demand of type 1 and the total demand is lower for higher CV's which explains why lower CV's originate higher percentages of reductions.

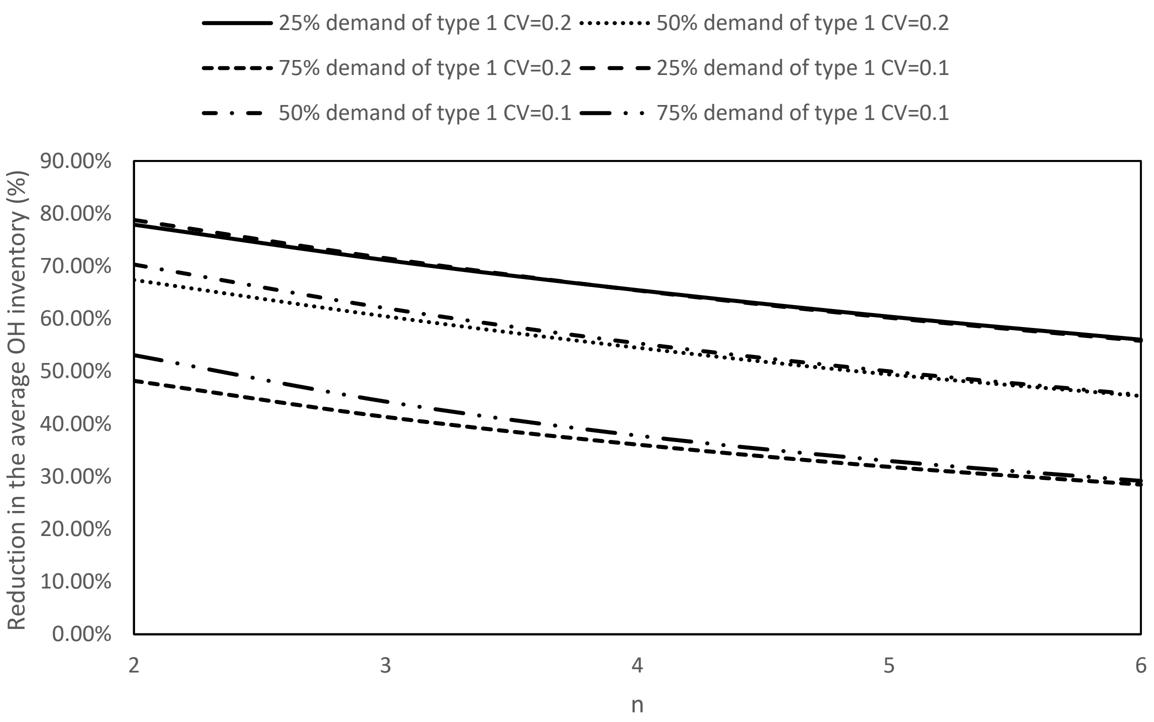

Figure 25 - Policies comparison for different percentages of d_1, different CV's and with different values of n when the OW follows a uniform distribution

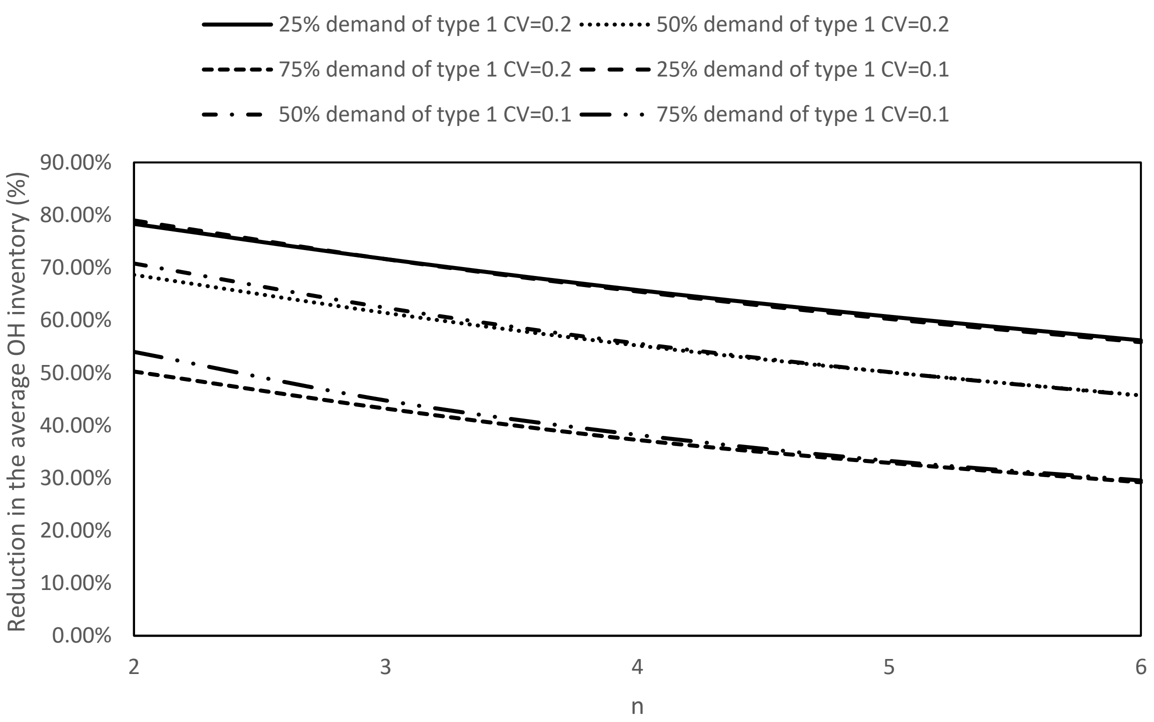

Figure 26 - Policies comparison for different percentages of d_1, different CV's and with different values of n when the OW follows an exponential distribution

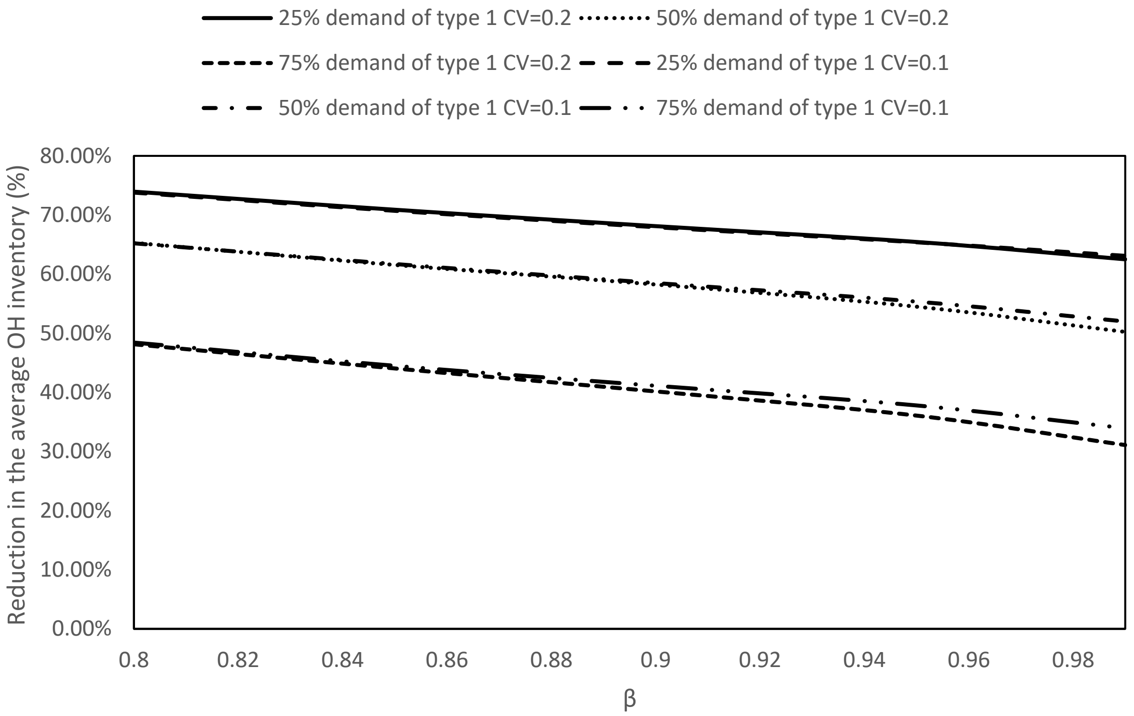

Figure 27 - Policies comparison for different percentages of d_1, different CV's and with different values of β when the OW follows a uniform distribution

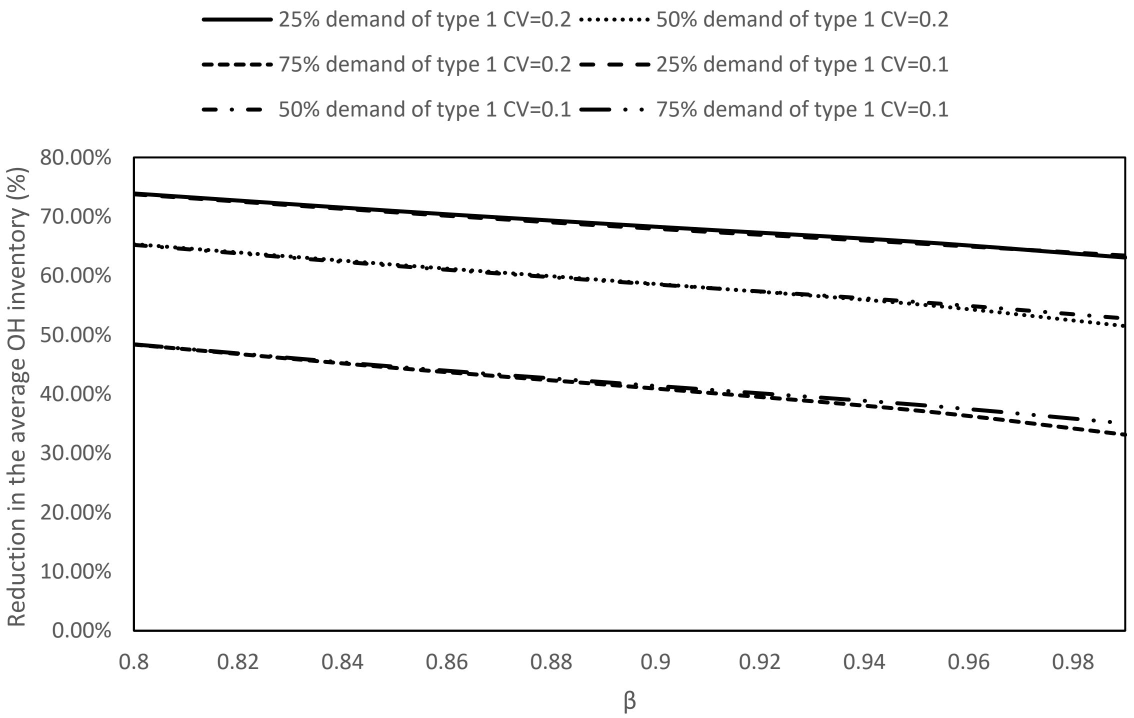

Figure 28 - Policies comparison for different percentages of d_1, different CV's and with different values of β when the OW follows an exponential distribution

The contribution of different order windows is also studied. We tested different scenarios where the CV varies between 0.1 and 0.25. The policies are compared when the order window has the same mean but different parameters or distributions. The values tested for the mean of the order window were: $0.25E[L+\tau]$, $0.5E[L+\tau]$ and $0.75E[L+\tau]$. In these experiments, the fill rate is 95% and the average time between consecutive replenishments is 4. Some of the results obtained in the experiments are presented in Figure 29 through Figure 31. Numerical results are presented in Annex F.

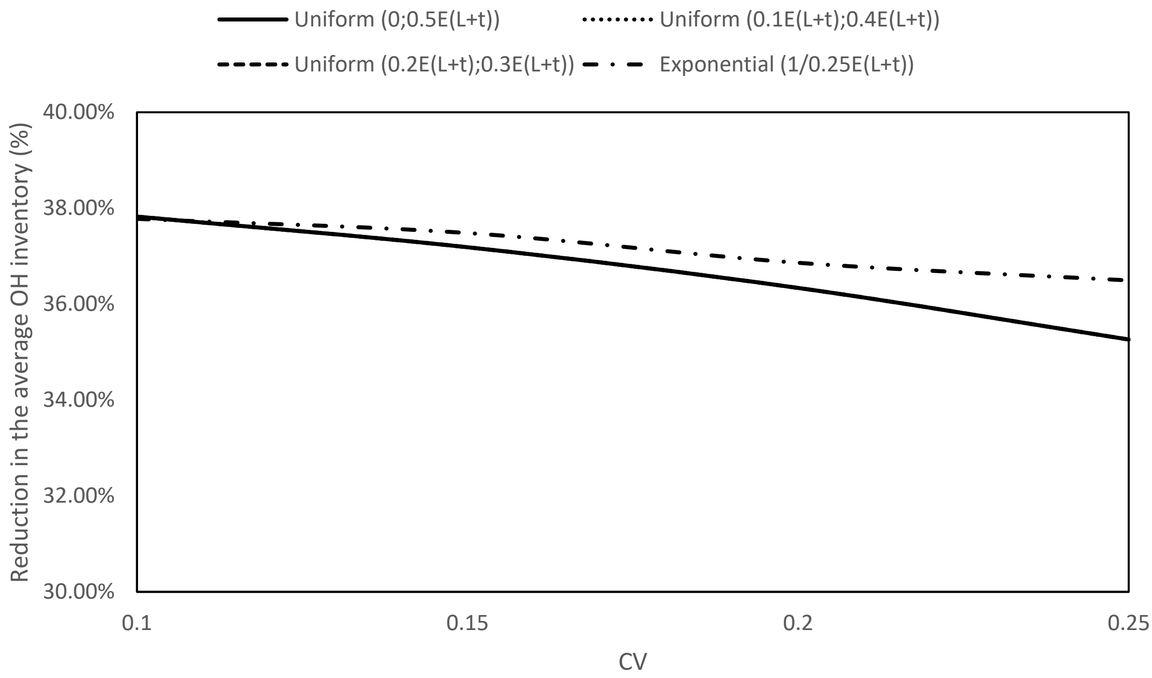

Figure 29 - Policies comparison for different order windows with mean $0.25E[L+\tau]$

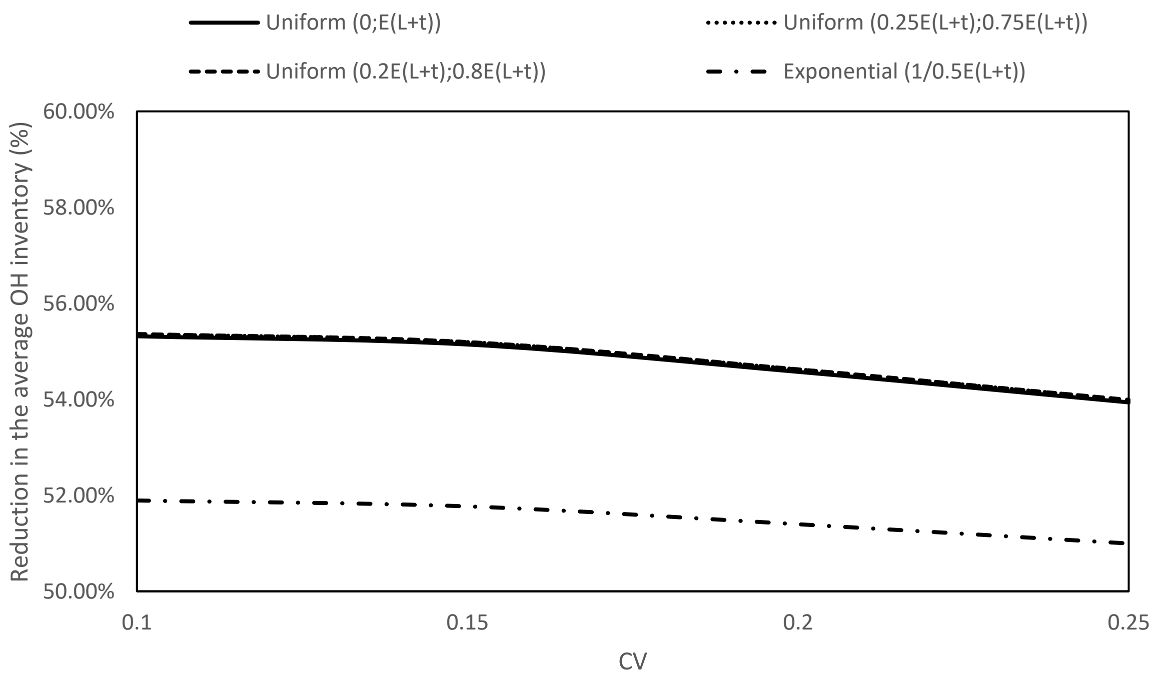

Figure 30 - Policies comparison for different order windows with mean $0.5E[L+\tau]$

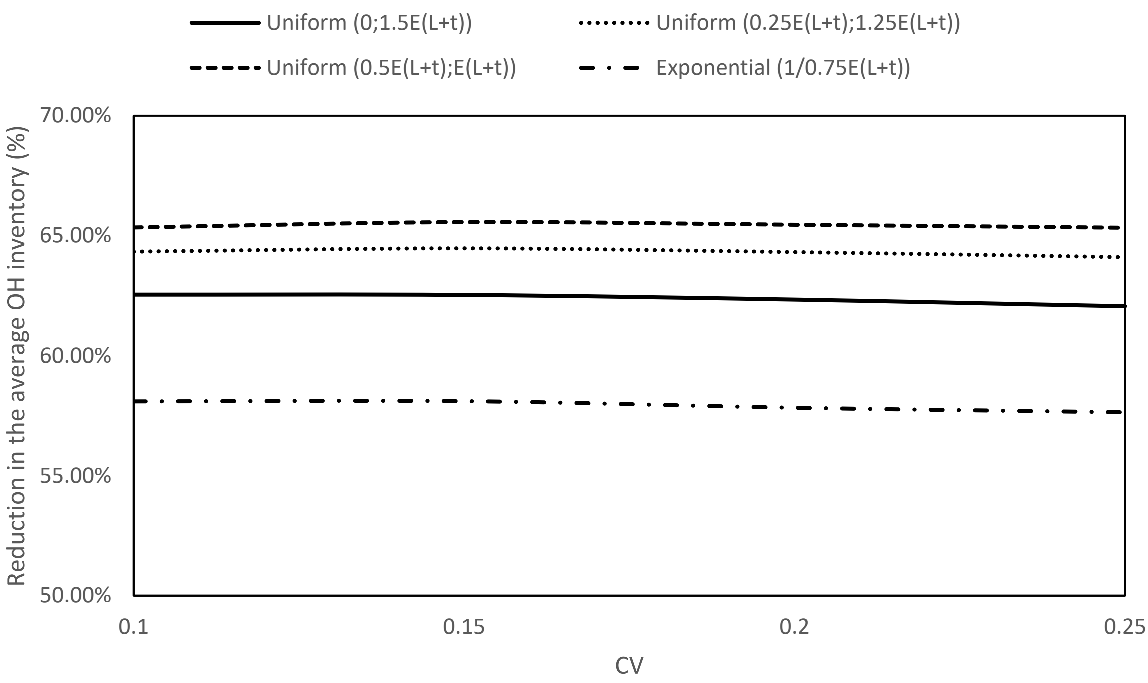

Figure 31 - Policies comparison for different order windows with mean $0.75E[L + \tau]$

For order windows with the same mean, an exponential order window provides lower percentages of reduction in the average OH inventory than uniform distributed order windows, except in the scenario with mean $0.25E[L + \tau]$. The exception is due to the higher differences in the calculation of the standard deviation between uniform and exponential order windows, when the percentage of demand of type 1 is higher, which corresponds to order windows with lower means. For other scenarios, an exponential distribution provides lower percentages of reduction because its mean is higher than the median.

Even order windows with the same distribution, can present different percentages of reductions due to differences in the parameters. For uniform distributed order windows, higher values of the mean and lower differences between the lower and upper bounds (lower variances), provide higher percentages of reduction. Concerning the CV's of the demand per unit time, higher values provide lower percentages of reduction.

6 Conclusions and future work

Retailers have been expanding business with the use of online channels, and have generated the online retailing industry. In every sector retailers are improving the ability of meeting customers' expectations with improved supply chains and inventory management policies. This work addresses a critical aspect related to the grocery sector where it is common for customers to choose a specific date of delivery because the presence of the customer is critical upon the delivery of goods. The time window provided by these orders provides additional flexibility and an opportunity for retailers to make better inventory management decisions.

We introduce the concept of commitment upon arrival, where we consider that retailers will only commit if there is available stock (on-hand and on-order, subtracted by already committed stock). This approach is what distinguishes our work from the existing ADI papers. ADI research motivation comes, specially, from the B2B setting, by motivating the members of the supply chain to share information among them. In this setting there exists a distinction on when the order is announced and when the order is due. Our motivation arises from the B2C, specifically from the grocery sector, where there is no such distinction. In the ADI research it is assumed that retailers commit to every customer orders irrespective of their product availability, and only afterwards decide which orders to fulfill. This is a very important aspect for customers, they would not like to get their orders cancelled or delayed when the delivery date is due. Our approach does not allow that to happen.

Our (s, Q) adapted policy for continuous review uses the order window to classify the demand in different types, which allows the retailer to take advantage of the flexibility provided by the order window. Our experiments show that, when compared to the traditional retail (s, Q) optimal policy, our policy provides savings that, on average, range between 11.91% and 28.50% with α service level metrics and between 11.91% and 28.36% with β service level metrics. The results were obtained for values in which the percentage of total demand that can use the in-transit stock to satisfy orders varies between 25% and 75% and for a given set of costs.

The influence of the order window and the costs on the savings was tested and our experiments show that the values for savings presented above can change significantly if the given costs are changed or when the order window changes.

Our experiments show that exponential distributed order windows provide lower percentage of savings than normal or uniform distributed order windows. We also concluded that higher percentages of savings are obtained when the mean value of the order window is higher, because there is higher probability of an order being demand of type 2 or type 3.

When the given costs are changed, we concluded that higher percentages of savings are obtained for lower values of A, higher values of B_1 and higher values of H when using α service level metrics. For β service level metrics, higher percentages of savings are obtained for lower values of A, higher values of B_2 and higher values of r and v.

If the supplier imposes the quantity that the retailer can order, then lower CV's and lower percentages of demand of type 1 provide higher percentages of savings for both service level metrics.

Our (R, s, S) adapted policy uses the order window to classify the demand in different types, which allows the retailer to take advantage of the flexibility provided by the order window.

Our experiments show that lower values of n combined with lower percentages of demand of type 1 provide higher percentages of reductions in the average on-hand inventory. We also concluded that lower values of β combined with lower percentages of demand of type 1 provide higher percentages of reductions in the average on-hand inventory.

When compared to the traditional (R, s, S) policy, our policy provides reductions in the average on-hand inventory that, on average, range between 38.31% and 68.32%. The results were obtained for values in which the percentage of total demand that can use the in-transit stock to satisfy orders varies between 25% and 75%.

Regarding the experiments with the (R, s, S) adapted policy, the contribution of different order windows was tested. The observations are similar to the observations in the (s, Q) adapted policy. Different observations may appear for higher values of the percentage of demand of type 1, which are related to the expression used for the standard deviation of demand of type 1 in the (R, s, S) policy.

When assessing the assumptions made for the (R, s, S), for the same scenario, higher values of L provide lower errors for the fill rate which may suggest that for higher values of L the real distribution of demand in the protection period comes closer to the normal distribution. The errors we obtained in the experiments, concerning the fill rate, range from 0.10% to 5.27% in the MOO scenario and from 0.59% to 9.47% in the SOO scenario.

Recall that, although the parameters of other types of demand other than demand of type 1 are not needed in our policy, these types of demand are very important for the retailer because it tells the retailer if he can or cannot commit to an order using the in-transit inventory.

There are several paths to continue and extend this line of research. Future work may include different distributions for the demand, variable lead time and non-stationary demand. This last extension is critical, since real data clearly indicates that demand significantly changes along the week, the month and the year. Indeed, the day of the week significantly impacts both the arrival of orders and the order window.

One limitation of our approach was considering Poisson demand to characterize the different types of demand and then approximate it to normal distributed demand. Although this approach is relatively common in the literature related to inventory management, it limited the values of the coefficients of variance we could test. Another approach to characterize the demand may be another step for future work, and different distributions for the order window can also be tested.

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ANNEX A: Another approach to obtain the parameters of demand during the lead time

As the time goes by from 0 to L, it is obvious that for a certain distribution of the *OW* the probability of an order being of type 1 decreases and the probability of being type 2 increases. Here we follow the same approach presented in 4.1.2, but using Figure A1, in order to obtain the parameters depending on time.

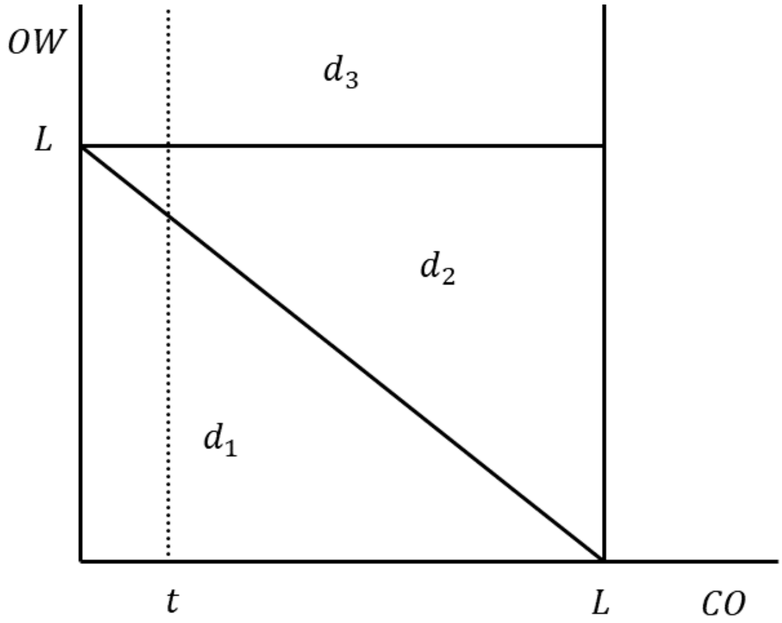

Figure A1 - 2D representation of the division of demand in a (s, Q) policy in *t*

Now, the *CO* can take values between t and L so it follows a uniform between t and L.

$$f_{CO}(CO) = \begin{cases} \dfrac{1}{L-t} & for\ CO \in [t; L] \\ 0 & otherwise \end{cases} \tag{1}$$

Multiplying the probabilities by the rate we obtain the rate of each type of demand:

$$\lambda(t) = \lambda_{CO} P(t)_{dx} \tag{2}$$

Then we can calculate the mean value of the demand through time *t* by:

$$m(t) = \int_0^t \lambda(s) ds \tag{3}$$

The expressions for the probabilities change accordingly to the distribution assumed for the *OW*.

a) *OW* ~Uniform[*a;b*]

$$f_{OW}(OW) = \begin{cases} \dfrac{1}{b-a} & for\ OW \in [a; b] \\ 0 & otherwise \end{cases} \tag{4}$$

In this case we need to divide the results:

- $b < L$

I. If $0 < t < L - b$

$$P(t)_{d1} = \int_a^b \int_t^L f_{OW}(OW) f_{CO}\,(CO) d_{CO}\, d_{OW} = 1 \tag{5}$$

$$P(t)_{d2} = \int_{L-t}^{L-t} \int_t^L f_{OW}(OW) f_{CO}\,(CO) d_{CO}\, d_{OW} = 0 \tag{6}$$

$$P(t)_{d3} = 0 \tag{7}$$

$$\mu_1 \rightarrow m(t) = \int_0^t \lambda(s) ds = \lambda_{CO} t \tag{8}$$

$$\mu_2 \rightarrow m(t) = 0 \tag{9}$$

$$\mu_3 \rightarrow m(t) = 0 \tag{10}$$

II. If $L - b < t < L - a$

$$P(t)_{d1} = \int_a^{L-t} \int_t^L f_{OW}(OW) f_{CO}\,(CO) d_{CO}\, d_{OW} = \frac{L - t - a}{b - a} \tag{11}$$

$$P(t)_{d2} = \int_{L-t}^{b} \int_t^L f_{OW}(OW) f_{CO}\,(CO) d_{CO}\, d_{OW} = \frac{b - L + t}{b - a} \tag{12}$$

$$P(t)_{d3} = 0 \tag{13}$$

$$\begin{aligned} \mu_1 \rightarrow m(t) &= (8) + \int_{L-b}^{t} \lambda(s) ds \\ &= (8) + \frac{(b - L + t)(2a - b - L + t)}{2(a - b)} \lambda_{CO} \end{aligned} \tag{14}$$

$$\mu_2 \rightarrow m(t) = \int_{L-b}^{t} \lambda(s) ds = \frac{(b - L + t)^2}{(b - a)}\ \lambda_{CO} \tag{15}$$

$$\mu_3 \rightarrow m(t) = 0 \tag{16}$$

III. If $L - a < t < L$

$$P(t)_{d1} = \int_{L-t}^{L-t} \int_t^L f_{OW}(OW) f_{CO}\,(CO) d_{CO}\, d_{OW} = 0 \tag{17}$$

$$P(t)_{d2} = \int_{L-t}^{b} \int_t^L f_{OW}(OW) f_{CO}\,(CO) d_{CO}\, d_{OW} = 1 \tag{18}$$

$$P(t)_{d3} = 0 \tag{19}$$

$$\mu_1 \rightarrow m(t) = (14) \tag{20}$$

$$\mu_2 \rightarrow m(t) = (15) + \int_{L-a}^{t} \lambda(s) ds = (15) + (t - L + a)\lambda_{CO} \tag{21}$$

$$\mu_3 \rightarrow m(t) = 0 \tag{22}$$

- $b > L \; and \; a < L$

I. If $0 < t < L - a$

$$P(t)_{d1} = \int_{a}^{L-t}\int_{t}^{L} f_{OW}(OW) f_{CO}\,(CO) d_{CO}\, d_{OW} = \frac{L-t-a}{b-a} \tag{23}$$

$$P(t)_{d2} = \int_{L-t}^{L}\int_{t}^{L} f_{OW}(OW) f_{CO}\,(CO) d_{CO}\, d_{OW} = \frac{t}{b-a} \tag{24}$$

$$P(t)_{d3} = \int_{L}^{b} f_{OW}(OW) d_{OW} = \frac{b-L}{b-a} \tag{25}$$

$$\mu_1 \to m(t) = \int_{0}^{t} \lambda(s) ds = \frac{t(2a-2L+t)}{2(a-b)} \lambda_{CO} \tag{26}$$

$$\mu_2 \to m(t) = \int_{0}^{t} \lambda(s) ds = \frac{t^2}{2(b-a)} \lambda_{CO} \tag{27}$$

$$\mu_3 \to m(t) = \int_{0}^{t} \lambda(s) ds = \frac{b-L}{b-a}\, t\, \lambda_{CO} \tag{28}$$

II. If $L - a < t < L$

$$P(t)_{d1} = \int_{a}^{L-t}\int_{t}^{L} f_{OW}(OW) f_{CO}\,(CO) d_{CO}\, d_{OW} = 0 \tag{29}$$

$$P(t)_{d2} = \int_{a}^{L}\int_{t}^{L} f_{OW}(OW) f_{CO}\,(CO) d_{CO}\, d_{OW} = \frac{L-a}{b-a} \tag{30}$$

$$P(t)_{d3} = \int_{L}^{b} f_{OW}(OW) d_{OW} = \frac{b-L}{b-a} \tag{31}$$

$$\mu_1 \to m(t) = (26) \tag{32}$$

$$\mu_2 \to m(t) = (27) + \int_{L-a}^{t} \lambda(s) ds = (27) + \frac{(L-a)(a-L+t)}{(b-a)} \lambda_{CO} \tag{33}$$

$$\mu_3 \to m(t) = (28) + \int_{L-a}^{t} \lambda(s) ds = (28) + \frac{b-L}{b-a}(t-L+a)\lambda_{CO} \tag{34}$$

b) OW ~Exponential(λ_{OW})

$$f_{OW}(OW) = \begin{cases} \lambda_{OW} e^{-\lambda_{OW} t} & for\ OW \geq 0 \\ 0 & for\ OW < 0 \end{cases} \tag{35}$$

The results obtained are:

$$P(t)_{d1} = \int_{0}^{L-t}\int_{t}^{L} f_{OW}(OW) f_{CO}\,(CO) d_{CO}\, d_{OW} = 1 - e^{\lambda_{OW}(t-L)} \tag{36}$$

$$P(t)_{d2} = \int_{L-t}^{L}\int_{t}^{L} f_{OW}(OW) f_{CO}\,(CO) d_{CO}\, d_{OW} = e^{-\lambda_{OW} L}(e^{\lambda_{OW} t} - 1) \tag{37}$$

$$P(t)_{d3} = \int_{L}^{\infty} f_{OW}(OW) d_{OW} = e^{-\lambda_{OW} L} \tag{38}$$

$$\mu_1 \to m(t) = \int_0^t \lambda(s)ds = \left(t - \frac{e^{-\lambda_{OW}L}\left(e^{t\,\lambda_{OW}} - 1\right)}{\lambda_{OW}}\right)\lambda_{CO} \tag{39}$$

$$\mu_2 \to m(t) = \int_0^t \lambda(s)ds = \left(\frac{e^{-\lambda_{OW}L}\left(e^{t\,\lambda_{OW}} - 1 - t\,\lambda_{OW}\right)}{\lambda_{OW}}\right)\lambda_{CO} \tag{40}$$

$$\mu_3 \to m(t) = \int_0^t \lambda(s)ds = \left(t\, e^{-\lambda_{OW}L}\right)\lambda_{CO} \tag{41}$$

c) OW ~Normal(μ_{OW}, σ_{OW})

$$f_{OW}(OW) = \frac{1}{\sqrt{2\pi(\sigma_{\mathrm{ow}})^2}} e^{-\frac{(OW-\mu_{\mathrm{ow}})^2}{2(\sigma_{\mathrm{ow}})^2}} \tag{42}$$

The results obtained are:

$$\begin{aligned} P(t)_{d1} &= \int_0^{L-t}\int_t^L f_{OW}(OW) f_{CO}\,(CO) d_{CO}\, d_{OW} \\ &= \frac{1}{2}\left(\mathrm{erf}\left(\frac{L-t-\mu_{OW}}{\sqrt{2}\sigma_{OW}}\right) - \mathrm{erf}\left(\frac{-\mu_{OW}}{\sqrt{2}\sigma_{OW}}\right)\right) \end{aligned} \tag{43}$$

$$\begin{aligned} P(t)_{d2} &= \int_{L-t}^{L}\int_t^L f_{OW}(OW) f_{CO}\,(CO) d_{CO}\, d_{OW} \\ &= \frac{1}{2}\left(\mathrm{erf}\left(\frac{L-\mu_{OW}}{\sqrt{2}\sigma_{OW}}\right) - \mathrm{erf}\left(\frac{L-t-\mu_{OW}}{\sqrt{2}\sigma_{OW}}\right)\right) \end{aligned} \tag{44}$$

$$P(t)_{d3} = \int_L^{\infty} f_{OW}(OW) d_{OW} = \frac{1}{2}\left(\mathrm{erfc}\left(\frac{L-\mu_{OW}}{\sqrt{2}\sigma_{OW}}\right)\right) \tag{45}$$

$$\begin{aligned} \mu_1 \to m(t) &= \int_0^t \lambda(s)ds \\ &= \frac{1}{2}\Bigg(\left(e^{-\frac{(L-\mu_{OW})^2}{2\sigma_{OW}^2}} - e^{-\frac{(-L+t+\mu_{OW})^2}{2\sigma_{OW}^2}}\right)\sqrt{\frac{2}{\pi}}\sigma_{OW} \\ &+ t\,\mathrm{erf}\left(\frac{\mu_{OW}}{\sqrt{2}\sigma_{OW}}\right) + (-L+\mu_{OW})\,\mathrm{erf}\left(\frac{-L+\mu_{OW}}{\sqrt{2}\sigma_{OW}}\right) \\ &- (-L+t+\mu_{OW})\,\mathrm{erf}\left(\frac{-L+t+\mu_{OW}}{\sqrt{2}\sigma_{OW}}\right)\Bigg)\lambda_{CO} \end{aligned} \tag{46}$$

$$\mu_2 \rightarrow m(t) = \int_0^t \lambda(s)ds$$
$$= \frac{1}{2}\left(\left(-e^{-\frac{(L-\mu_{OW})^2}{2\sigma_{OW}^2}} + e^{-\frac{(-L+t+\mu_{OW})^2}{2\sigma_{OW}^2}}\right)\sqrt{\frac{2}{\pi}}\sigma_{OW}\right.$$
$$+ t \operatorname{erf}\left(\frac{L-\mu_{OW}}{\sqrt{2}\sigma_{OW}}\right) + (L-u) \operatorname{erf}\left(\frac{-L+\mu_{OW}}{\sqrt{2}\sigma_{OW}}\right)$$
$$\left. + (-L+t+\mu_{OW}) \operatorname{erf}\left(\frac{-L+t+\mu_{OW}}{\sqrt{2}\sigma_{OW}}\right)\right)\lambda_{CO} \tag{47}$$

$$\mu_3 \rightarrow m(t) = \int_0^t \lambda(s)ds = \frac{1}{2}\ t \operatorname{erfc}\left(\frac{L-\mu_{OW}}{\sqrt{2}\sigma_{OW}}\right)\lambda_{CO} \tag{48}$$

These results confirm the results obtained in the direct approach, by calculating the mean values at $t = L$ and obtaining the same expressions. The advantage of using this approach instead of the direct approach is that we obtain $\lambda(t)$ of each type of demand, and consequently we know how each type of demand changes with time: d_x~Poisson($\lambda_x(t)$). Although the total demand has constant rate, the different types of demand have variable rates that this approach allows to obtain. Figure A2 and Figure A3 are two examples of the variation of the demand rate if OW follows a uniform distribution. In all Figures, the lead time and the order windows are in UT and the demand rate in $\frac{units}{UT}$.

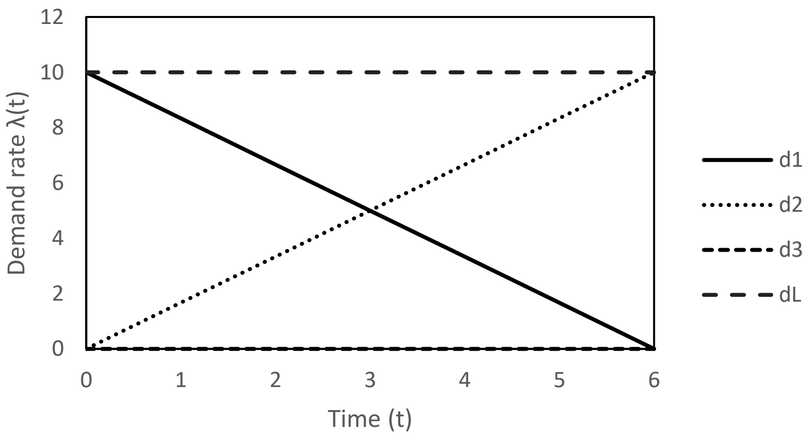

Figure A2 – Demand rate $\lambda(t)$ of each type of demand for a uniform OW~[0;6], $L = 6$ and $\lambda_{CO} = 10$

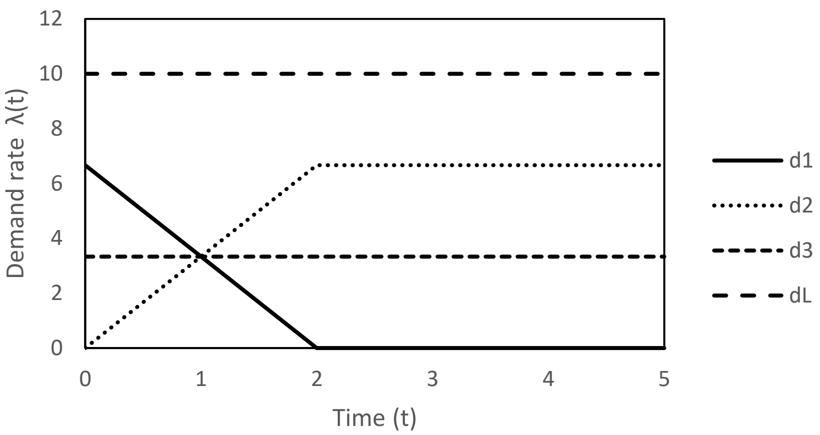

Figure A3 – Demand rate $\lambda(t)$ of each type of demand for a uniform OW~[3;6], $L = 5$ and $\lambda_{CO} = 10$

Figure A4 and Figure A5 are two examples of the variation of the demand rate if OW follows an exponential distribution.

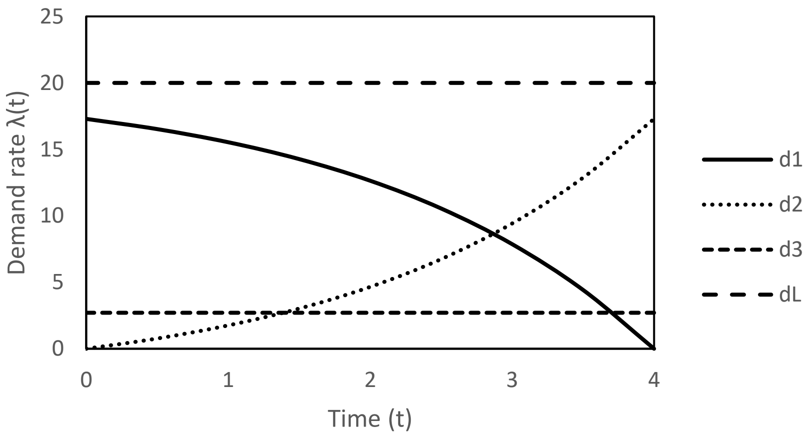

Figure A4 - Demand rate $\lambda(t)$ of each type of demand for an exponential OW~[$\lambda = 0.5$], $L = 4$ and $\lambda_{CO} = 20$

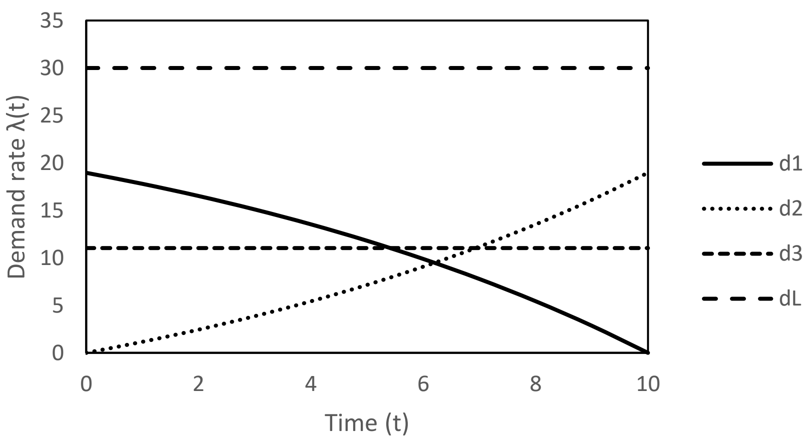

Figure A5 - Demand rate $\lambda(t)$ of each type of demand for an exponential OW~[$\lambda = 0.1$], $L = 10$ and $\lambda_{CO} = 30$

Figure A6 and Figure A7 are two examples of the variation of the demand rate if OW follows a normal distribution.

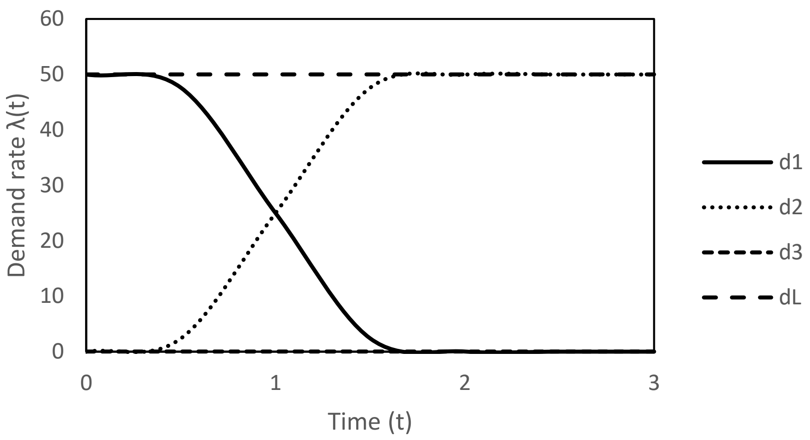

Figure A6 - Demand rate λ(t) of each type of demand for a normal *OW*~[2;0.3], $L = 3$ and $\lambda_{CO} = 50$

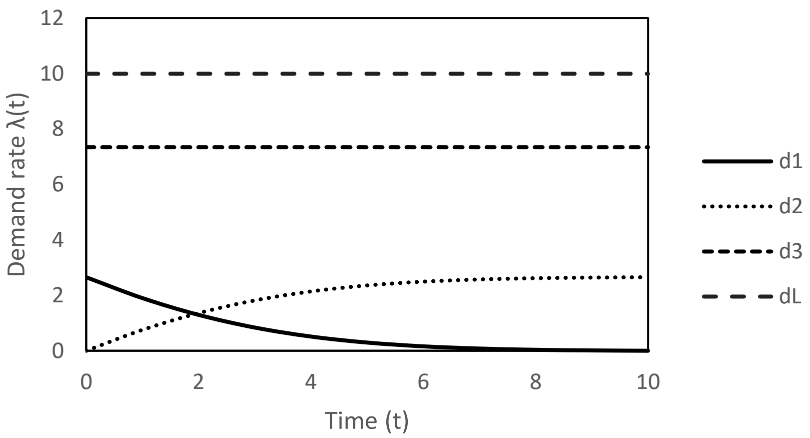

Figure A7 - Demand rate $\lambda(t)$ of each type of demand for a normal *OW*~[12.5;4], $L = 10$ and $\lambda_{CO} = 10$

As expected, during the lead time, the demand rate of d_1 decreases with time and the demand rate of d_2 increases with time because for the same parameters of the OW, as we get closer to the end of the lead time, higher is the probability that the customer delivery date is after the lead time. The demand rate of d_3 is constant with time because it only depends on the OW.

ANNEX B: Derivations for the iterative procedure

Here we present in more detail the derivations and considerations made to obtain the expressions used in the iterative procedure.

I. α service level

The expression for the ETRC is:

$$ETRC\ (k,Q) = A\frac{D}{Q} + \left[\frac{Q}{2} + k\sigma_1\right]vr + B_1\frac{D}{Q}P_{u_1\geq}(k) \tag{1}$$

A necessary condition (unless we are at a boundary) for the minimization of a function of two variables is that the partial derivative with respect to each variable be set to zero.

Partial derivative with respect to k:

$$\frac{\delta ETRC\ (k,Q)}{\delta k} = 0\ (=)\ \sigma_1 vr - B_1\frac{D}{Q}f_{u_1}(k) = 0\ (=)$$

$$(=)\ k = \sqrt{2\,ln\left(\frac{DB_1}{\sqrt{2\pi}Qvr\sigma_1}\right)} \tag{2}$$

$$If\ \ \frac{DB_1}{\sqrt{2\pi}Qvr\sigma_1} < 1\ set\ k\ at\ its\ lowest\ allowable\ value$$

We assume the lowest allowable value is $k = 0$

To obtain the result in equation (2), the following facts were used (Silver et al. 1998):

1) $f_u(k) = \frac{1}{\sqrt{2\pi}}e^{-\frac{k^2}{2}}$

2) $\frac{dP_{u\geq}(k)}{dk} = -\,f_u(k)$

Partial derivative with respect to Q:

$$\frac{\delta ETRC\ (k,Q)}{\delta Q} = 0\ (=) - \frac{AD}{Q^2} + \frac{1}{2}vr - \frac{B_1DP_{u_1\geq}(k)}{Q^2} = 0\ (=)$$

$$(=)\ Q = EOQ\ \sqrt{1 + \frac{B_1}{A}P_{u_1\geq}(k)} \tag{3}$$

Where:

$$EOQ = \sqrt{\frac{2AD}{vr}}$$

<u>Using Excel functions (listed in upper case):</u>

$$P_{u\geq}(k) = 1 - NORM.S.DIST(k, TRUE)$$

If we are interested in a certain service level, equation (4.43) can be modified to obtain k:

$$k = NORM.S.INV(\alpha)$$

II. β service level

The expression for the ETRC is:

$$ETRC\ (k, Q) = A\frac{D}{Q} + \left[\frac{Q}{2} + k\sigma_1\right]vr + \frac{B_2 v\sigma_1 G_{u_1}(k)D}{Q} \tag{4}$$

A necessary condition (unless we are at a boundary) for the minimization of a function of two variables is that the partial derivative with respect to each variable be set to zero.

Partial derivative with respect to k:

$$\frac{\delta ETRC\ (k, Q)}{\delta k} = 0\ (=)\ \sigma_1 vr - \frac{B_2 Dv\sigma_1}{Q}P_{u_1 \geq}(k) = 0\ (=)$$

$$(=)\ P_{u\geq}(k) = \frac{rQ}{B_2 D} \tag{5}$$

To obtain the result in equation (5), the following fact was used (Silver et al. 1998):

$$3)\ \ \frac{dG_u(k)}{dk} = -\ P_{u\geq}(k)$$

Partial derivative with respect to Q:

$$\frac{\delta ETRC\ (k, Q)}{\delta Q} = 0\ (=) - \frac{AD}{Q^2} + \frac{1}{2}vr - \frac{B_2 Dv\sigma_1 G_{u_1}(k)}{Q^2}\ = 0\ (=)$$

$$(=)\ Q = EOQ\ \sqrt{1 + \frac{B_2\sigma_1 v}{A}G_{u_1}(k)} \tag{6}$$

Where:

$$EOQ = \sqrt{\frac{2AD}{vr}}$$

<u>Using Excel functions (listed in upper case):</u>

Equation (5) used in the iterative procedure can be modified to obtain k:

$$k = \ NORM.S.INV\left(1 - \frac{rQ}{DB_2}\right)$$

$$If\ \frac{rQ}{DB_2} > 1\ set\ k\ as\ its\ lowest\ allowable\ value$$

If we are interested in a certain service level, equation (4.48) can be modified to obtain k:

$$f_{u_1}(k) - kP_{u_1\geq}(k) = \ \frac{Q}{\sigma_1}\left(\frac{1-\beta}{\beta}\right)(=)$$

$$(=) NORM.S.DIST(k, FALSE) - k\big(1 - NORM.S.DIST(k, TRUE)\big) = \frac{Q}{\sigma_1}\left(\frac{1-\beta}{\beta}\right)$$

Using the SOLVER in Excel we can obtain k.

To obtain this expression, the following fact was used (Silver et al. 1998):

$$4)\ \ G_{u_1}(k) = f_{u_1}(k) - kP_{u_1\geq}(k)$$

ANNEX C: Additional results obtained in the experiments for the (s, Q) policy

I. Experiments with α service level

Here we present additional results obtained in the experiments that are not in the main body of the thesis. Table C1 shows the cost reduction that provides the use of the new policy with α service level for the online retail. For each scenario, the optimal values of the pair (s, Q) are presented and the costs compared (OR - online retail policy, TR - traditional retail policy).

Table C1 - Policies comparison with α service level with optimal pairs (s, Q)

		CV						
		0.05		0.1		0.2		
Percentage of Demand of type 1		OR	TR	OR	TR	OR	TR	Average
75%	s	333	436	95	122	30	38	
	Q	258	260	254	255	252	252	
	OC	348.8	346.2	354.3	352.9	357.1	357.1	
	HC	464.0	766.1	420.7	502.6	394.1	418.3	
	SC	22.2	0.0	9.7	0.0	4.4	0.0	
	TC	835.1	1112.2	784.8	855.5	755.6	775.5	
	% savings	**24.92%**		**8.26%**		**2.56%**		**11.91%**
50%	s	228	436	67	122	22	38	
	Q	257	260	253	255	252	252	
	OC	350.2	346.2	355.7	352.9	357.1	357.1	
	HC	450.1	1054.1	411.1	574.6	389.2	436.3	
	SC	17.3	0.0	7.7	0.0	3.5	0.0	
	TC	817.6	1400.2	774.5	927.5	749.8	793.5	
	% savings	**41.61%**		**16.49%**		**5.50%**		**21.20%**
25%	s	122	436	38	122	14	38	
	Q	255	260	252	255	251	252	
	OC	352.9	346.2	357.1	352.9	358.6	357.1	
	HC	428.8	1342.1	398.1	646.6	381.0	454.3	
	SC	11.5	0.0	5.2	0.0	2.4	0.0	
	TC	793.2	1688.2	760.4	999.5	741.9	811.5	
	% savings	**53.02%**		**23.92%**		**8.57%**		**28.50%**

Numerical results for the cost reduction from using the new policy when the retailer is limited by the supplier to a certain order quantity Q are presented, for three values of Q, in Table C2.

Table C2 - Policies comparison with α service level for different values of the order quantity, different CV's and percentages of d_1

Savings (%)									
CV	$\boldsymbol{Q} = 100$			$\boldsymbol{Q} = 250$			$\boldsymbol{Q} = 400$		
	$\mathbf{75\%}\boldsymbol{d_1}$	$\mathbf{50\%}\ \boldsymbol{d_1}$	$\mathbf{25\%}\boldsymbol{d_1}$	$\mathbf{75\%}\boldsymbol{d_1}$	$\mathbf{50\%}\boldsymbol{d_1}$	$\mathbf{25\%}\boldsymbol{d_1}$	$\mathbf{75\%}\boldsymbol{d_1}$	$\mathbf{50\%}\boldsymbol{d_1}$	$\mathbf{25\%}\boldsymbol{d_1}$
0.05	19.39%	33.89%	44.68%	24.82%	41.55%	52.98%	22.99%	39.18%	50.42%
0.075	10.01%	19.38%	27.54%	13.23%	25.00%	34.64%	11.99%	23.03%	32.18%
0.1	6.04%	12.36%	18.31%	8.07%	16.33%	23.79%	7.22%	14.83%	21.75%
0.15	2.89%	6.30%	9.78%	3.87%	8.48%	13.05%	3.41%	7.57%	11.72%
0.2	1.70%	3.87%	6.17%	2.28%	5.23%	8.32%	1.98%	4.63%	7.39%

The contribution of different order windows in the savings was also studied. Policies were compared when the order window has the same mean but different parameters or distributions. Numerical results are presented in Table C3 through Table C5.

Table C3 – Policies comparison with α service level for different order windows with mean $0.25L$

Savings (%)						
CV	**OW**					
	U[0;0.5L]	U[0.1L;0.4L]	U[0.2L;0.3L]	Exp$\left(\frac{1}{0.25L}\right)$	N(0.25L,0.025L)	N(0.25L,0.075L)
0.05	24.92%	24.92%	24.92%	24.54%	24.92%	24.96%
0.1	8.26%	8.26%	8.26%	8.10%	8.26%	8.28%
0.15	3.90%	3.90%	3.90%	3.81%	3.90%	3.91%
0.2	2.56%	2.56%	2.56%	2.50%	2.56%	2.56%

Table C4 - Policies comparison with α service level for different order windows with mean $0.5L$

Savings (%)						
CV	**OW**					
	U[0;L]	U[0.25L;0.75L]	U[0.2L;0.8L]	Exp$\left(\frac{1}{0.5L}\right)$	N(0.5L,0.05L)	N(0.5L,0.15L)
0.05	41.61%	41.61%	41.61%	37.77%	41.61%	41.63%
0.1	16.49%	16.49%	16.49%	14.37%	16.49%	16.51%
0.15	8.50%	8.50%	8.50%	7.27%	8.50%	8.51%
0.2	5.50%	5.50%	5.50%	4.70%	5.50%	5.51%

Table C5 - Policies comparison with α service level for different order windows with mean $0.75L$

Savings (%)						
CV	**OW**					
	U[0;1.5L]	U[0.25L;1.25L]	U[0.5L;L]	Exp$\left(\frac{1}{0.75L}\right)$	N(0.75L,0.075L)	N(0.75L,0.225L)
0.05	49.60%	51.77%	53.02%	44.31%	53.02%	52.44%
0.1	21.50%	23.02%	23.92%	18.10%	23.92%	23.50%
0.15	11.53%	12.49%	13.07%	9.45%	13.07%	12.80%
0.2	7.52%	8.17%	8.57%	6.12%	8.57%	8.38%

II. Experiments with β service level

Here we present additional results obtained in the experiments that are not in the main body of the thesis. Table C6 shows the cost reduction that provides the use of the new policy with β service level for the online retail. For each scenario, the optimal values of the pair (s, Q) are presented and the costs compared.

Table C6 – Policies comparison with β service level with optimal pairs (s, Q)

Percentage of Demand of type 1		CV 0.05 OR	CV 0.05 TR	CV 0.1 OR	CV 0.1 TR	CV 0.2 OR	CV 0.2 TR	Average
75%	s	325	428	88	114	25	33	
	Q	259	260	254	255	252	253	
	OC	347.5	346.2	354.3	352.9	357.1	355.7	
	HC	442.2	743.0	400.6	479.5	380.3	405.4	
	SC	22.7	0.0	11.3	0.0	5.7	0.0	
	TC	812.3	1089.2	766.3	832.5	743.2	761.1	
	% savings	25.42%		7.95%		2.36%		11.91%
50%	s	220	428	60	114	18	33	
	Q	257	260	254	255	252	253	
	OC	350.2	346.2	354.3	352.9	357.1	355.7	
	HC	426.7	1031.0	394.2	551.5	377.2	423.4	
	SC	18.5	0.0	9.2	0.0	4.6	0.0	
	TC	795.5	1377.2	757.8	904.5	738.9	779.1	
	% savings	42.24%		16.22%		5.16%		21.21%
25%	s	114	428	33	114	10	33	
	Q	255	260	253	255	252	253	
	OC	352.9	346.2	355.7	352.9	357.1	355.7	
	HC	407.4	1319.0	384.5	623.5	373.0	441.4	
	SC	13.1	0.0	6.5	0.0	3.3	0.0	
	TC	773.4	1665.2	746.7	976.5	733.4	797.1	
	% savings	53.55%		23.53%		7.99%		28.36%

Numerical results for the cost reduction from using the new policy when the retailer is limited by the supplier to a certain order quantity Q are presented, for three values of Q, in Table C7.

Table C7 - Policies comparison with β service level for different values of the order quantity, different CV's and percentages of d_1

Savings (%)									
CV	$\boldsymbol{Q} = 100$			$\boldsymbol{Q} = 250$			$\boldsymbol{Q} = 400$		
	$\mathbf{75\%}\boldsymbol{d_1}$	$\mathbf{50\%}\ \boldsymbol{d_1}$	$\mathbf{25\%}\boldsymbol{d_1}$	$\mathbf{75\%}\boldsymbol{d_1}$	$\mathbf{50\%}\boldsymbol{d_1}$	$\mathbf{25\%}\boldsymbol{d_1}$	$\mathbf{75\%}\boldsymbol{d_1}$	$\mathbf{50\%}\boldsymbol{d_1}$	$\mathbf{25\%}\boldsymbol{d_1}$
0.05	19.66%	34.25%	44.97%	25.37%	42.21%	53.54%	23.55%	39.85%	50.99%
0.075	10.02%	19.42%	27.49%	13.32%	25.19%	34.74%	12.05%	23.19%	32.25%
0.1	5.93%	12.22%	18.05%	7.94%	16.21%	23.52%	7.06%	14.67%	21.45%
0.15	2.72%	6.04%	9.36%	3.61%	8.11%	12.47%	3.12%	7.17%	11.11%
0.2	1.54%	3.60%	5.74%	1.99%	4.81%	7.66%	1.68%	4.18%	6.72%

The contribution of different order windows in the savings was also studied. Policies were compared when the order window has the same mean but different parameters or distributions. Numerical results are presented in Table C8 through Table C10.

Table C8 - Policies comparison with β service level for different order windows with mean $0.25L$

Savings (%)						
CV	**OW**					
	U[0;0.5L]	U[0.1L;0.4L]	U[0.2L;0.3L]	Exp$\left(\frac{1}{0.25L}\right)$	N(0.25L,0.025L)	N(0.25L,0.075L)
0.05	25.42%	25.42%	25.42%	25.03%	25.42%	25.45%
0.1	7.95%	7.95%	7.95%	7.79%	7.95%	7.97%
0.15	3.69%	3.69%	3.69%	3.60%	3.69%	3.70%
0.2	2.36%	2.36%	2.36%	2.31%	2.36%	2.36%

Table C9 - Policies comparison with β service level for different order windows with mean $0.5L$

Savings (%)						
CV	**OW**					
	U[0;L]	U[0.25L;0.75L]	U[0.2L;0.8L]	Exp$\left(\frac{1}{0.5L}\right)$	N(0.5L,0.05L)	N(0.5L,0.15L)
0.05	42.24%	42.24%	42.24%	38.40%	42.24%	42.26%
0.1	16.22%	16.22%	16.22%	14.09%	16.22%	16.23%
0.15	8.18%	8.18%	8.18%	6.99%	8.18%	8.19%
0.2	5.16%	5.16%	5.16%	4.40%	5.16%	5.16%

Table C10 - Policies comparison with β service level for different order windows with mean $0.75L$

Savings (%)						
CV	**OW**					
	U[0;1.5L]	U[0.25L;1.25L]	U[0.5L;L]	Exp$\left(\frac{1}{0.75L}\right)$	N(0.75L,0.075L)	N(0.75L,0.225L)
0.05	50.19%	52.33%	53.55%	44.94%	53.55%	52.99%
0.1	21.17%	22.65%	23.53%	17.81%	23.53%	23.12%
0.15	11.09%	11.99%	12.54%	9.10%	12.54%	12.28%
0.2	7.03%	7.63%	7.99%	5.74%	7.99%	7.82%

ANNEX D: Expression derivation for demand characterization in the $(\boldsymbol{R}, \boldsymbol{s}, \boldsymbol{S})$ policy and auxiliary notation

Here we present the approach followed to obtain the expressions needed to the characterization of demand.

The ratio between a certain type of demand and the total demand for a given value of τ is given by:

$$\int \int f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \tag{1}$$

The limits of integration change accordingly to the type of demand we are calculating.

This ratio should be weighted because higher values of τ are related to higher values of total demand, so:

$$\int \left[\int \int f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \right] \frac{(L+\tau)\mu}{\int (L+\tau)\mu\, f_\tau(\tau) d_\tau} f_\tau(\tau) d_\tau \tag{2}$$

Where:

$$\frac{(L+\tau)\mu}{\int (L+\tau)\mu\, f_\tau(\tau) d_\tau} f(\tau)_\tau d_\tau = \frac{(L+\tau)}{L \int f_\tau(\tau) d_\tau + \int \tau\, f_\tau(\tau) d_\tau} f_\tau(\tau) d_\tau = \\ = \frac{(L+\tau)}{L + E[\tau]} f_\tau(\tau) d_\tau \tag{3}$$

Because we assume τ follows a uniform distribution:

$$f_{\tau w}(\tau) = \frac{(L+\tau)}{L+E[\tau]} f_\tau(\tau) = \frac{(L+\tau)}{\left(L + \frac{R}{2}\right) R} \tag{4}$$

Equation (2) becomes:

$$\int \left[\int \int f_{OW}(OW) f_{CO}(CO) d_{CO} d_{OW} \right] f_{\tau w}(\tau) d_\tau \tag{5}$$

Equation (5) is the expression we use to obtain the probability of each type of demand.

The auxiliary notation, used to minimize the length of the expressions obtained, is presented in Table D1.

Table D1 – Auxiliary variables used to characterize the demand in the (R, s, S) policy for uniform OW

$B = min(b; L)$

$C = min(L + R; T)$

$D = min(L; max(a; R))$

$E = min(R; T)$

$F = min(L + R; max(a; R))$

$G = min(a; L)$

$H = min(L + R; max(b; L))$

$I = min(L + R; max(b; R))$

$J = min(L; max(b; R))$

$K = min\ (max(b - (L + R); 0); R)$

$M = max(L + R; a)$

$N = min(R; b)$

$O = min(a; R)$

$P = max(a - (L + R); 0)$

$Q = min(R; max(b; L))$

$T = max(a; L)$

$W = min(L; b)$

$X = min(max(b - L; 0); R)$

$Y = min(max(a - L; 0); R)$

$Z = min(P; R)$

ANNEX E: Assessment of the assumptions made in the (R, s, S) policy

The objective of this Annex is to evaluate the assumptions regarding the distribution for demand in the protection period and the expression for the standard deviation of demand of type 1. With the author permission, we made some modifications to the simulator developed by Zein (2017) that replicates what would happen in actual application of a (R, s, S) system for online retail. Then we compare the results obtained in the simulation with the expected results and take some conclusions.

I. Simulation design

Here we describe briefly the simulator used. A thorough explanation of the simulator is beyond the scope of this thesis. The simulator developed in the on-going work of Zein (2017) is used to simulate the behavior of customer orders in the online retail and how a (R, s, S) policy deals with the inventory management in this type of systems. The simulator is based in the approach presented by Silver et al. (2009), summarized in 2.3, adapted for the online retail scenario.

In the simulation, the inputs are the CV, the desired value of the fill rate (β in our notation) and the desired number of review intervals between replenishments (n). Then the approach presented in 2.3 is adapted and used to calculate s and S:

1) Obtain E[τ] and Var[τ] from Table 2 and Table 3
2) Calculate the mean and variance of d_1

$$E[d_1] = \mu_1 = (E[\tau] + L)\mu P(d_1) \tag{1}$$

$$Var[d_1] = \sigma_1^2 = min\left[\left((E[\tau] + L)\sigma^2 P(d_1) + \mu^2 Var[\tau]\, P(d_1)\right) C^2;\ \sigma_L^2\right] \tag{2}$$

3) Calculate the target allowed (average) units short per replenishment cycle (AUSPRC)

$$AUSPRC = (1 - \beta)E[Q] \tag{3}$$

4) Choose k to satisfy

$$G_{u_1}(k) = \frac{(1-\beta)E[Q]}{\sigma_1} = \frac{(1-\beta)n\mu}{\sigma_1} \tag{4}$$

5) Calculate the reorder point

$$s = \mu_1 + k\sigma_1 \tag{5}$$

6) Calculate the order-up-to level

$$S = s + n\mu - E[\tau]\mu \tag{6}$$

For a given scenario, the values of the parameters (R, s, S) are obtained and, the simulation is instantiated 4000 times (4000 orders) to be able to compute the fill rate and the average time between replenishments. This procedure is repeated 100 times and the average fill rate and average time between replenishments are calculated. These values are compared to the desired values given as inputs to the simulator.

This simulator was originally developed for fixed *OW* and the time between orders was fixed in each of the 100 iterations and generated by a normal distribution. We modify the simulator to deal with exponential distributed time between orders and to deal with *OW* following an exponential or uniform distribution.

The values used as inputs are presented in Table E1. We used CV = 0.1 and $\mu = 100$ because we assume the demand per unit time follows a Poisson distribution in our analytical approach. The experiments were made for 25%, 50% and 75% of demand of type 1 and $R = 1$.

Table E1 - Inputs used in the simulation

Scenario	SOO		MOO	
$\boldsymbol{n}$	2	4	2	4
$\boldsymbol{L}$	1	2	3	6
$\boldsymbol{\mu}$	100	100	100	100
CV	0.1	0.1	0.1	0.1
$\boldsymbol{\beta}$	0.95	0.95	0.95	0.95

II. Results

Here we present and discuss the simulated results. Table E2 and Table E3 present the mean absolute percentage error (MAPE) between the simulated results and the expected results (input values) if the OW follows a uniform distribution. Table E4 and Table E5 present the results if the OW follows an exponential distribution.

Table E2 - Simulation results with uniform OW and MOO scenario

% $\boldsymbol{d_1}$	$\boldsymbol{L}$	**Expected results**		**Simulated results**		**Error (%)**	
		n	β	n	β	n	β
25 %	6	4	0.95	4.334	0.949	8.36%	0.10%
25 %	3	2	0.95	2.299	0.943	14.96%	0.69%
50 %	6	4	0.95	4.099	0.976	2.47%	2.77%
50 %	3	2	0.95	2.059	0.977	2.93%	2.92%
75 %	6	4	0.95	4.031	0.979	0.78%	3.04%
75 %	3	2	0.95	2.001	0.987	0.04%	3.88%

Table E3 - Simulation results with uniform OW and SOO scenario

% $\boldsymbol{d_1}$	$\boldsymbol{L}$	**Expected results**		**Simulated results**		**Error (%)**	
		n	β	n	β	n	β
25 %	2	4	0.95	4.188	0.929	4.69%	2.24%
25 %	1	2	0.95	2.067	0.930	3.37%	2.16%
50 %	2	4	0.95	4.163	0.956	4.09%	0.65%
50 %	1	2	0.95	2.067	0.962	3.37%	1.30%
75 %	2	4	0.95	4.026	0.967	0.66%	1.78%
75 %	1	2	0.95	2.007	0.984	0.34%	3.61%

Table E4 - Simulation results with exponential OW and MOO scenario

% d_1	***L***	**Expected results**		**Simulated results**		**Error (%)**	
		n	β	n	β	n	β
25 %	6	4	0.95	4.324	0.922	8.09%	2.97%
25 %	3	2	0.95	2.301	0.900	15.06%	5.27%
50 %	6	4	0.95	4.118	0.965	2.96%	1.57%
50 %	3	2	0.95	2.058	0.967	2.92%	1.84%
75 %	6	4	0.95	4.026	0.977	0.65%	2.83%
75 %	3	2	0.95	1.991	0.986	0.44%	3.81%

Table E5 - Simulation results with exponential OW and SOO scenario

% d_1	***L***	**Expected results**		**Simulated results**		**Error (%)**	
		n	β	n	β	n	β
25 %	2	4	0.95	4.259	0.880	6.47%	7.42%
25 %	1	2	0.95	2.102	0.860	5.09%	9.47%
50 %	2	4	0.95	4.201	0.941	5.03%	0.99%
50 %	1	2	0.95	2.079	0.956	3.97%	0.59%
75 %	2	4	0.95	4.050	0.960	1.24%	1.04%
75 %	1	2	0.95	1.989	0.979	0.55%	3.10%

We believe there are three reasons for the inaccuracy of the simulated results. The first one is the assumption that demand follows a normal distribution in the protection period. This assumption seems to produce worst results for lower percentages of demand of type 1 and an exponential distributed OW. Looking at the results, we observe that for the same scenario, generally higher values of L provide lower errors for the fill rate which may suggest that for higher values of L the real distribution of demand comes closer to the normal distribution. Regarding the errors related to the average time between replenishments, they are not so critical, because even if the average is equal to the desired value, there would still be occasions where the time between replenishments would differ, so the time between replenishments cannot be assured to be constant.

We believe the second reason for the inaccuracy of the results is the approximation used for the standard deviation of demand of type 1, which should be improved in future work.

The third reason is related to numerical errors that happen in the simulator, and that, on our experiments, are more often for lower percentages of demand of type 1.

ANNEX F: Additional results obtained in the experiments for the $(\boldsymbol{R}, \boldsymbol{s}, \boldsymbol{S})$ policy

Here we present additional results obtained in the experiments that are not in the main body of the thesis. Table F1 and Table F2 present the percentages of reduction in the average on-hand inventory from using the adapted policy for different percentages of demand of type 1, different CV's and different values of n. Table F3 and Table F4 present the percentages of reduction in the average on-hand inventory from using the adapted policy for different percentages of demand of type 1, different CV's and different values of β.

Table F1 - Policies comparison for different percentages of d_1, different CV's and with different values of n when the OW follows a uniform distribution

Reduction in the average OH inventory (%)						
$\boldsymbol{n}$	CV = 0.1			CV = 0.2		
	$25\%\boldsymbol{d_1}$	$50\%\ \boldsymbol{d_1}$	$75\%\boldsymbol{d_1}$	$25\%\boldsymbol{d_1}$	$50\%\boldsymbol{d_1}$	$75\%\boldsymbol{d_1}$
2	78.78%	70.31%	53.08%	77.88%	67.37%	48.19%
3	71.51%	62.01%	44.25%	71.14%	60.45%	41.33%
4	65.38%	55.34%	37.79%	65.44%	54.49%	36.07%
5	60.20%	50.00%	32.96%	60.40%	49.41%	31.83%
6	55.80%	45.51%	29.17%	56.02%	45.31%	28.46%
$\%\boldsymbol{d_1}$	**25%**		**50%**		**75%**	
Average	66.25%		56.02%		38.31%	

Table F2 - Policies comparison for different percentages of d_1, different CV's and with different values of n when the OW follows an exponential distribution

Reduction in the average OH inventory (%)						
$\boldsymbol{n}$	CV = 0.1			CV = 0.2		
	$25\%\boldsymbol{d_1}$	$50\%\ \boldsymbol{d_1}$	$75\%\boldsymbol{d_1}$	$25\%\boldsymbol{d_1}$	$50\%\boldsymbol{d_1}$	$75\%\boldsymbol{d_1}$
2	78.97%	70.78%	53.98%	78.31%	68.62%	50.24%
3	71.61%	62.35%	44.74%	71.63%	61.38%	43.19%
4	65.46%	55.57%	38.20%	65.73%	55.19%	37.23%
5	60.24%	50.13%	33.25%	60.68%	50.11%	32.87%
6	55.81%	45.62%	29.53%	56.17%	45.70%	29.18%
$\%\boldsymbol{d_1}$	**25%**		**50%**		**75%**	
Average	66.46%		56.55%		39.24%	

Table F3 - Policies comparison for different percentages of d_1, different CV's and with different values of β when the OW follows a uniform distribution

Reduction in the average OH inventory (%)						
$\boldsymbol{\beta}$	**CV = 0.1**			**CV = 0.2**		
	$\mathbf{25\%d_1}$	$\mathbf{50\%\ d_1}$	$\mathbf{75\%d_1}$	$\mathbf{25\%d_1}$	$\mathbf{50\%d_1}$	$\mathbf{75\%d_1}$
0.8	73.78%	65.21%	48.42%	73.95%	65.28%	48.13%
0.85	70.70%	61.71%	44.49%	70.89%	61.55%	44.04%
0.9	67.93%	58.49%	41.11%	68.11%	58.24%	40.16%
0.95	65.38%	55.34%	37.79%	65.44%	54.49%	36.07%
0.99	63.09%	51.99%	33.93%	62.50%	50.23%	31.09%
$\mathbf{\%d_1}$	**25%**		**50%**		**75%**	
Average	68.18%		58.25%		40.52%	

Table F4 - Policies comparison for different percentages of d_1, different CV's and with different values of β when the OW follows an exponential distribution

Reduction in the average OH inventory (%)						
$\boldsymbol{\beta}$	**CV = 0.1**			**CV = 0.2**		
	$\mathbf{25\%d_1}$	$\mathbf{50\%\ d_1}$	$\mathbf{75\%d_1}$	$\mathbf{25\%d_1}$	$\mathbf{50\%d_1}$	$\mathbf{75\%d_1}$
0.8	73.77%	65.22%	48.41%	73.90%	65.35%	48.37%
0.85	70.72%	61.67%	44.63%	70.95%	61.87%	44.43%
0.9	67.94%	58.53%	41.38%	68.28%	58.62%	40.93%
0.95	65.46%	55.57%	38.20%	65.73%	55.19%	37.23%
0.99	63.41%	52.78%	35.05%	63.08%	51.50%	33.13%
$\mathbf{\%d_1}$	**25%**		**50%**		**75%**	
Average	68.32%		58.63%		41.18%	

The contribution of different order windows in the reductions in the average OH inventory was also studied. Policies were compared when the order window has the same mean but different parameters or distributions. Numerical results are presented in Table F5 through Table F7.

Table F5 - Policies comparison for different order windows with mean $0.25E[L+\tau]$

Reduction in the average OH inventory (%)				
CV	**OW**			
	U[0;0.5$E[L+\tau]$]	U[0.1$E[L+\tau]$; 0.4$E[L+\tau]$]	U[0.2$E[L+\tau]$; 0.3$E[L+\tau]$]	Exp$\left(\frac{1}{0.25E[L+\tau]}\right)$
0.1	37.82%	37.82%	37.82%	37.78%
0.15	37.19%	37.19%	37.19%	37.48%
0.2	36.34%	36.34%	36.34%	36.86%
0.25	35.26%	35.26%	35.26%	36.49%

Table F6 - Policies comparison for different order windows with mean $0.5E[L+\tau]$

Reduction in the average OH inventory (%)				
CV	**OW**			
	U[0;$E[L+\tau]$]	U[0.25$E[L+\tau]$; 0.75$E[L+\tau]$]	U[0.2$E[L+\tau]$; 0.8$E[L+\tau]$]	Exp$\left(\frac{1}{0.5E[L+\tau]}\right)$
0.1	55.31%	55.36%	55.36%	51.89%
0.15	55.14%	55.19%	55.19%	51.77%
0.2	54.57%	54.62%	54.62%	51.40%
0.25	53.94%	53.99%	53.99%	51.00%

Table F7 - Policies comparison for different order windows with mean $0.75E[L+\tau]$

Reduction in the average OH inventory (%)				
CV	**OW**			
	U[0;1.5$E[L+\tau]$]	U[0.25$E[L+\tau]$; 1.25$E[L+\tau]$]	U[0.5$E[L+\tau]$; $E[L+\tau]$]	Exp$\left(\frac{1}{0.75E[L+\tau]}\right)$
0.1	62.54%	64.34%	65.34%	58.09%
0.15	62.53%	64.47%	65.57%	58.10%
0.2	62.34%	64.32%	65.46%	57.83%
0.25	62.06%	64.10%	65.33%	57.64%