

$$\sum_{\mathcal{O}} \begin{array}{c} \mathcal{O}_1 \quad \mathcal{O}_3 \\ \diagdown \quad \diagup \\ \text{---} \mathcal{O} \text{ ---} \\ \diagup \quad \diagdown \\ \mathcal{O}_2 \quad \mathcal{O}_4 \end{array} = \sum_{\mathcal{O}'} \begin{array}{c} \mathcal{O}_1 \quad \mathcal{O}_3 \\ \diagdown \quad \diagup \\ \text{---} \mathcal{O}' \text{ ---} \\ \diagup \quad \diagdown \\ \mathcal{O}_2 \quad \mathcal{O}_4 \end{array}$$

Bootstrapping Conformal Field Theories

Emilio Trevisani

Doutoramento em Física

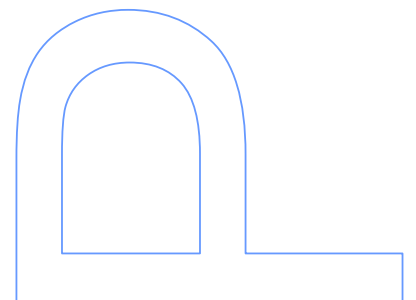
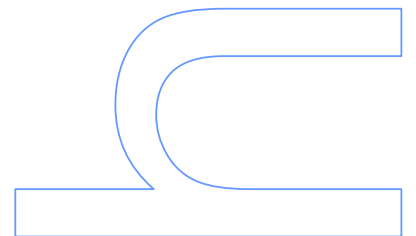
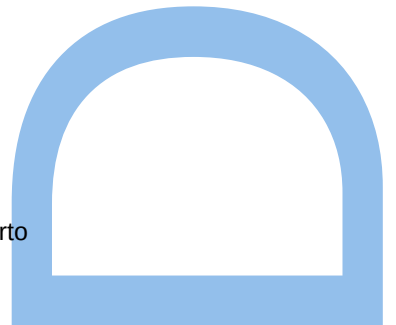
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Abstract

The bootstrap program for conformal field theories in d dimensions had an impressive development in the last few years. The fundamental ingredients to apply bootstrap techniques are some kinematical functions known as conformal blocks, which can be thought as the contribution of one conformal multiplet to a four point function. Despite the importance of these functions, little was known about them for generic four point functions in generic spacetime dimensions. In this thesis we introduce two systematic methods to compute any conformal block. The first one is based on a perturbative expansion in one of their variables. The second one is based on a recursion formula which naturally arises from representation theoretical considerations. We exemplify these methods in order to obtain many conformal blocks. Finally we use this technology in order to implement the numerical bootstrap study for the four point function of conserved currents in three spacetime dimensions.

Resumo

O programa bootstrap para teorias de campo conformes em d dimensões teve um desenvolvimento impressionante nos últimos anos. Os ingredientes fundamentais para aplicar técnicas de bootstrap são algumas funções cinemáticas conhecidas como conformal blocks, que podem ser pensadas como a contribuição de um multiplete conforme a uma função de quatro pontos. Apesar da importância dessas funções, pouco se sabe sobre elas para funções genéricas de quatro pontos em dimensões de espaço-tempo genéricas. Nesta tese apresentamos dois métodos sistemáticos para computar qualquer conformal block. A primeira é baseada em uma expansão perturbativa em uma das suas variáveis. O segundo é baseado em uma fórmula de recursão que surge naturalmente de considerações de teoria de representação. Nós exemplificamos esses métodos para obter muitos conformal blocks. Enfim, utilizamos esta tecnologia para implementar um estudo de bootstrap numérico para a função de quatro pontos de correntes conservadas em três dimensões do espaço-tempo.

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Chapter 1

Introduction

1.1 Conformal field theories

Symmetries play a fundamental role in physics. Their presence always simplifies the description of a problem and sometimes may also lead to its complete solution. On the other hand, very symmetric models may not always describe systems which can actually be realized in nature. Conformal field theories (CFTs) are rare examples of very symmetric models which are also physically relevant.

Despite the amount of symmetry, CFTs are very rich theories which control the behavior of a vast set of physical systems: they indeed arise simply by studying a system from very far apart. This remarkable fact descends naturally from the renormalization group (RG) flow. The RG is based on the idea that if we are given a model defined at a certain scale a , we can conveniently use it at a scale $L > a$ once we integrate out all its features which depend on scales smaller than L . By iterating this procedure we can obtain an infinite set of new coarse grained effective descriptions valid at larger and larger scales. This process ends when we reach a fixed point of the RG flow, where the new effective description at very large scales do not differ from the old ones. These fixed points are therefore scale invariant. The scale symmetry of the fixed points in many relevant cases is enhanced to conformal symmetry. Therefore, by construction, the RG flow often generates CFTs which describe a system at large distances.

The philosophy of the RG flow is particularly useful in the context of quantum field theories (QFTs). In fact a QFT can be thought of itself as an RG flow, which can have both an infrared (IR) and an ultraviolet (UV) scale invariant fixed point. Moreover in the framework of QFT it is generally possible to prove that scale symmetry does imply conformal symmetry¹. Therefore a UV-complete quantum field theory can be usually described as flow between two CFTs as pictured in figure 1.1. This fact also means that CFTs may be very important tools to understand the landscape of QFTs.

¹This statement is not proved in general. However there exist proofs in specific settings under mild assumptions [1, 2].

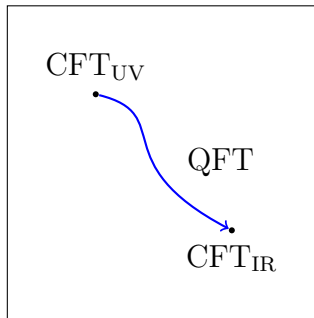


Figure 1.1: Quantum field theories and the renormalization group flow.

CFTs are also important for statistical physics and condensed matter applications. In fact, when a system is reaching a second order phase transition, its correlation length approaches infinity. In this regime the system becomes scaleless and its critical behavior is described in terms of a CFT. A very important example of this, is given by the critical point of the Ising model, which is well known to correspond to a CFT. Very interestingly, the critical point of this very simple model actually describes the behavior of a class of real phenomena: the magnets close to the Curie temperature and the critical opalescent liquids. This may sound surprising, but in fact it is again a consequence of the RG flow. Since the coarse graining procedure induced by the RG flow smoothens the microscopic details of the model, it is often the case that the CFT_{IR} has universal properties. Therefore, many different systems in the UV, both from the real world (like magnets and water) and from the theoretical one (like statistical models defined on a lattice and continuous quantum field theories), may have the same universal IR description. From this point of view we can consider CFTs as the way to label these universality classes.

Finally, CFTs are also important in order to study quantum theories of gravity. In fact quantum gravity in a $d + 1$ dimensional anti-de Sitter (AdS) space has a dual description in terms of a CFT on its d dimensional boundary. We can therefore hope that, by understanding CFTs, we will be eventually able to shed light on the long-standing problem of quantizing gravity.

From the theoretical point of view CFTs are very interesting models since their amount of symmetry allows to apply the “bootstrap” philosophy, the idea of “pull oneself over a fence by one’s bootstraps”. In other words, the idea of trying to solve a non trivial dynamical problem just by thoughtfully imposing all the constraints which naturally arise from symmetries and other rigorous considerations.

We will return to this point later, after defining some basic features of CFTs in d spacetime dimensions.

1.2 Correlation functions and OPE

Conformal field theories are quantum field theories which enjoy extra symmetries. Beside the usual symmetry under the Poincaré group they are also invariant under scale transformations and special conformal transformations (which are just compositions of translations and inversions). The spectrum of the theory is encoded in the set of all possible local operators $\mathcal{O}(x)$, which can be conveniently classified in terms of how they transform under scaling and $SO(d)$ rotation. Therefore we associate two quantum labels to $\mathcal{O}(x)$: the scaling (or conformal) dimension Δ and the $SO(d)$ spin. It is also convenient to organize the operators in conformal multiplets. The highest weight of the multiplet is called “primary”, while all the other operators are “descendants”. The action of the generator of the translations can be used to go from the primary operator to any descendant in the multiplet. Therefore it is convenient to just consider primary operators since we can obtain any descendant by applying derivatives on them.

In any CFT, the symmetries are so constraining that they fix automatically the correlation functions with one, two and three operator insertions up to some coefficients. For example if we consider scalar primary operators $\mathcal{O}_i$ with conformal dimensions Δ_i we obtain

$$\langle \mathcal{O}(x) \rangle = \delta_{\mathcal{O}\mathbb{I}} \quad (1.1)$$

$$\langle \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) \rangle = \frac{\delta_{\mathcal{O}_i \mathcal{O}_j}}{(x_{12}^2)^{\Delta_i}} \quad (1.2)$$

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle = \frac{c_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3}}{(x_{12}^2)^{\frac{\Delta_1 + \Delta_2 - \Delta_3}{2}} (x_{23}^2)^{\frac{\Delta_2 + \Delta_3 - \Delta_1}{2}} (x_{31}^2)^{\frac{\Delta_3 + \Delta_1 - \Delta_2}{2}}} \quad (1.3)$$

where $x_{ij}^\mu \equiv x_i^\mu - x_j^\mu$, and $\mathbb{I}$ is the identity operator. The constants $c_{\mathcal{O}_i \mathcal{O}_j \mathcal{O}_k}$ are called OPE coefficients (or structure constants). The labels of the operators $\mathcal{O}_i$ and the coefficients $c_{\mathcal{O}_i \mathcal{O}_j \mathcal{O}_k}$ depend on the specific theory considered and therefore are the only dynamical data entering in formulae (1.1-1.3).

Another remarkable property shared by all the CFTs, is that they possess an underlying algebraic structure, which comes from the so called “operator product expansion” (OPE). In particular we can express a product of two operators $\mathcal{O}_1(x_1) \mathcal{O}_2(x_2)$ (for simplicity we consider here just scalar operators) in terms of an infinite sum of local primary operators

$$\underbrace{\mathcal{O}_1(x_1) \mathcal{O}_2(x_2)} = \sum_{\mathcal{O}} c_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}} \mathcal{K}_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}}(x_{12}, \partial_{x_2}) \mathcal{O}(x_2) , \quad (1.4)$$

which can be also encoded in the following diagrammatic notation

$$\sum_{\mathcal{O}} \begin{array}{c} \mathcal{O}_1 \\ \diagdown \\ \bigcirc \\ \diagup \\ \mathcal{O}_2 \end{array} = \mathcal{O} , \quad (1.5)$$

where the circles represent the OPE coefficients $c_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}}$. The differential operator $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}}(x_{12}, \partial_{x_2})$ can be reconstructed by replacing (1.4) into all the three point functions involving $\mathcal{O}_1$, $\mathcal{O}_2$ and any third operator $\mathcal{O}$. For example when the operator $\mathcal{O}$ is scalar, we can obtain the structure of $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}}(x_{12}, \partial_{x_2})$ by replacing (1.4) in equation (1.3),

$$\mathcal{K}_{\mathcal{O}_i\mathcal{O}_j\mathcal{O}}(x, \partial_y) = \frac{1}{(x^2)^{\frac{\Delta_i+\Delta_j-\Delta_{\mathcal{O}}}{2}}} [1 + \#x \cdot \partial_y + O(x^2)] . \quad (1.6)$$

In the square brackets there is a differential operator written in a power series of x^μ . All the coefficients of the series can be reconstructed by matching the shape of the three point functions (1.3). The operator $\mathcal{K}$ is therefore completely fixed by kinematics.

Actually an OPE can be formally written in any quantum field theory. However, since the OPE is written in terms of a series, it is crucial to understand which is its radius of convergence. If we consider a CFT, any finite radius of convergence would introduce a length scale in the theory, breaking therefore scale invariance. Therefore the radius of convergence has to be infinite. A finite radius of convergence appears only when we use the OPE in a correlation function with some spectator operators. In this case it is possible to define a convergent OPE of $\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)$ only if one can draw a sphere around x_1 and x_2 which does not contain any of the other operator insertions in the correlation function.

1.3 CFT data

Correlations functions with $n > 3$ operator insertions are less constrained by symmetry. In fact it is possible to build conformal invariant cross ratios out of four or more points,

$$s_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2} . \quad (1.7)$$

In particular with n points it is possible to define $\frac{n(n-3)}{2}$ independent conformal cross ratios. We can see this in the example of a 4-point function, where there are only two independent cross ratios usually defined as

$$u \equiv \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} , \quad v \equiv \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} . \quad (1.8)$$

In this case it is easy to check that all the other choices of s_{ijkl} can always be expressed in terms of u and v (for example $s_{1243} = u/v$). From this observation we therefore conclude that n -point functions are determined up to unknown functions of $\frac{n(n-3)}{2}$ variables. This may sound discouraging. However we can express an n -point functions in terms of a $(n-1)$ -point functions

point functions by means of the OPE,

$$\langle \underbrace{\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)} \mathcal{O}_3(x_3) \dots \mathcal{O}_n(x_n) \rangle = \quad (1.9)$$

$$= \sum_{\mathcal{O}} c_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}} \mathcal{K}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}}(x_{12}, \partial_{x_2}) \langle \mathcal{O}(x_2)\mathcal{O}_3(x_3) \dots \mathcal{O}_n(x_n) \rangle . \quad (1.10)$$

Using iteratively the OPE, any n -point function can be written in terms of two and three point functions. The complete dynamical data entering this reduction is again the spectrum of the theory and the set of OPE coefficients. This very fact leads us to an alternative definition of (the sector of all local observables of) a CFT, which does not make use of a Lagrangian description and it is fully non-perturbative. CFTs are defined by the CFT data: the set of local primary operators (*i.e.* the spectrum of the theory) and the OPE coefficients,

$$\text{CFT data} = \{\mathcal{O}_i, c_{\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k}\} . \quad (1.11)$$

Any CFT can be therefore specified by an (infinite) list of numbers. This is a great simplification with respect to what is known for a generic quantum field theory, where it is usually not possible to make any sort of statement about the theory at finite coupling.

1.4 Conformal bootstrap

It is very important to stress that a generic infinite list of numbers does not define a good CFT. In fact CFTs are expected to be quite rare lists of numbers which have very special relations between themselves. The “conformal bootstrap” approach to CFTs is the idea of constraining the landscape of allowed CFT data by imposing on generic lists of numbers all the possible requirements that we are aware of.

For example a natural requirement that we can impose is that the CFT is a unitary theory. This simple demand, which will be reviewed in detail in section 4.5, already gives a very important bound on the CFT data

$$\text{Unitary CFT data} = \{\mathcal{O}_i : \Delta_i \geq \Delta_i^*, c_{\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k} \in \mathbb{R}\} , \quad (1.12)$$

where the value of Δ_i^* only depends on the $SO(d)$ representation of the operator $\mathcal{O}_i$. For example when $\mathcal{O}_i$ is a scalar operator we find $\Delta_i^* = \frac{d-2}{2}$, while if $\mathcal{O}_i$ is a traceless and symmetric operator with l indices $\Delta_i^* = d + l - 2$.

The most important requirement, which can be imposed in any CFT, is the associativity of the OPE, namely $\underbrace{\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)}\mathcal{O}_3(x_3) = \mathcal{O}_1(x_1)\underbrace{\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)}$. A very useful way to implement this constraint, is by applying it to a four point function. The result is the following crossing equation

$$\langle \underbrace{\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)}\underbrace{\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)} \rangle = \langle \underbrace{\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)}\mathcal{O}_4(x_4) \rangle . \quad (1.13)$$

Since formula (1.13) plays a crucial role in the conformal bootstrap program, we try now to write it in more explicit way which should make manifest its features.

1.5 Conformal blocks

Using the definition (1.4) we can write the double OPEs of (1.13) in terms of differential operators acting on two point functions,

$$\begin{aligned} \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle &= \\ &= \sum_{\mathcal{O}} c_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}} c_{\mathcal{O}_3 \mathcal{O}_4 \mathcal{O}} \underbrace{\mathcal{K}_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}}(x_{12}, \partial_{x_2}) \mathcal{K}_{\mathcal{O}_3 \mathcal{O}_4 \mathcal{O}}(x_{34}, \partial_{x_4}) \langle \mathcal{O}(x_2) \mathcal{O}(x_4) \rangle}_{\equiv G_{\mathcal{O}}(x_1, x_2, x_3, x_4)} . \end{aligned} \quad (1.14)$$

Almost everything in formula (1.14) can be obtained by kinematics. In fact both the differential operators $\mathcal{K}$ and the 2-point function on which they act, are completely fixed by symmetry. It is therefore convenient to define a function $G_{\mathcal{O}}$ that carries all the kinematical information of formula (1.14).

The functions $G_{\mathcal{O}}$ are called conformal partial waves. From formula (1.14) it is natural to associate a diagram to $G_{\mathcal{O}}$ which takes the following form

$$c_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}}c_{\mathcal{O}_3\mathcal{O}_4\mathcal{O}}G_{\mathcal{O}}(x_1,x_2,x_3,x_4)\equiv \begin{array}{c} \mathcal{O}_1 \qquad \qquad \mathcal{O}_3 \\ \diagdown \quad \diagup \\ \bigcirc \quad \equiv \quad \bigcirc \\ \diagup \quad \diagdown \\ \mathcal{O}_2 \qquad \qquad \mathcal{O}_4 \end{array} \quad . \quad (1.15)$$

The conformal partial wave can in principle be fixed in any d -dimensional theory just by using (1.14). However in practice this may not be the most efficient way to compute them. Since conformal partial waves are functions of four points we can exploit conformal invariance in order to fix them up to a function $q_{\mathcal{O}}$ of two cross ratios

$$G_{\mathcal{O}}(x_1, x_2, x_3, x_4) = \frac{\left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_{12}}{2}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_{34}}{2}}}{(x_{12}^2)^{\frac{\Delta_1 + \Delta_2}{2}} (x_{34}^2)^{\frac{\Delta_3 + \Delta_4}{2}}} g_{\mathcal{O}}(u, v) . \quad (1.16)$$

The functions $g_{\mathcal{O}}$ are called conformal blocks (CBs). The CBs are functions which explicitly depend on the spacetime dimension d , two cross ratios and the labels of the operators $\mathcal{O}$, $\mathcal{O}_i$. As an example we write the result for the CBs in $d = 4$, when $\mathcal{O}_i$ are scalar operators with conformal dimensions Δ_i and $\mathcal{O}$ is a spin l traceless and symmetric operator with conformal dimensions Δ ,

$$g_{\mathcal{O}}(u, v) = (-2)^{-l} \frac{z\bar{z}}{z - \bar{z}} [k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - (z \leftrightarrow \bar{z})] , \quad (1.17)$$

$$k_\beta(x) \equiv x^{\beta/2} {}_2F_1\left(\frac{\beta - \Delta_{12}}{2}, \frac{\beta + \Delta_{34}}{2}, \beta; x\right), \quad (1.18)$$

where $\Delta_{ij} \equiv \Delta_i - \Delta_j$ and the two new conformal cross ratios z and $\bar{z}$ are related to the u and v of (1.8) by $u = z\bar{z}$ and $v \equiv (1 - z)(1 - \bar{z})$. Unfortunately the computation of the CBs is not always straightforward. In fact a big part of this thesis will be devoted to the search of systemic strategies to conveniently compute the CBs.

1.6 Comments and motivation

Using the diagrammatic notation for the conformal partial waves we can conveniently represent the crossing equation as follows

$$\sum_{\mathcal{O}} \begin{array}{c} \mathcal{O}_1 \quad \mathcal{O}_3 \\ \diagdown \quad \diagup \\ \text{---} \mathcal{O} \text{---} \\ \diagup \quad \diagdown \\ \mathcal{O}_2 \quad \mathcal{O}_4 \end{array} = \sum_{\mathcal{O}'} \begin{array}{c} \mathcal{O}_1 \quad \mathcal{O}_3 \\ \diagdown \quad \diagup \\ \text{---} \mathcal{O}' \text{---} \\ \diagup \quad \diagdown \\ \mathcal{O}_2 \quad \mathcal{O}_4 \end{array} . \quad (1.19)$$

It is also instructive to spell out the form of the crossing equation after we rewrite the conformal partial waves in terms of the CBs using (1.16). In this way the crossing equation loses its dependence on the coordinate x_i and can be fully expressed in terms of the cross ratios. For example when all the operators are equal scalars ϕ we obtain the following simple formula

$$\sum_{\mathcal{O}} (c_{\phi\phi\mathcal{O}})^2 (v^{\Delta_{\phi}} g_{\mathcal{O}}(u, v) - u^{\Delta_{\phi}} g_{\mathcal{O}}(v, u)) = 0 . \quad (1.20)$$

It is important to make some remarks on the crossing equation. As we already stressed, the unknowns of the equation are the OPE coefficients and the spectrum of the theory. The number of unknowns is therefore infinite. On the other hand it is possible to obtain infinitely many equations from one crossing relation by fixing the value of the two real parameters u and v to some numbers. Also it is possible to consider different choices of the external operators, obtaining an infinite set of inequivalent crossing equations. Therefore even if the number of unknowns is infinite, one may expect that the infinite number of equations could be powerful enough to fix them.

It is also interesting to notice how the equations depend on the unknowns. The OPE coefficients appear always in squares in (1.19). The labels of the exchanged operator $\mathcal{O}$ appear as variables inside the conformal blocks, which are usually quite complicated functions, *e.g.* (1.17). The structure of the equation (1.19) is therefore very non-linear in its unknowns.

From these remarks it is clear that the crossing relation should be very hard to solve. In fact these equations were known since the early works of [3, 4]. However for many years there have not been many attempts to make use of them. An important exception is the two dimensional case, where the bootstrap program was fundamental to classify the rational $2d$ CFTs. This extraordinary result was however possible because the conformal group in $d = 2$ is enhanced to

the infinite dimensional group of transformations generated by the Virasoro algebra. In higher dimensions the problem is much harder and we are just at the beginning of the discovery of its real power. The first breakthrough was the work of [5] from 2008, where the problem of searching for the CFT data of a unitary CFT which respects the crossing equations was made tractable with numerical techniques. This was the birth of the “numerical bootstrap program”. The bootstrap program is now having a new spring and since 2008 there have been very many efforts to constrain the landscape of allowed CFTs in many dimensions and with possible extra symmetries [6–28]. Between these works it certainly stands out the result on the 3 dimensional critical Ising model, where at the moment the numerical bootstrap techniques give the best estimates for its low lying CFT data (which in turns gives the best predictions for the critical exponents of the model). These striking results are strengthening the hope that by using the “conformal bootstrap” philosophy it will be eventually possible to obtain a full characterization of the landscape of CFTs. The motivation to study the subject is therefore very strong.

The only ingredients which are needed in order to use the crossing equation (1.19) are the conformal blocks. If we conjecture that the crossing equation can in fact fully constrain the CFT data, we also are lead to conclude that the CBs should know everything about the landscape of the CFTs. From this point of view the crossing equations can be seen as a way to ask to the CBs which theories are allowed. Therefore, even if new strategy will be developed in the future in order to constrain/find the CFT data, it is very likely that they will all boil down to some operation done on CBs.

1.7 Outline

This thesis is mostly devoted to find CBs. One of the main goals is to obtain a systematic algorithm that can be used to build any CB in any spacetime dimension. In fact we propose two algorithms and one application. We begin this work by reviewing, in section 2, preliminary techniques and definitions for CFTs in d dimensions. In section 3 we explain how to write any conformal block in a convenient series expansion. In section 4 we describe an alternative algorithm to build any conformal block, from a recursion relation. This is based on the idea that the CBs have poles in Δ with residue proportional to other conformal blocks. Finally in section 5 we will use the CBs in order to implement a numerical bootstrap study for the case of four external conserved currents in three dimensions.

Chapter 2

CFT Preliminaries

2.1 Conformal algebra and representations

2.1.1 Conformal Algebra

The conformal group is the set of transformations $x^\mu \rightarrow x'^\mu$ on $\mathbb{R}^d$ which preserves the metric up to a local rescaling,

$$g'^{\mu\nu} = \Omega(x)g^{\mu\nu} . \quad (2.1)$$

It is possible to show that such transformations are of four different types, which we represent in the following table,

$x^{\mu'} = \alpha x^\mu$	Dilations
$x^{\mu'} = \Lambda^\mu_\nu x^\nu$	Rotations
$x^{\mu'} = x^\mu + a^\mu$	Translations
$x^{\mu'} = \frac{x^\mu + b^\mu x^2}{1 + 2b \cdot x + b^2 x^2}$	Special Conformal Transformations

where $\alpha \in \mathbb{R}$, $a^\mu, b^\mu \in \mathbb{R}^d$ and $\Lambda^\mu_\nu \in SO(d)$ are the parameters of the transformations. The special conformal transformations (SCTs) may look exotic, however they can be obtained by the consecutive action of three familiar transformations: an inversion which maps zero to infinity a translation and another inversion. It is however important to define the SCT, since the inversion is a transformation which is not connected with the identity and therefore it does not need to be a symmetry of a generic CFT (in fact only parity invariant CFTs are symmetric under inversion).

When we consider infinitesimal transformations we obtain a representation for the generators $D, P_\mu, K_\mu, L_{\mu\nu}$ respectively of dilatation, translations, special conformal transformations and rotations,

$$\begin{aligned} D &= -ix \cdot \partial , & P^\mu &= -i\partial^\mu , \\ L^{\mu\nu} &= -i(x^\mu \partial^\nu - x^\nu \partial^\mu) , & K^\mu &= -i(2x^\mu x \cdot \partial - x^2 \partial^\mu) . \end{aligned} \quad (2.2)$$

One can check that the generators above form an algebra, the conformal algebra, which is defined

by the following commutation relations

$$\begin{aligned}
[D, P_\mu] &= iP_\mu, & [D, K_\mu] &= -iK_\mu, \\
[P_\mu, L_{\nu\rho}] &= i(\eta_{\mu\nu}P_\rho - \eta_{\mu\rho}P_\nu), & [K_\mu, L_{\nu\rho}] &= i(\eta_{\mu\nu}K_\rho - \eta_{\mu\rho}K_\nu), \\
[K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D - L_{\mu\nu}), \\
[L_{\mu\nu}, L_{\rho\sigma}] &= i(\eta_{\nu\rho}L_{\mu\sigma} \pm \text{perm}),
\end{aligned} \tag{2.3}$$

where the other possible commutations are zero. The conformal group is isomorphic to $SO(d+1, 1)$. One easy way to see this is by grouping the generators as follows

$$J_{0d+1} \equiv D, \quad J_{\mu\nu} \equiv L_{\mu\nu}, \quad J_{0\mu} \equiv \frac{1}{2}(P_\mu - K_\mu), \quad J_{d+1\mu} \equiv \frac{1}{2}(P_\mu + K_\mu).$$

From these definitions one finds that J_{AB} (with $A = 0, 1, \dots, d, d+1$) has the in fact same commutation relations as the generator of $SO(d+1, 1)$

$$[J_{AB}, J_{CD}] = i(\eta_{AD}J_{BC} + \eta_{BC}J_{AD} - \eta_{AC}J_{BD} - \eta_{BD}J_{AC}).$$

The definition of the generator J_{AB} is also convenient in order to find the Casimir operator of the algebra. In fact, since J_{AB} is the usual generator of Lorentz transformations, we obtain for free that $J^2 = J_{AB}J^{AB}$ is the conformal quadratic Casimir.

2.1.2 Labeling the spectrum of a CFT

One important feature of the CFTs is the existence of a map between the local operators and the states of the theory. This map can be seen by choosing a special quantization for the theory, called “radial quantization”. This quantization is defined by foliating the spacetime in concentric $SO(d-1)$ spheres (instead of choosing surfaces of equal Minkowski time). The reason of this different choice is due to the presence of the dilation generator in the algebra. In fact D generates the motion from a sphere to another and therefore plays the role of the Hamiltonian. Another useful way to understand radial quantization is by mapping the radial coordinate r as follows $\tau = \log r$, so that the points $r = 0, \infty$ are respectively mapped to $\tau = -\infty, \infty$ and τ can be interpreted as a time variable ($\tau \in \mathbb{R}$). The resulting geometry is thus the one of a cylinder $\mathbb{R} \times SO(d-1)$. Moreover the foliation in equal concentric spheres is mapped in the usual foliation in equal time τ slices of the cylinder as represented in figure 2.1. Therefore the Hamiltonian on the cylinder H_{cyl} is exactly given by the dilation operator D . We will use this language in the chapter 3.

The most important consequence of radial quantization is that any state of the quantized theory has a description in terms of local operators and vice-versa. In fact given a local operator $\mathcal{O}(x)$ we can simply define a state

$$\mathcal{O}(0) \longrightarrow \mathcal{O}(0)|0\rangle \equiv |\mathcal{O}\rangle. \tag{2.4}$$

Equivalently given a state defined on a given sphere we can evolve it back to the origin of the

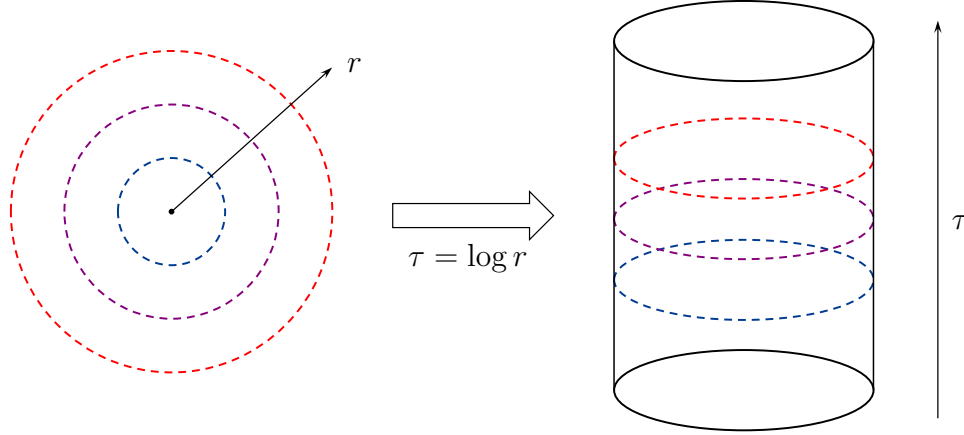


Figure 2.1: Radial quantization and cylinder configuration.

sphere obtaining a local operator

$$|\mathcal{O}\rangle \longrightarrow \mathcal{O}(0)|0\rangle . \quad (2.5)$$

Since the state-operator map is one to one, we can interchangeably use one language or the other. In what follows we characterize the spectrum of the states $|\mathcal{O}\rangle$ of the theory (but of course everything we will say could be equally written in terms of operators).

Since the states $|\mathcal{O}\rangle$ are defined on spheres, it is natural to classify them in terms of their behavior under scaling and rotation. Therefore a state $|\mathcal{O}\rangle$ has two quantum numbers: the conformal dimension Δ and the $SO(d)$ spin. A bosonic state¹ can be written as $|\Delta, \nu_1 \dots \nu_l\rangle$, where the l indices correspond to an irreducible tensor of $SO(d)$ (possibly with mixed symmetry properties). From the algebra (2.3) it is clear that P^μ and K^μ respectively increase and decrease by one unit the conformal dimension of the eigenstates of the dilation operator D . Consequently they can be considered as the creation and annihilation operators of the algebra. We can therefore define an highest weight state² with respect to the action of K^μ , which we call *primary state*. In the tensor representation, the action of the generators on the primary state is given by

$$D|\Delta, \nu_1 \dots \nu_l\rangle = i\Delta|\Delta, \nu_1 \dots \nu_l\rangle , \quad K_\mu|\Delta, \nu_1 \dots \nu_l\rangle = 0 , \quad (2.6)$$

$$J_{\alpha\beta}|\Delta, \nu_1 \dots \nu_l\rangle = \sum_{k=1}^l [M_{\alpha\beta}]_{\mu}^{\nu_k} |\Delta, \nu_1 \dots \nu_{k-1} \mu \nu_{k+1} \dots \nu_l\rangle , \quad (2.7)$$

where

$$[M_{\alpha\beta}]_{\mu}^{\nu} = i (\delta_{\beta}^{\nu} \eta_{\alpha\mu} - \delta_{\alpha}^{\nu} \eta_{\beta\mu}) . \quad (2.8)$$

¹In this work we will not consider fermionic representations of $SO(d)$ but everything could be also generalized to take them in account.

²To be precise, in this context physicists conventionally define a lowest weight of the algebra. This is of course equivalent. In fact the two choices are interchangeably obtained using the map $\Delta \leftrightarrow -\Delta$. For this reason, depending on the context, we will decide to use one notation or the other.

The action of P_μ on a primary creates the so called *descendant* states. If we act n times with P^μ on $|\Delta, \nu_1 \dots \nu_l\rangle$ we obtain a descendant state with dimension $\Delta + n$, which may transform in a different representation of $SO(d)$. A primary state $|\mathcal{O}\rangle$ together with the full set of descendants states $P \dots P|\mathcal{O}\rangle$ form a so called *conformal multiplet* or *conformal representation*.

Actually the notation $|\Delta, \nu_1 \dots \nu_l\rangle$ is schematic, since the tensor indices written in this way do not really specify which $SO(d)$ irreducible representation we are considering. A better way to define an $SO(d)$ irreducible tensor representation is by means of a Young tableau. This is specified by a set of integers $(l_1, \dots, l_N)$, where l_k represents the number of boxes of the k -th row. Every box is filled with one tensor index and the indices in the same row are symmetric while the ones in the columns are antisymmetric. We also remove all possible traces of the tensor. The resulting representation can be written as follows,

$$\mathbf{Y} = \begin{array}{c} \begin{array}{|c|c|c|c|c|c|c|} \hline \mu_1^1 & \cdots & \cdots & \cdots & \cdots & \cdots & \mu_{l_1}^1 \\ \hline \mu_1^2 & \cdots & \cdots & \cdots & \mu_{l_2}^2 & & \\ \hline \vdots & \vdots & \vdots & & & & \\ \hline \mu_1^N & \cdots & \mu_{l_N}^N & & & & \\ \hline \end{array} \\ \end{array} . \quad (2.9)$$

In the following we will label the states either with the list of number $\Delta, l_1, \dots, l_N$ or by making explicit the tensor indices $\Delta, \mathbf{Y}$.

2.1.3 Representations of $SO(d)$

The representations of $SO(d)$ play a fundamental role in CFTs since they label the states in the spectrum of any theory. It is therefore important to find convenient technologies in order to simplify expressions and computations involving such representations.

2.1.3.1 Traceless symmetric representations

A very important role is played by the traceless and symmetric representation of $SO(d)$, which is specified by a simple one-row Young tableau

$$\mathbf{Y} = \begin{array}{|c|c|c|c|} \hline \mu_1 & \mu_2 & \cdots & \mu_l \\ \hline \end{array} . \quad (2.10)$$

This is in fact the only class of $SO(d)$ representations which can appear in the OPE of two scalar operators. Since this representation is ubiquitous in the CFT literature, we will often refer to it shortly as the spin l representation. It is convenient to introduce an index-free notation for such representation [29–31]. In fact a symmetric and traceless tensor $\mathcal{T}^{\mu_1 \dots \mu_l}$ can be encoded into a polynomial $\mathcal{T}(z)$ by

$$\mathcal{T}(z) \equiv \mathcal{T}^{\mu_1 \dots \mu_l} z_{\mu_1} \cdots z_{\mu_l} , \quad (2.11)$$

where z^μ is an auxiliary vector of $\mathbb{C}^d$ that satisfy $z \cdot z = 0$. We can recover the tensor from the polynomial by the relation

$$\mathcal{T}^{\mu_1 \dots \mu_l} = \frac{1}{l!(h-1)_l} D_z^{\mu_1} \dots D_z^{\mu_l} \mathcal{T}(z) . \quad (2.12)$$

where $(a)_l \equiv \Gamma(a+l)/\Gamma(a)$ is the Pochhammer symbol, $h \equiv d/2$ and D_z^μ is the differential operator defined by [29–31]

$$D_z^\mu \equiv \left(h - 1 + z \cdot \frac{\partial}{\partial z} \right) \frac{\partial}{\partial z_\mu} - \frac{1}{2} z^\mu \frac{\partial^2}{\partial z \cdot \partial z} . \quad (2.13)$$

In this formalism, we denote a primary operator with spin l and conformal dimension Δ by

$$\mathcal{O}(x, z) \equiv \mathcal{O}^{\mu_1 \dots \mu_l}(x) z_{\mu_1} \dots z_{\mu_l} . \quad (2.14)$$

It is often useful to know how to write a projector $\pi(\mathbf{Y}, \mathbf{Y}')$ into an $SO(d)$ representation $\mathbf{Y}$. In the traceless and symmetric case, the projector can be written in terms of the auxiliary vectors z^μ and the differential operators D_z^μ as follows

$$\pi(\boxed{\mu_1 \dots \mu_l}, \boxed{\nu_1 \dots \nu_l}) = \frac{1}{l!(h-1)_l} D_z^{\mu_1} \dots D_z^{\mu_l} z^{\nu_1} \dots z^{\nu_l} . \quad (2.15)$$

It is also possible to obtain a closed form for the projector contracted with l vectors $x_{\mu_1} \dots x_{\mu_l}$ and l vectors $y_{\nu_1} \dots y_{\nu_l}$,

$$\pi(\boxed{x \dots x}, \boxed{y \dots y}) = \frac{l!}{2^l (h-1)_l} (x^2 y^2)^{l/2} C_l^{h-1}(\hat{x} \cdot \hat{y}) . \quad (2.16)$$

where $C_l^\nu(x)$ is the Gegenbauer polynomial and $\hat{v}^\mu \equiv v^\mu/|v|$.

2.1.3.2 Mixed symmetric representations

The traceless and symmetric representations of $SO(d)$ are actually enough to study CFTs in three spacetime dimensions. However when $d > 3$ one may need to consider new representations. It is therefore useful to extend the index-free notation also to more general cases [32]. In particular, if we consider a Young tableau $\mathbf{Y}$ specified by $(l_1, l_2, \dots, l_N)$, it is convenient to contract the i -th row of $\mathbf{Y}$ with l_i auxiliary vectors z_i^μ ,

$$\begin{array}{cccccc} \boxed{z_1} & \cdots & \cdots & \cdots & \cdots & \cdots & \boxed{z_1} \\ \boxed{z_2} & \cdots & \cdots & \cdots & \boxed{z_2} & & \\ \vdots & \vdots & \vdots & & & & \\ \boxed{z_N} & \cdots & \boxed{z_N} & & & & \end{array} . \quad (2.17)$$

The N different auxiliary vectors $z_1^\mu, \dots, z_N^\mu$ are again null $z_i^2 = 0$ in order to ensure the tracelessness of each row. This representation is redundant since it does not implement the anti-symmetrization and the tracelessness of the columns. To control the redundancy it is particularly important to find ways to restore the indices with the correct symmetrization.

To do so, one can construct a generalized version of the operator D_z of (2.13). This was done for the case of Young tableaux with two rows (l_1, l_2) [33]. Unfortunately, already for this case, the expressions of the differential operators are quite lengthy so we will not present them here. However they are defined in formula (E.5) in the appendix.

The other strategy which one can generalize makes use of the projectors $\pi(\mathbf{Y}, \mathbf{Y}')$. In [33] the generalization of formula (2.16) was found for four classes of representations with two rows (l_1, l_2) ,

$$\begin{array}{|c|c|c|c|c|c|} \hline & & & & & \dots \\ \hline & & & & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|c|c|} \hline & & & & & \dots \\ \hline & & & & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|c|c|} \hline & & & & & \dots \\ \hline & & & & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|c|c|} \hline & & & & & \dots \\ \hline & & & & & \\ \hline \end{array}, \quad (2.18)$$

and for four classes of representations with tree rows (l_1, l_2, l_3) ,

$$\begin{array}{|c|c|c|c|c|c|} \hline & & & & & \dots \\ \hline & & & & & \\ \hline & & & & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|c|c|} \hline & & & & & \dots \\ \hline & & & & & \\ \hline & & & & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|c|c|} \hline & & & & & \dots \\ \hline & & & & & \\ \hline & & & & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|c|c|} \hline & & & & & \dots \\ \hline & & & & & \\ \hline & & & & & \\ \hline \end{array}. \quad (2.19)$$

All these projectors are known in a closed form for any value of l_1 . Again because of the length of the expressions we will not write them explicitly here (one explicit example will however be presented in equation (3.56) in the following). The representations (2.18) and (2.19) are actually important in relevant cases. In fact they constitute the full set of $SO(d)$ representations which can appear in the OPE $\mathcal{O}_1 \times \mathcal{O}_2$, of operators $\mathcal{O}_i$ of at most spin 2.

2.1.4 Conformal Algebra in Cartan-Weyl Basis

As we already explained, the conformal group is isomorphic to $SO(d+1, 1)$. For some applications it may be useful to know how to cast the conformal algebra in terms of the Cartan-Weyl basis. In this language even and odd d present different features, so we will treat them separately. Consider first the case of odd dimension $d = 2N + 1$. It is convenient to introduce d auxiliary vectors

$$z_j^\eta \equiv \frac{1}{\sqrt{2}} \underbrace{(0, 0, \dots)}_1, \dots, \underbrace{(0, 0, 1, -\eta i)}_{j-1}, \underbrace{(0, 0)}_j, \underbrace{(0, 0)}_{j+1}, \dots, \underbrace{(0, 0, 0)}_N, \quad z_{N+1} \equiv (0, \dots, 0, 1), \quad (2.20)$$

with $\eta = \pm$ and $j = 1, 2, \dots, N$. We can then formulate the conformal algebra in a Cartan-Weyl

basis:

$$\begin{aligned}
E_{\alpha_0^+} &\equiv K \cdot z_{N+1} , & E_{\alpha_{0j}^{+\eta}} &\equiv \frac{1}{\sqrt{2}} K \cdot z_j^\eta , & H_0 &\equiv iD , \\
E_{\alpha_0^-} &\equiv P \cdot z_{N+1} , & E_{\alpha_{0j}^{-\eta}} &\equiv \frac{1}{\sqrt{2}} P \cdot z_j^\eta , & H_j &\equiv i z_j^- \cdot M \cdot z_j^+ , \\
E_{\alpha_j^\eta} &\equiv \sqrt{2} z_j^\eta \cdot M \cdot z_{N+1} , & E_{\alpha_{jk}^{\eta_1 \eta_2}} &\equiv z_j^{\eta_1} \cdot M \cdot z_k^{\eta_2} , & &
\end{aligned} \tag{2.21}$$

with $1 \leq j < k \leq N$. The generators $H_{j=0,1,\dots,N}$ commute among themselves and form the Cartan subalgebra. From this definition and using the commutation relations (2.3) it is easy to check that

$$[H_k, E_\alpha] = (\alpha)^k E_\alpha , \quad [E_\alpha, E_{-\alpha}] = \frac{2}{\langle \alpha, \alpha \rangle} (\alpha)^k H_k . \tag{2.22}$$

where α stands for any root of $\mathfrak{so}(d+2)$ and $(\alpha)^k$ denotes its k -th coordinate. The root system of $\mathfrak{so}(d+2)$ is given by $(N+1)$ -dimensional vectors α_j^η and $\alpha_{ij}^{\eta_1 \eta_2}$,

$$\alpha_j^\eta = (\underbrace{0, \dots, 0}_0, \underbrace{\eta, 0, \dots, 0}_j, \underbrace{0, \dots, 0}_N) , \quad \alpha_{jk}^{\eta_1 \eta_2} = (\underbrace{0, \dots, 0}_0, \underbrace{\eta_1, 0, \dots, 0}_j, \underbrace{0, \eta_2, 0, \dots, 0}_k, \underbrace{0, \dots, 0}_N) . \tag{2.23}$$

In the tensor representation, a generic primary state is defined in terms of Δ and a Young tableau $\mathbf{Y}$. Contracting all the indices of the i -th row of $\mathbf{Y}$ with vectors z_i^+ , we obtain an index-free Young tableau $\mathbf{Y}^+$

$$\mathbf{Y}^+ \equiv \begin{array}{c} \begin{array}{|c|c|c|c|c|c|c|} \hline z_1^+ & \cdots & \cdots & \cdots & \cdots & \cdots & z_1^+ \\ \hline z_2^+ & \cdots & \cdots & \cdots & z_2^+ & & \\ \hline \vdots & \vdots & \vdots & & & & \\ \hline \end{array} \\ \begin{array}{|c|c|c|} \hline z_N^+ & \cdots & z_N^+ \\ \hline \end{array} \end{array} . \tag{2.24}$$

which together with Δ specifies an highest weight state $|\lambda\rangle$ of the full $\mathfrak{so}(d+2)$ algebra. In this notation it is convenient to define $\lambda \equiv (-\Delta, l_1, \dots, l_N)$ because the state with positive Δ is actually a lowest weight with respect to the action of P^μ . Using (2.21) and (2.6-2.7), it is easy to show that $|\lambda\rangle$ is annihilated by all positive roots

$$E_{\alpha_j^+} |\lambda\rangle = 0 , \quad E_{\alpha_{ij}^{+\eta}} |\lambda\rangle = 0 . \tag{2.25}$$

Moreover, one can read off the weights by acting with H_i ,

$$H_0 |\lambda\rangle = -\Delta |\lambda\rangle , \tag{2.26}$$

$$H_i |\lambda\rangle = l_i |\lambda\rangle \quad (1 \leq i \leq N) . \tag{2.27}$$

The quadratic Casimir is given by

$$\mathcal{C} = \sum_{k=0}^N H_k H_k + \frac{1}{2} \sum_{\alpha \in \Phi} \langle \alpha, \alpha \rangle E_\alpha E_{-\alpha} , \tag{2.28}$$

where Φ stands for the set of all roots of $\mathfrak{so}(d+2)$. It is useful to classify the states in terms of

their eigenvalues under the action of the Casimir,

$$\mathcal{C}|\lambda\rangle = c_\lambda|\lambda\rangle . \quad (2.29)$$

In the Cartan-Weyl basis it is trivial to evaluate the Casimir on a the highest weight state $|\lambda\rangle$ by means of (2.28) and (2.22). We can therefore obtain the general form of the eigenvalue c_λ as follows

$$c_\lambda = \langle \lambda, \lambda + 2\rho \rangle = \Delta(\Delta - d) + \sum_{i=1}^N l_i(l_i + d - 2i) , \quad (2.30)$$

where the Weyl vector $\rho \equiv \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$, is the half sum of the positive roots. In this case, it is given by

$$\rho = \left(N + \frac{1}{2}, N - \frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2} \right) . \quad (2.31)$$

The analysis is similar for even dimensions ($d = 2N$). The Cartan subalgebra is the same but we do not have the vector z_{N+1} and the associated generators $E_{\alpha_k^\eta}$. This difference has important consequences when we define the highest weight states. Since now $E_{\alpha_j^+}$ is absent, we can also build an highest weight state contracting the indices of the last line of the Young tableau with z_N^- . As a result, we obtain two highest weight representations associated to each Young tableau with $l_N > 0$,

$$\begin{array}{c} \begin{array}{|c|c|c|c|c|c|c|} \hline z_1^+ & \cdots & \cdots & \cdots & \cdots & \cdots & z_1^+ \\ \hline z_2^+ & \cdots & \cdots & \cdots & z_2^+ & & \\ \hline \vdots & \vdots & \vdots & & & & \\ \hline z_N^+ & \cdots & z_N^+ & & & & \\ \hline \end{array} & , & \begin{array}{|c|c|c|c|c|c|c|} \hline z_1^+ & \cdots & \cdots & \cdots & \cdots & \cdots & z_1^+ \\ \hline z_2^+ & \cdots & \cdots & \cdots & z_2^+ & & \\ \hline \vdots & \vdots & \vdots & & & & \\ \hline z_N^- & \cdots & z_N^- & & & & \\ \hline \end{array} \end{array} . \quad (2.32)$$

It is easy to see that the eigenvalue of H_N is given by l_N for the first case and by $-l_N$ in the second case. Therefore, we will label these two representations with the weight vectors $\lambda = (-\Delta, l_1, \dots, l_N)$ and $\lambda = (-\Delta, l_1, \dots, -l_N)$.

2.2 Correlation functions and tensor structures

2.2.1 The embedding space formalism

As we already mentioned, the conformal group is isomorphic to $SO(d+1, 1)$ and its action on $\mathbb{R}^d$ is given in terms of some non linear transformations. On the other hand $SO(d+1, 1)$ acts in a simple linear way in $\mathbb{M}^{d+2}$. The idea behind the embedding space formalism [31] is to uplift each point in the physical space to a point on the null cone of the embedding space $\mathbb{M}^{d+2}$ defined by

$$P = (P^0, P^\mu, P^{d+1}) \in \mathbb{R}^{1, d+1} , \quad P^2 = -(P^0)^2 + (P^{d+1})^2 + \delta_{\mu\nu} P^\mu P^\nu = 0 . \quad (2.33)$$

The physical space is then described by the set of rays $P \sim \alpha P$ (with $\alpha > 0$). In particular one can obtain $x \in \mathbb{R}^d$ by projecting P onto the Poincaré section $P^0 + P^{d+1} = 1$, parametrized by

$$P_{\text{Poincaré}} = \left(\frac{1+x^2}{2}, x^\mu, \frac{1-x^2}{2} \right). \quad (2.34)$$

Interestingly one can also consider a projection onto different sections. This would correspond to the choice of a different conformally flat physical space. For example in the chapter 3 we will focus on the section $\delta_{\mu\nu} P^\mu P^\nu = 1$. This is naturally parametrized by an Euclidean time $\tau \in \mathbb{R}$ and a unit vector $n^\mu \in S^{d-1} \subset \mathbb{R}^d$,

$$P_{\text{cyl}} = (\cosh \tau, n^\mu, \sinh \tau). \quad (2.35)$$

The induced metric on this section is just the metric on the cylinder $\mathbb{R} \times S^{d-1}$, with line element $ds^2 = d\tau^2 + d\Omega_{S^{d-1}}^2$.

We can extend the definition of local operators from the physical space to the full light cone $\mathcal{O}(x) \rightarrow \mathcal{O}(P)$, requiring the following homogeneity property

$$\mathcal{O}(\lambda P) = \lambda^\Delta \mathcal{O}(P), \quad \lambda \in \mathbb{R}. \quad (2.36)$$

In addition, if one operator is a tensor with l indices in any representation of $SO(d)$, it needs to satisfy the following transversality condition

$$P_{A_1} \mathcal{O}^{A_1 \dots A_l}(P) = 0. \quad (2.37)$$

The operator can be projected back to physical space by means of

$$\mathcal{O}_{\mu_1 \dots \mu_l}(x) = \left. \frac{\partial P^{A_1}}{\partial x^{\mu_1}} \dots \frac{\partial P^{A_l}}{\partial x^{\mu_l}} \mathcal{O}_{A_1 \dots A_l}(P) \right|_{P=P(x)}. \quad (2.38)$$

In order to define an index-free notation it is useful to introduce auxiliary polarization vectors Z^A directly in the embedding space. Contracting a traceless and symmetric operator with Z^A , we obtain

$$\mathcal{O}(P, Z) \equiv Z^{A_1} \dots Z^{A_l} \mathcal{O}_{A_1 \dots A_l}(P). \quad (2.39)$$

The new polarization vectors need to satisfy $Z^2 = 0 = Z \cdot P$ to ensure the tracelessness and the transversality conditions. Very conveniently we can encode all the requirements for operators with dimension Δ and spin l into the following simple relation

$$\mathcal{O}(\alpha P, \beta Z + \gamma P) = \alpha^\Delta \beta^l \mathcal{O}(P, Z), \quad (\alpha, \beta, \gamma \in \mathbb{R}). \quad (2.40)$$

One can also lift the definition of the differential operator D_z^μ as D_Z^A . It is of course possible to define an index-free notation for any mixed symmetry tensor operator, however we will not review it here because it will not be used in this work.

An important reason why we introduced the notion of the embedding space, is because it helps

in constructing correlation functions for operators with spin. In fact we can fix the shape of a correlation function by asking that (2.40) holds for any operator insertion $\mathcal{O}_i(P_i, Z_i)$. This can be done very conveniently by using the building blocks H_{ij} and $V_{i,jk}$ of [31]

$$H_{ij} = \frac{(Z_i \cdot Z_j)(P_i \cdot P_j) - (Z_i \cdot P_j)(P_i \cdot Z_j)}{(P_i \cdot P_j)} , \quad (2.41)$$

$$V_{i,jk} = \frac{(Z_i \cdot P_j)(P_i \cdot P_k) - (Z_i \cdot P_k)(P_i \cdot P_j)}{\sqrt{-2(P_i \cdot P_j)(P_j \cdot P_k)(P_k \cdot P_i)}} , \quad (2.42)$$

which are defined in order to be automatically invariant under the transformation $P_i \rightarrow \lambda_i P_i$ and $Z_i \rightarrow Z_i + \beta_i P_i$. In order to return to the real space we simply have to choose a section of the embedding space. For example in the Poincaré section (2.34),

$$P_i \cdot P_j \rightarrow -\frac{1}{2}x_{ij}^2 , \quad Z_i \cdot P_j \rightarrow -z_i \cdot x_{ij} , \quad Z_i \cdot Z_j \rightarrow z_i \cdot z_j . \quad (2.43)$$

Replacing (2.43) into (2.41-2.42) we can therefore obtain the following conformal invariant building blocks in $\mathbb{R}^d$,

$$H_{ij} \rightarrow \mathfrak{h}_{ij} \equiv z_i \cdot I(x_{ij}) \cdot z_j , \quad (2.44)$$

$$V_{i,jk} \rightarrow \mathfrak{v}_{i,jk} \equiv (z_i \cdot \hat{x}_{ij}) \frac{|x_{ik}|}{|x_{jk}|} - (z_i \cdot \hat{x}_{ik}) \frac{|x_{ij}|}{|x_{jk}|} , \quad (2.45)$$

where we defined $I_{\mu\nu}(x) \equiv \eta_{\mu\nu} - 2\hat{x}_\mu \hat{x}_\nu$ and $\hat{x}^\mu \equiv x^\mu/|x|$. In the following we will give few examples of how to use this technology in order to build correlation functions.

2.2.2 Two point functions

Now that we introduce all the important background we can use it to construct correlation functions. In the case of scalar operators, the two, three and four point functions were already analyzed in the introduction. Here we will focus on operators with $SO(d)$ spin. The first non-trivial case that we want to study is the two-point function.

In the embedding space we can easily obtain the two point function of (canonically normalized) primary operators with dimension Δ and spin l ,

$$\langle \mathcal{O}(P_1, Z_1) \mathcal{O}(P_2, Z_2) \rangle = \frac{H_{12}^l}{P_{12}^\Delta} , \quad (2.46)$$

where $P_{ij} \equiv -2P_i \cdot P_j$. It is trivial to fix this expression in the embedding space by using (2.40). In fact the factor P_{12}^Δ is obtained imposing the correct scaling of $\mathcal{O}(\alpha P) = \alpha^\Delta \mathcal{O}(P)$ and the tensor structures are encoded in the building blocks (2.41-2.42).

The same correlation function can be projected onto the Poincaré section (2.34),

$$\langle \mathcal{O}(x_1, z_1) \mathcal{O}(x_2, z_2) \rangle = \frac{(z_1 \cdot I(x_{12}) \cdot z_2)^l}{x_{12}^{2\Delta}} . \quad (2.47)$$

It is actually easy to write down the most generic bosonic two point function of operators in the representation $\mathbf{Y}$ of $SO(d)$. In fact this can also be obtained in real space without the use of any polarization vector,

$$\langle \mathcal{O}(x_1, \mathbf{Y}) \mathcal{O}(x_2, \mathbf{Y}') \rangle = \frac{\pi (I(x_{12}) \cdot \mathbf{Y}, \mathbf{Y}')}{(x_{12}^2)^\Delta}, \quad (2.48)$$

where the compact notation $I(x_{12}) \cdot \mathbf{Y}$ means that we contract one tensor $I_{\mu\nu}(x_{12})$ with each index of the Young tableau $\mathbf{Y}$. For example if $\mathbf{Y}$ is defined as in (2.9), we define

$$I(x) \cdot \mathbf{Y} \equiv \prod_{j=1}^N \prod_{i=1}^{l_j} I_{\mu_i^j \nu_i^j}(x) \quad \begin{array}{|c|c|c|c|c|c|c|} \hline \nu_1^1 & \cdots & \cdots & \cdots & \cdots & \cdots & \nu_{l_1}^1 \\ \hline \nu_1^2 & \cdots & \cdots & \cdots & \nu_{l_2}^2 & & \\ \hline \vdots & \vdots & \vdots & & & & \\ \hline \nu_1^N & \cdots & \nu_{l_N}^N & & & & \\ \hline \end{array}. \quad (2.49)$$

Of course (2.48) can be also written in the embedding space and/or by using auxiliary vectors [32], however the expression is simple enough not to require the use of more sophisticated technology.

2.2.3 Three point functions

In the present section we analyze the three point functions of operators with spin.

The three point function of two scalar primaries $\mathcal{O}_1, \mathcal{O}_2$ and one spin l primary operator $\mathcal{O}$ is completely fixed up to an overall constant. In the embedding space this can be easily written as

$$\langle \mathcal{O}_1(P_1) \mathcal{O}_2(P_2) \mathcal{O}(P_3, Z_3) \rangle = c_{12\mathcal{O}} \frac{V_{3,21}^l}{(P_{12})^{\frac{\Delta_1+\Delta_2-\Delta}{2}} (P_{13})^{\frac{\Delta+\Delta_{12}}{2}} (P_{23})^{\frac{\Delta-\Delta_{12}}{2}}}, \quad (2.50)$$

where from now on we will use the notation $c_{12\mathcal{O}} \equiv c_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}}$. Again it is trivial to obtain this expression by using the property (2.40) and the building blocks (2.41-2.42). When the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ are not both scalars, the three point functions are not fixed anymore up to a single constant. As an example of this we can consider $\mathcal{O}_1$ to be a spin one operator,

$$\langle \mathcal{O}_1(P_1, Z_1) \mathcal{O}_2(P_2) \mathcal{O}(P_3, Z_3) \rangle = \frac{c_{12\mathcal{O}}^{(1)} V_{1,23} V_{3,12}^l + c_{12\mathcal{O}}^{(2)} H_{13} V_{3,12}^{l-1}}{(P_{12})^{\frac{\Delta_1+\Delta_2-\Delta}{2}} (P_{13})^{\frac{\Delta+\Delta_{12}}{2}} (P_{23})^{\frac{\Delta-\Delta_{12}}{2}}}, \quad (2.51)$$

where the two OPE coefficients $c_{12\mathcal{O}}^{(p)}$ are not fixed by kinematics and are both part of the CFT data. The general three point function of operators with spin l_1, l_2, l_3 can be constructed by using all the possible linear combinations of the conformal invariants

$$\frac{\prod_{i=1}^3 V_i^{m_i} \prod_{i<j} H_{ij}^{n_{ij}}}{(P_{12})^{\frac{\Delta_1+\Delta_2-\Delta}{2}} (P_{13})^{\frac{\Delta+\Delta_{12}}{2}} (P_{23})^{\frac{\Delta-\Delta_{12}}{2}}} \quad (2.52)$$

which respect $m_i + \sum_{j \neq i} n_{ij} = l_i$, where we used the notation

$$V_1 \equiv V_{1,23} , \quad V_2 \equiv V_{2,31} , \quad V_3 \equiv V_{3,12} . \quad (2.53)$$

In fact these are the only independent building blocks, since $V_{i,jk} = -V_{i,kj}$ and $H_{ij} = H_{ji}$. In front of each combination (2.52), specified by the numbers m_i and n_{ij} , there is an independent OPE coefficient which belongs to the CFT data.

2.2.4 The leading OPE

In this section we aim at generalizing the study of the OPE for operators with spin. We will be interested in particular to the leading term of the OPE expansion, which we will use as an alternative method to classify the tensor structures in the three-point functions [31, 34, 35].

The OPE of two scalar operators $\mathcal{O}_1(x_1)$ and $\mathcal{O}_2(x_2)$ was introduced in (1.4). The leading OPE arises by considering the leading term of the expansion for $x_1 \rightarrow x_2$. For example the leading OPE $\mathcal{O}_1 \times \mathcal{O}_2$ of equation (1.4-1.6) for the exchange of a scalar operator $\mathcal{O}$ is

$$\mathcal{O}_1(x)\mathcal{O}_2(0) \sim c_{12\mathcal{O}} \frac{1}{(x^2)^{\frac{\Delta_1+\Delta_2-\Delta}{2}}} \mathcal{O}(x) . \quad (2.54)$$

The symbol $\sim$ in (2.54) means both that we are considering the limit $x \rightarrow 0$ and that we are only considering the exchange of one multiplet labeled by $\mathcal{O}$. Notice that the limit $x \rightarrow 0$ of the differential operator (1.6) has the effect of eliminating all the derivatives of $\mathcal{K}_{12\mathcal{O}}$, which in turns means that we are taking in account only the contribution of the primary, neglecting all its descendants. It is easy to see that the structure $(x^2)^{-\frac{\Delta_1+\Delta_2-\Delta}{2}}$ of (2.54) is actually completely fixed by the scaling properties of the operators. In fact we can use the leading OPE in order to fix the tensor structures appearing in the three point functions.

As an example we consider the case of $\mathcal{O}_1, \mathcal{O}_2$ scalar and $\mathcal{O}$ of spin l . The leading OPE is restricted to be

$$\mathcal{O}_1(x)\mathcal{O}_2(0) \sim \frac{1}{l!} c_{12\mathcal{O}} \frac{\mathcal{O}(0, \partial_z)}{(x^2)^{\frac{\Delta_1+\Delta_2-\Delta+l}{2}}} (x \cdot z)^l , \quad (2.55)$$

where all the traceless and symmetric indices of $\mathcal{O}$ are contracted with derivatives ∂_z (therefore it is not necessary to use D_z). We introduced a factor of $1/l!$ which could be absorbed in the definition of $c_{12\mathcal{O}}$ in order to have the same definition of OPE coefficients given in (2.50). In fact one can project (2.50) onto the Poincaré section and check that the limit $x_1 \rightarrow x_2$ implies that the leading OPE (2.55) holds. This also gives a concrete example that the normalization of the OPEs coefficients may differ in different conventions, so it is fundamental to declare in each case which convention we are choosing.

In section (2.2.3) we explained that the three point functions of operators with spin have in general many conformal invariant tensor structures, which multiply different OPE coefficients. From this fact we can conclude that of the OPE diagram (1.5) needs to be generalized as follows

$$\sum_{\mathcal{O}} \sum_q \begin{array}{c} \mathcal{O}_1 \\ \diagdown \\ \textcircled{q} \\ \diagup \\ \mathcal{O}_2 \end{array} = \mathcal{O} \quad , \quad (2.56)$$

where the circle decorated with the letter q represents the OPE coefficient $c_{12\mathcal{O}}^{(q)}$. To each q corresponds a tensor structure multiplied by a differential operator $\mathcal{K}_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}}^{(q)}$ which generalizes the one of (1.6). As in the scalar case we can study the leading OPE which corresponds to consider just the contribution of the primary.

We first consider three traceless and symmetric primary operators $\mathcal{O}_i$ with conformal dimensions Δ_i and spins l_i . We can define the following leading OPE³

$$\mathcal{O}_1(x_1, z_1) \mathcal{O}_2(0, z_2) \sim \frac{\mathcal{O}_3(0, \partial_{z_3})}{(x^2)^{\frac{\Delta_1 + \Delta_2 - \Delta_3 + l_1 + l_2 + l_3}{2}}} \sum_q c_{123}^{(q)} t^{(q)}(x, z_1, z_2, z_3) . \quad (2.57)$$

The tensor structures $t^{(q)}(x, z_1, z_2, z_3)$ must be Lorentz invariant and satisfy

$$t^{(q)}(\mu x, \lambda_1 z_1, \lambda_2 z_2, \lambda_3 z_3) = \mu^{l_1 + l_2 + l_3} \lambda_1^{l_1} \lambda_2^{l_2} \lambda_3^{l_3} t^{(q)}(x, z_1, z_2, z_3) . \quad (2.58)$$

All the structures $t^{(q)}$ are therefore generated by

$$\prod_i |x|^{l_i - m_i} (x \cdot z_i)^{m_i} \prod_{i < j} (z_i \cdot z_j)^{n_{ij}} , \quad (2.59)$$

such that $m_i + \sum_{j \neq i} n_{ij} = l_i$. Notice that the counting is that same as in (2.52), as it should. From (2.63) one can also write the leading OPE in the other channels. They can be all obtained using the structures $t^{(q)}$ of (2.57), but with the parameters x, z_1, z_2, z_3 appropriately transformed, for example

$$\mathcal{O}_2(x, z_2) \mathcal{O}_3(0, z_3) \sim \frac{\mathcal{O}_1(0, \partial_{z_1})}{(x^2)^{\frac{\Delta_2 + \Delta_3 - \Delta_1 + l_1 + l_2 + l_3}{2}}} \sum_q c_{123}^{(q)} t^{(q)}(-x, z_1, I(x) \cdot z_2, z_3) . \quad (2.60)$$

Since the only non trivial part of (2.60) is the transformation of the variables in the structures $t^{(q)}$, we help the reader by showing the following table,

OPE channel	Structure
$\mathcal{O}_1 \times \mathcal{O}_2 \sim \mathcal{O}_3$	$t^{(q)}(x, z_1, z_2, z_3)$
$\mathcal{O}_2 \times \mathcal{O}_1 \sim \mathcal{O}_3$	$t^{(q)}(-x, z_1, z_2, z_3)$
$\mathcal{O}_3 \times \mathcal{O}_2 \sim \mathcal{O}_1$	$t^{(q)}(x, z_1, I(x) \cdot z_2, z_3)$

(2.61)

³In chapter 4 we make use of (2.57) in a non-conventional channel. Everything written here will hold after the replacement $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3 \rightarrow \mathcal{O}, \mathcal{O}_1, \mathcal{O}_2$.

which represents the structures $t^{(q)}$ in the channel (2.57) and the two other channels where we exchange $1 \leftrightarrow 2$ and $1 \leftrightarrow 3$. Composing the transformations presented in (2.61) one can construct all the possible OPE channels. For example (2.60) is a composition of $1 \leftrightarrow 3$ and $1 \leftrightarrow 2$.

A very important feature of the leading OPE tensor structures $t^{(q)}$ is that they can be simply lifted in order to reconstruct the full three point function

$$\langle \mathcal{O}_1(x_1, z_1) \mathcal{O}_2(x_2, z_2) \mathcal{O}_3(x_3, z_3) \rangle = \sum_q c_{123}^{(q)} \mathcal{T}^{(p)}(\{x_i, z_i\}) , \quad (2.62)$$

by defining [31, 35]

$$\mathcal{T}^{(p)}(\{x_i, z_i\}) \equiv \frac{t^{(q)} \left(\frac{x_{13}x_{23}^2 - x_{23}x_{13}^2}{|x_{12}||x_{23}||x_{13}|}, I(x_{13}) \cdot z_1, I(x_{23}) \cdot z_2, z_3 \right)}{(x_{12}^2)^{\frac{\Delta_1 + \Delta_2 - \Delta_3}{2}} (x_{23}^2)^{\frac{\Delta_2 + \Delta_3 - \Delta_1}{2}} (x_{13}^2)^{\frac{\Delta_1 + \Delta_3 - \Delta_2}{2}}} . \quad (2.63)$$

In fact taking the appropriate OPE limits of (2.63) it is easy to show that the table (2.61) holds. Another interesting case that is useful to mention is the leading OPE of two traceless symmetric operators $\mathcal{O}_1, \mathcal{O}_2$ with spin l_1, l_2 which are exchanging an operator $\mathcal{O}_3$ in a generic $SO(d)$ representation $\mathbf{Y}_3$ labelled by $(l_{3,1}, l_{3,2}, l_{3,3})$ (this is the most generic representation that can appear in this OPE),

$$\mathcal{O}_1(x, z_1) \mathcal{O}_2(0, z_2) \sim \frac{\mathcal{O}_3(0, \mathbf{Y}_3)}{(x_{12}^2)^{\frac{\Delta_1 + \Delta_2 - \Delta_3}{2}}} \sum_q c_{123}^{(q)} t^{(q)}(\hat{x}, z_1, z_2, \mathbf{Y}_3) , \quad (2.64)$$

where the notation with the two sets of indices $\mathbf{Y}_3$ on the right hand side of (2.64) means that they are contracted. The tensor structures defined in this way are not index-free. They are generated by distributing the indices of $\mathbf{Y}_3$ in the vectors $\hat{x}, z_1, z_2$, such that,

$$t^{(q)}(\alpha x, \alpha_1 z_1, \alpha_2 z_2, \mathbf{Y}_3) = \alpha^{l_1 + l_2 + l_{3,1}} \alpha_1^{l_1} \alpha_2^{l_2} t^{(q)}(\hat{x}, z_1, z_2, \mathbf{Y}_3) . \quad (2.65)$$

It is of course possible to define a similar leading OPE for three operators in generic $SO(d)$ representations, however this will not be useful in this work.

2.2.5 Four point functions

A four point functions of traceless and symmetric operators $\mathcal{O}_i$ with dimensions Δ_i and spin l_i

$$\mathcal{F}_4(\{P_i, Z_i\}) \equiv \langle \mathcal{O}_1(P_1, Z_1) \mathcal{O}_2(P_2, Z_2) \mathcal{O}_3(P_3, Z_3) \mathcal{O}_4(P_4, Z_4) \rangle \quad (2.66)$$

can be written in the embedding space as

$$\mathcal{F}_4(\{P_i, Z_i\}) = \frac{\left(\frac{P_{24}}{P_{14}}\right)^{\frac{\Delta_1 - \Delta_2}{2}} \left(\frac{P_{14}}{P_{13}}\right)^{\frac{\Delta_3 - \Delta_4}{2}}}{(P_{12})^{\frac{\Delta_1 + \Delta_2}{2}} (P_{34})^{\frac{\Delta_3 + \Delta_4}{2}}} \sum_s \mathbf{f}_s(u, v) Q_s(\{P_i, Z_i\}) , \quad (2.67)$$

where Q_s are all the possible scalar structures constructed with P_i and Z_i with the following property

$$Q_s(\{\lambda_i P_i, \alpha_i Z_i + \beta_i P_i\}) = Q_s(\{P_i, Z_i\}) \prod_i (\alpha_i)^{l_i} . \quad (2.68)$$

As usual, we generate Q_s by using the building blocks $V_{i,jk}$ and H_{ij} of (2.41-2.42). The building blocks are not completely independent, they satisfy the identity

$$(P_2 \cdot P_3)(P_1 \cdot P_4)V_{1,23} + (P_2 \cdot P_4)(P_1 \cdot P_3)V_{1,42} + (P_3 \cdot P_4)(P_1 \cdot P_2)V_{1,34} = 0 , \quad (2.69)$$

which means that for each point i we only have two independent choices of $V_{i,jk}$. A convenient choice is $W_i \equiv V_{i,i+1,i+2}$ and $\bar{W}_i \equiv V_{i,i-1,i+2}$, where the subscripts i, j, k of $V_{i,jk}$ are defined modulo 4. The structures $Q_s(\{P_i, Z_i\})$ are therefore generated by

$$\prod_{i=1}^4 W_i^{m_i} \bar{W}_i^{\bar{m}_i} \prod_{i < j} H_{ij}^{n_{ij}} , \quad (2.70)$$

such that $m_i + \bar{m}_i + \sum_{j \neq i} n_{ij} = l_i$. For example, in the case of three scalars and one vector, the resulting basis for independent structures is just

$$\begin{cases} Q_1(\{P_i\}, Z_1) = V_{1,23} \\ Q_2(\{P_i\}, Z_1) = V_{1,34} \end{cases} . \quad (2.71)$$

There are other identities among these structures. For example in $d = 3$ one can define

$$W_{ij}^{A_1 \dots A_6} = Z_i^{[A_1} Z_j^{A_2} P_1^{A_3} P_2^{A_4} P_3^{A_5} P_4^{A_6]} = 0 . \quad (2.72)$$

The contraction $W_{(12)(34)} = \eta_{A_1 B_1} \dots \eta_{A_6 B_6} W_{12}^{A_1 \dots A_6} W_{34}^{B_1 \dots B_6}$ can be written as a linear combination of the tensor structures that form a basis for four-point functions. This gives rise to a linear relation between the tensor structures Q_s . Using the 3 invariants $W_{(12)(34)}$, $W_{(13)(24)}$ and $W_{(14)(23)}$ we obtain 2 independent relations between the structures Q_s in $d = 3$.⁴

More examples will be written in the following chapters.

Instead of the cross ratio u and v we will often use the so called “radial coordinates” of [36]. These are defined by performing a conformal transformation to map x_i to the configuration shown in figure 2.2, where r and $\eta = \cos \theta$. To state the exact relation between the radial coordinates r, η and the standard cross ratios

$$r e^{i\theta} = \frac{z}{(1 + \sqrt{1 - z})^2} , \quad r e^{-i\theta} = \frac{\bar{z}}{(1 + \sqrt{1 - \bar{z}})^2} , \quad (2.73)$$

where z and $\bar{z}$ were already defined in the introduction by $u = z\bar{z}$ and $v = (1 - z)(1 - \bar{z})$.

⁴One can check the identity $W_{(12)(34)} + W_{(13)(24)} - W_{(14)(23)} = 0$ for any d .

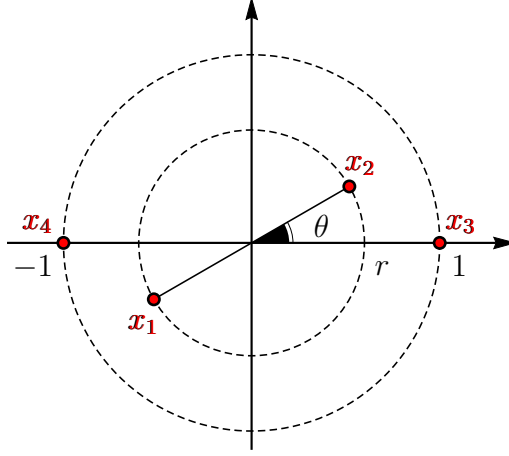


Figure 2.2: Radial coordinates of [36].

2.2.6 Conformal Blocks

2.2.6.1 Tensor structures and radial coordinates

In this section we define the main conventions for the conformal blocks, which will be used in the next chapters.

Let us now consider the conformal partial wave decomposition of the four point function $\mathcal{F}_4(\{P_i, Z_i\})$ defined in (2.66). If we consider the OPE channel $\mathcal{O}_1 \times \mathcal{O}_2$ and $\mathcal{O}_3 \times \mathcal{O}_4$ we obtain

$$\mathcal{F}_4(\{P_i, Z_i\}) = \sum_{\mathcal{O}} \sum_{p,q} c_{12\mathcal{O}}^{(p)} c_{34\mathcal{O}}^{(q)} G_{\mathcal{O}}^{(p,q)}(\{P_i, Z_i\}) \quad (2.74)$$

$$= \sum_{\mathcal{O}} \sum_{p,q} \begin{array}{c} \mathcal{O}_1 \\ \diagdown \quad \diagup \\ \textcircled{p} \text{---} \textcircled{q} \\ \diagup \quad \diagdown \\ \mathcal{O}_2 \quad \mathcal{O}_3 \\ \mathcal{O}_4 \end{array} . \quad (2.75)$$

Notice that in this case $\mathcal{O}$ is not fixed to be traceless and symmetric. It is labeled by Δ and a Young tableau $\mathbf{Y}$ (with at most three rows). Using (2.67) we can encode the conformal partial wave into a sum of CBs which just depend on the cross ratios

$$G_{\mathcal{O}}^{(p,q)}(\{P_i, Z_i\}) = \frac{\left(\frac{P_{24}}{P_{14}}\right)^{\frac{\Delta_1 - \Delta_2}{2}} \left(\frac{P_{14}}{P_{13}}\right)^{\frac{\Delta_3 - \Delta_4}{2}}}{(P_{12})^{\frac{\Delta_1 + \Delta_2}{2}} (P_{34})^{\frac{\Delta_3 + \Delta_4}{2}}} \sum_s g_{\mathcal{O},s}^{(p,q)}(r, \eta) Q_s(\{P_i, Z_i\}) . \quad (2.76)$$

In (2.76), instead of u and v , we use the radial coordinates r, η of [36].

2.2.6.2 Casimir equation

Another reason to use the embedding formalism is that it simplifies the computations for the Casimir equation. In fact in the embedding space the generators of conformal transformations

are simple rotations

$$J^{MN} = -i (P^M \partial_P^N - P^N \partial_P^M + Z^M \partial_Z^N - Z^N \partial_Z^M) , \quad (2.77)$$

and the Casimir is just $J^{MN} J_{NM}$. Let us now consider a four point function where we are performing the OPE in the channels 1-2 and 3-4. Acting with the Casimir on the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ has the effect of reading the label $c_{\mathcal{O}}$ of the exchanged operator. Therefore the CBs solve the following differential equation [37]

$$\frac{1}{2}(J_1 + J_2)^2 G_{\mathcal{O}}^{(p,q)}(\{P_i, Z_i\}) = c_{\mathcal{O}} G_{\mathcal{O}}^{(p,q)}(\{P_i, Z_i\}) , \quad (2.78)$$

$$c_{\mathcal{O}} = \Delta(\Delta - 2h) + \sum_{i=1}^{[h]} l_i(l_i + 2h - 2i) . \quad (2.79)$$

as we found in (2.30). By replacing the definition (2.76), one can convert (2.79) into a set of coupled partial differential equations for the functions $g_{\mathcal{O},s}^{(p,q)}(r, \eta)$. In particular the structure of such equations can be schematically expressed as

$$\sum_{s'} \mathcal{D}_{ss'}(r, \eta) g_{\mathcal{O},s'}^{(p,q)}(r, \eta) = c_{\mathcal{O}} g_{\mathcal{O},s}^{(p,q)}(r, \eta) , \quad (2.80)$$

where $\mathcal{D}_{ss'}(r, \eta)$ is a differential operator of the second order in r and η (since we are using the quadratic Casimir). This set of differential equations is usually too complicated to be solved analytically. However it may simplify in some special cases.

For example in the case of equal external scalar operators we can considerably simplify formula (2.80) since the four point function has a single tensor structure. We can conveniently cast the differential equation in z and $\bar{z}$ coordinates as follows

$$\mathcal{D}(z, \bar{z}) g_{\Delta l}(z, \bar{z}) = c_{\Delta l} g_{\Delta l}(z, \bar{z}) , \quad (2.81)$$

where $\mathcal{D}$ is defined by

$$\begin{aligned} \mathcal{D}(z, \bar{z}) \equiv & 2[z^2(1-z)\partial_z^2 - z^2\partial_z] + 2[\bar{z}^2(1-\bar{z})\partial_{\bar{z}}^2 - \bar{z}^2\partial_{\bar{z}}] \\ & + 2(d-2)\frac{z\bar{z}}{z-\bar{z}}[(1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}] . \end{aligned} \quad (2.82)$$

This differential equation can be exactly solved when d is even. For example one can check that (1.17) is a solution of (2.81) when $d = 4$. More examples of the Casimir equation can be found in the following chapters.

Chapter 3

Radial Expansion for Conformal Blocks

In this chapter we develop a method to compute any bosonic conformal block as a series expansion in the optimal radial coordinate introduced by Hogervorst and Rychkov. The method reduces to the known result when the external operators are all the same scalar operator, but it allows to compute conformal blocks for external operators with spin. Moreover, we explain how to write closed form recursion relations for the coefficients of the expansions. We study three examples of four point functions in detail: one vector and three scalars; two vectors and two scalars; two spin 2 tensors and two scalars. Finally, for the case of two external vectors, we also provide a more efficient way to generate the series expansion using the analytic structure of the blocks as a function of the scaling dimension of the exchanged operator.

3.1 Review of the scalar conformal block

Let us start by introducing a four point function of scalar operators written in the embedding space. We can expand it in conformal blocks as follows

$$\langle \mathcal{O}_1(P_1) \mathcal{O}_2(P_2) \mathcal{O}_3(P_3) \mathcal{O}_4(P_4) \rangle = \sum_{\mathcal{O}} c_{12\mathcal{O}} c_{34\mathcal{O}} G_{\Delta,l}(P_i) , \quad (3.1)$$

where the sum is over primary operators $\mathcal{O}$ with dimension Δ and spin l . We will focus on the cylinder section (2.35) parametrized by $P_{\text{cyl}} = (\cosh \tau, n^\mu, \sinh \tau)$, where $\tau \in \mathbb{R}$ and $n^\mu \in S^{d-1} \subset \mathbb{R}^d$ is a unit vector. The following choice of configuration leads to the radial coordinates of [38],

$$\begin{aligned} P_1 &= (1, n, 0) , & P_3 &= (\cosh \tau, -n', \sinh \tau) , \\ P_2 &= (1, -n, 0) , & P_4 &= (\cosh \tau, n', \sinh \tau) , \end{aligned} \quad (3.2)$$

with n and n' unit vectors in $\mathbb{R}^d$. This choice of null vectors in the embedding space corresponds to the configuration on the cylinder $\mathbb{R} \times S^{d-1}$ shown in figure 3.1. Substituting the choice (3.2) for P_i in $G_{\Delta,l}$ we obtain the conformal block on the cylinder,

$$c_{12\mathcal{O}} c_{34\mathcal{O}} G_{\Delta,l}(P_i) \rightarrow \mathcal{G}_{\Delta,l}(r, \eta) . \quad (3.3)$$

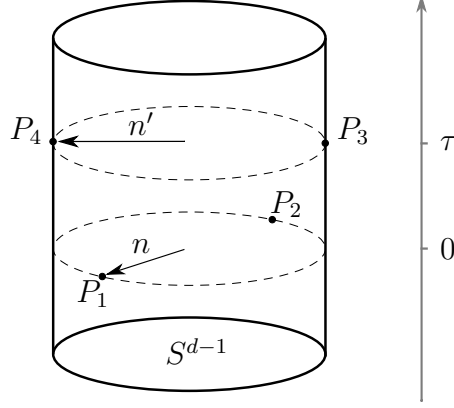


Figure 3.1: Cylinder configuration that leads to the radial coordinates of [38].

where we introduced the variables r and η of [38],

$$r = e^{-\tau} , \quad \eta = n \cdot n' . \quad (3.4)$$

It is trivial to check that r and η are related to the usual cross ratios as follows

$$u = \frac{P_{12}P_{34}}{P_{13}P_{24}} = \frac{16r^2}{(1 + 2\eta r + r^2)^2} , \quad v = \frac{P_{14}P_{23}}{P_{13}P_{24}} = \frac{(1 - 2\eta r + r^2)^2}{(1 + 2\eta r + r^2)^2} , \quad (3.5)$$

where $P_{ij} = -2P_i \cdot P_j$.

3.1.1 A natural series expansion

On the cylinder it is also possible to write

$$\mathcal{G}_{\Delta,l}(r, \eta) = \langle \mathcal{O}_4(n') \mathcal{O}_3(-n') | r^{H_{cyl}} \mathcal{P}_l | \mathcal{O}_2(-n) \mathcal{O}_1(n) \rangle , \quad (3.6)$$

where H_{cyl} is the Hamiltonian on the cylinder and $\mathcal{P}_l$ is the projector into the conformal family with primary $\mathcal{O}_{\Delta,l}$. It is natural to rewrite the projector as a sum over a complete set of states

$$\mathcal{G}_{\Delta,l}(r, \eta) = \sum_{m=0}^{\infty} r^{\Delta+m} \sum_{j=\max(0,l-m)}^{l+m} \sum_a \langle \mathcal{O}_4(n') \mathcal{O}_3(-n') | m, [\mu_1 \mu_2 \dots \mu_j], a \rangle \langle m, [\mu_1 \mu_2 \dots \mu_j], a | \mathcal{O}_2(-n) \mathcal{O}_1(n) \rangle , \quad (3.7)$$

where the state $|m, [\mu_1 \mu_2 \dots \mu_j], a\rangle$ represents descendants of the operator $\mathcal{O}$ at the level m and with spin j . The label a distinguishes states at the same level and with the same spin. In this case, the only states that contribute are symmetric traceless representations of $SO(d)$ because the inner products in (3.7) vanish for more general representations. The inner product can be written as

$$\langle m, [\mu_1 \mu_2 \dots \mu_j], a | \mathcal{O}_2(-n) \mathcal{O}_1(n) \rangle = u(m, j, a) n_{(\mu_1} \dots n_{\mu_l)} , \quad (3.8)$$

where the parenthesis select only the traceless and symmetric part of the tensor. We may choose to set $u(0, l) = c_{12\mathcal{O}}$.¹ From these definitions we automatically get

$$\mathcal{G}_{\Delta, l}(r, \eta) = \sum_{m=0}^{\infty} r^{\Delta+m} \sum_{j=\max(0, l-m)}^{l+m} w(m, j) \mathcal{C}_j(n \cdot n'), \quad (3.9)$$

where $w(m, j) \equiv \sum_a u(m, j, a) \tilde{u}(m, j, a)$, and $\tilde{u}(m, j, a)$ comes from the inner product involving operators $\mathcal{O}_3$ and $\mathcal{O}_4$. The function $\mathcal{C}_j$ is a scalar spherical harmonic defined by

$$\mathcal{C}_j(n \cdot n') = n_{\alpha_1} \dots n_{\alpha_j} \pi(\boxed{\alpha_1 \alpha_2 \dots \alpha_j}, \boxed{\beta_1 \beta_2 \dots \beta_j}) n'_{\beta_1} \dots n'_{\beta_j}, \quad (3.10)$$

where π is the projector into traceless and symmetric tensors with j indices. Moreover, the harmonic function $\mathcal{C}_l(\eta)$ can be explicitly written in terms of Gegenbauer polynomials,

$$\mathcal{C}_l(\eta) \equiv \frac{l!}{(2h-2)_l} C_l^{(h-1)}(\eta), \quad (3.11)$$

as obtained in (2.16). Notice that the normalization in front of the Gegenbauer polynomial is such that $\mathcal{C}_l(1) = 1$ as expected from formula (3.10) when $n = n'$. The crucial point that makes the expansion (3.9) useful is that the coefficients $w(m, j)$ inherit some properties from the structure of the conformal representation exchanged in the conformal block. For example, we automatically get that at level $m = 0$ in the expansion there exists a single non-zero coefficient. It corresponds to the exchange of the primary, namely

$$w(0, j) = c_{12\mathcal{O}} c_{34\mathcal{O}} \delta_{j, l}. \quad (3.12)$$

Moreover, all level m descendants must have spin j in the interval $[\max(0, l-m), l+m]$, since they are created by the action of m derivatives, as each one of them can at most increase or decrease the spin of the operator by one. Therefore the coefficients have the following structure

$$w(m, j) = 0, \quad |j - l| > m, \quad (3.13)$$

which is depicted in figure 3.2. In addition, if $\mathcal{O}_4 = \mathcal{O}_1$ and $\mathcal{O}_3 = \mathcal{O}_2$ then unitarity implies that all coefficients are non-negative, $w(m, j) \geq 0$.

3.1.2 A recurrence relation for the expansion coefficients

We described the radial expansion for the conformal blocks, but we still need to fix the coefficients $w(m, j)$. In the following we explain how to use the conformal Casimir equation to obtain a recurrence relation for $w(m, j)$. To do so we first exploit the conformal symmetry to fix the conformal partial wave in terms of a function depending only on the cross ratios as we have seen

¹At level $m = 0$ there is no degeneracy, so we can drop the label a .

in formula (1.16),

$$G_{\Delta,l}(P_i) = \frac{\left(\frac{P_{24}}{P_{14}}\right)^{\frac{\Delta_1-\Delta_2}{2}} \left(\frac{P_{14}}{P_{13}}\right)^{\frac{\Delta_3-\Delta_4}{2}}}{(P_{12})^{\frac{\Delta_1+\Delta_2}{2}} (P_{34})^{\frac{\Delta_3+\Delta_4}{2}}} g_{\Delta,l}(r, \eta) . \quad (3.14)$$

We then write the conformal Casimir equation in the embedding space as written in (2.79),

$$(J_1 + J_2)^2 G_{\Delta,l}(P_i) = c_{\Delta,l} G_{\Delta,l}(P_i) , \quad (3.15)$$

in such a way to obtain a second order differential equation for $g_{\Delta,l}(r, \eta)$. To constrain the coefficients $w(m, j)$ we finally need to relate $\mathcal{G}_{\Delta,l}(r, \eta)$ with $g_{\Delta,l}(r, \eta)$. This is trivial, we simply have to write (3.14) using the definitions (3.2) for the points P_i on the cylinder,

$$\mathcal{G}_{\Delta,l}(r, \eta) = c_{12\mathcal{O}} c_{34\mathcal{O}} \mathcal{P}(r, \eta) g_{\Delta,l}(r, \eta) , \quad (3.16)$$

where

$$\frac{\left(\frac{P_{24}}{P_{14}}\right)^{\frac{\Delta_1-\Delta_2}{2}} \left(\frac{P_{14}}{P_{13}}\right)^{\frac{\Delta_3-\Delta_4}{2}}}{(P_{12})^{\frac{\Delta_1+\Delta_2}{2}} (P_{34})^{\frac{\Delta_3+\Delta_4}{2}}} \rightarrow \mathcal{P}(r, \eta) \equiv \frac{1}{2^{\sum_i \Delta_i}} \left(\frac{r^2 + 2\eta r + 1}{r^2 - 2\eta r + 1} \right)^{\frac{1}{2}(\Delta_{12}-\Delta_{34})} . \quad (3.17)$$

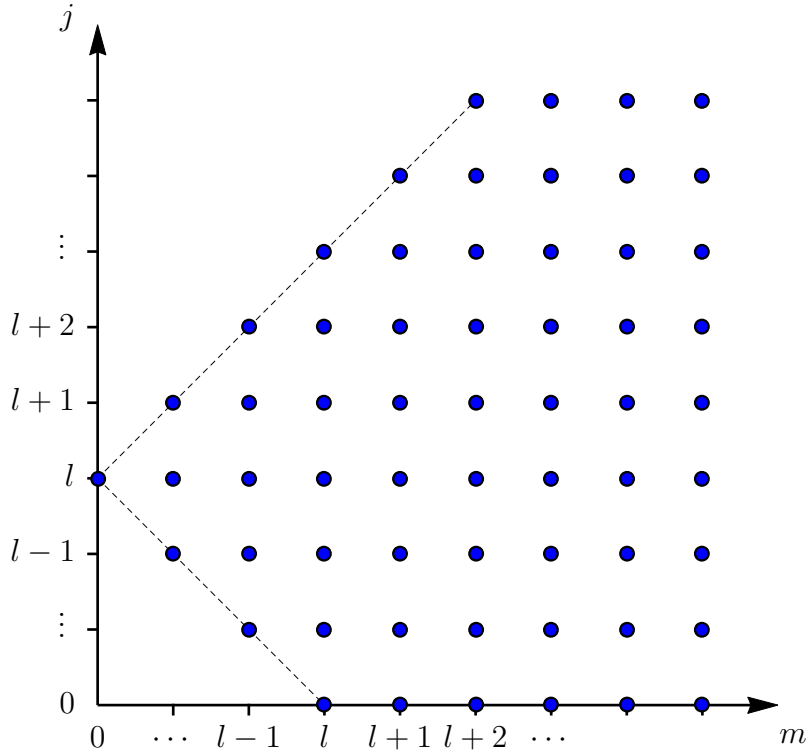


Figure 3.2: Support of the coefficients $w(m, j)$ in the expansion of $\mathcal{G}_{\Delta,l}$.

Using these definitions we obtain a differential equation on $\mathcal{G}_{\Delta,l}$,

$$\begin{aligned}
& (r^2 - 1) \left[f_1^2 f_2^2 c_{\Delta,l} - 4r \left((\Delta_{12}^2 + \Delta_{34}^2) f_3 + \Delta_{12} \Delta_{34} f_4 \eta (r^2 + 1) \right) \right] \mathcal{G}_{\Delta,l} \\
& + f_1 f_2 \eta (r^2 - 1) \left(f_4 + 8\eta^2 h r^2 - 2h (r^2 + 1)^2 \right) \partial_\eta \mathcal{G}_{\Delta,l} - f_1^2 f_2^2 (\eta^2 - 1) (r^2 - 1) \partial_\eta^2 \mathcal{G}_{\Delta,l} \\
& - f_1 f_2 r \left[4\eta^2 r^2 (-2h (r^2 + 1) + r^2 + 3) + (r^2 + 1) \left(2h (r^2 + 1)^2 + r^4 - 8r^2 - 1 \right) \right] \partial_r \mathcal{G}_{\Delta,l} \\
& - f_1^2 f_2^2 r^2 (r^2 - 1) \partial_r^2 \mathcal{G}_{\Delta,l} = 0 ,
\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}
f_1 &= (1 + r^2 - 2r\eta) , & f_2 &= (1 + r^2 + 2r\eta) , \\
f_3 &= r \left((r^2 + 1)^2 - 2\eta^2 (r^4 + 1) \right) , & f_4 &= r^2 (4\eta^2 + r^2 - 6) + 1 .
\end{aligned} \tag{3.19}$$

We now explain how to convert any Casimir differential equation into closed form recurrence relations for the coefficients of the expansion of the conformal blocks. One can convince oneself that any Casimir equation can be casted in the form²

$$p(r, \eta, \partial_r, \partial_\eta) \mathfrak{F}(r, \eta) = 0 , \tag{3.20}$$

where $p(x_1, x_2, x_3, x_4)$ is a polynomial in the variables x_i and we choose the order of the variables such that the derivatives are on the right. As an example, (3.18) is of the form (3.20). We want to show that if we expand $\mathfrak{F}(r, \eta)$ as follows

$$\mathfrak{F}(r, \eta) = \sum_{m,j=0}^{\infty} a_{m,j} \mathfrak{f}_{m,j}(r, \eta) , \quad \mathfrak{f}_{m,j}(r, \eta) = r^m \mathcal{C}_j(\eta) , \tag{3.21}$$

we are able to recast the differential equation (3.20) as a closed form algebraic relation on the coefficients $a_{m,j}$. To do so we need to know the action of η , ∂_η , r and ∂_r on the basis $\mathfrak{f}_{m,j}(r, \eta)$, that is

$$\begin{aligned}
r \mathfrak{f}_{m,l}(r, \eta) &= \mathfrak{f}_{m+1,l}(r, \eta) , & \partial_r \mathfrak{f}_{m,l}(r, \eta) &= m \mathfrak{f}_{m-1,l}(r, \eta) , \\
\eta \mathfrak{f}_{m,l}(r, \eta) &= \frac{1}{2(h+l-1)} (l \mathfrak{f}_{m,l-1}(r, \eta) + (2h+l-2) \mathfrak{f}_{m,l+1}(r, \eta)) , \\
\partial_\eta \mathfrak{f}_{m,l}(r, \eta) &= \frac{1}{(1-\eta^2)} \frac{l(2h+l-2)}{2(h+l-1)} (\mathfrak{f}_{m,l-1}(r, \eta) - \mathfrak{f}_{m,l+1}(r, \eta)) .
\end{aligned} \tag{3.22}$$

Using these formulas recursively one can obtain the action of any polynomial $p(r, \eta, \partial_r, \partial_\eta)$ on $\mathfrak{f}_{m,l}$. The only subtlety comes from the derivatives in η , since they bring a factor $(1 - \eta^2)$ in the denominator. This is not a problem since we can multiply the full equation by $(1 - \eta^2)$ each time we use the last equation of (3.22). This is enough to show that we can rewrite (3.20) as a

²Notice that the Casimir equations may contain rational functions of r and η . This is not a problem since one can always multiply the full expression by all the denominators.

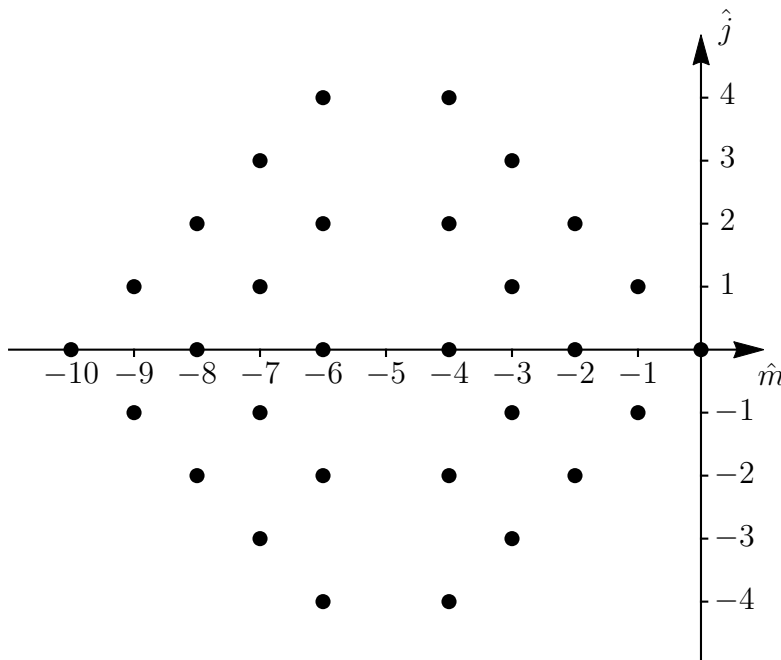


Figure 3.3: The set of points $\mathcal{S}$ involved in the recurrence relation for the coefficients $w(m, j)$ of the scalar conformal block.

closed form algebraic relation for the coefficients $a_{m,j}$. Nevertheless, in appendix A we explain useful manipulations to simplify the algebraic relations.

Applying this technology to the differential equation (3.18) we obtain that a linear combination of a finite number of coefficients $w(m, j)$ with some shifts in m and j is zero, schematically

$$\sum_{(\hat{m}, \hat{j}) \in \mathcal{S}} c(\hat{m}, \hat{j}) w(m + \hat{m}, j + \hat{j}) = 0 . \quad (3.23)$$

The set $\mathcal{S}$ that we obtain contains 30 points for example $(0, 0), (-1, 1), (-1, -1), \dots$, as pictured in figure 3.3, and $c(\hat{m}, \hat{j})$ are known functions of the variables $\Delta_{12}, \Delta_{34}, \Delta, l, h$, and m and j . We can therefore express the coefficient $w(m, j)$ with maximal m (the point at the origin of the axes in figure 3.3) as a linear combination of $w(m', j')$ with lower m' .

Moreover, if we set $m = 0$ in (3.23) we obtain a constraint on the possible initial conditions

$$(j - l)w(0, j) = 0 , \quad (3.24)$$

which is solved imposing $w(0, j) = 0$ for $j \neq l$. For $j = l$, instead, the coefficient $w(0, l)$ is left unrestricted. This is consistent with the initial condition (3.12) obtained from the structure of the conformal families.

Notice also that the restriction on the initial condition, together with the shape of figure 3.3, imply that the coefficients of the expansion organize as pictured in figure 3.2, which was obtained from the properties of conformal families.

Equation (3.23) with the initial condition (3.12) is enough to determine recursively the coefficients $w(m, j)$ for any m and j . We will not present here explicitly the relation (3.23) since the formula is very long. Instead, it is defined in a `Mathematica` file. As an example we present the

first coefficients of the expansion for generic parameters

$$\begin{aligned} w(1, l-1) &= \frac{\Delta_{12}\Delta_{34}l}{(h+l-1)(-\Delta+2h+l-2)} , \\ w(1, l+1) &= -\frac{\Delta_{12}\Delta_{34}(2h+l-2)}{(h+l-1)(\Delta+l)} . \end{aligned} \tag{3.25}$$

Notice that they go to zero when $\Delta_{34} = 0 = \Delta_{12}$, as expected from the results of [38].

Formula (3.23) is a new result, since in [38] it was not obtained a closed form recurrence relation for the coefficients of the expansion. The main reason for this is that the authors of [38] kept some rational functions of r and η (which have infinite expansion in small r) in the Casimir differential equation. Thus the associated recurrence relation would have had a set $\mathcal{S}$ containing an infinite number of points. On the other hand, multiplying the Casimir equation by all the denominators it becomes possible to find a recurrence relation with a finite set $\mathcal{S}$. Notice that this example explicitly shows that there exist many possible equivalent recurrence relations, which however are not equivalently efficient. In fact we expand on this point in appendix A, where we explain some strategies to further reduce the number of points in $\mathcal{S}$ with respect to the algorithm proposed in the main text. We would like to stress that while some features of the figure 3.3 have a physical meaning (they are related to the structure of conformal families as mentioned above), some others come just as a result of the computation and we do not know why they should hold from general principles. For example it is not clear why a priori we should expect a symmetric picture (with respect to the axis $\hat{m} = -5$) and why the points at $\hat{m} = -5$ are absent. In fact it would be interesting to understand which features are necessary and which is the minimal set $\mathcal{S}$.

As a last comment we mention that the recurrence relation (3.23) can further simplify when we specialize the value of some parameters. For example when $\Delta_{12} = 0 = \Delta_{34}$ all the coefficients $c(\hat{m}, \hat{j})$ for $\hat{m}$ odd vanish, therefore the actual support $\mathcal{S}$ reduces to a set of 18 points. Since the recurrence relation is written in an analytic form it is also possible to access limit cases when some parameters (like Δ , l , h) are large. For more details about the simplification of the recurrence relation in special cases, the reader may check the `Mathematica` files where all the definitions are presented explicitly.

3.2 Spinning conformal blocks

In this section we present the general framework to obtain the radial expansion of conformal blocks for external bosonic operators with spin. Initially we exemplify the formalism applying it to the simpler case of one external vector operator and three scalars, for which the allowed exchanged operators are still symmetric and traceless. After that, in separate subsections, we apply the method to the cases of two external vectors and two scalars, and of two spin 2 operators and two scalars. In these cases there are new mixed tensor operators that can be exchanged.

3.2.1 General formalism and example of one vector and three scalars

The conformal block expansion in the embedding formalism of a four point function of operators $\mathcal{O}_i$ with spin l_i and conformal dimension Δ_i was introduced in section 2.2.5. As an example, for one external vector we have

$$\langle \mathcal{O}_1(P_1, Z_1) \mathcal{O}_2(P_2) \mathcal{O}_3(P_3) \mathcal{O}_4(P_4) \rangle = \sum_{\mathcal{O}} \sum_{p=1}^2 c_{12\mathcal{O}}^{(p)} c_{34\mathcal{O}} G_{\Delta, l}^{(p)}(P_i, Z_1) . \quad (3.26)$$

The two possible values $p = 1, 2$ correspond to the two possible tensor structures of the three-point function $\langle \mathcal{O}_1(P_1, Z_1) \mathcal{O}_2(P_2) \mathcal{O}(P, Z) \rangle$ between external operators $\mathcal{O}_1$, $\mathcal{O}_2$ and the exchanged operator $\mathcal{O}$, which in this example can only be symmetric traceless of generic spin l . As in the scalar case, we are interested in writing the conformal blocks in the cylinder configuration. To do so we can use formulae (3.2) and the following definition for the embedding space polarization vectors

$$\begin{aligned} Z_1 &= (z_1 \cdot n, z_1, z_1 \cdot n) , & Z_3 &= (-z_3 \cdot n' e^\tau, z_3, -z_3 \cdot n' e^\tau) , \\ Z_2 &= (-z_2 \cdot n, z_2, -z_2 \cdot n) , & Z_4 &= (z_4 \cdot n' e^\tau, z_4, z_4 \cdot n' e^\tau) , \end{aligned} \quad (3.27)$$

where z_i are polarization vectors in $\mathbb{R}^d$ which obey $z_i^2 = 0$. Notice that $Z_i^2 = Z_i \cdot P_i = 0$, as it should [39]. With these definitions we get

$$\sum_{p,q} c_{12\mathcal{O}}^{(p)} c_{34\mathcal{O}}^{(q)} G_{\lambda}^{(p,q)}(P_i, Z_i) \rightarrow \mathcal{G}_{\lambda}(r, \eta, n \cdot z_i, n' \cdot z_i, z_i \cdot z_j) . \quad (3.28)$$

Therefore $\mathcal{G}_{\lambda}^{(p,q)}$ can be expanded in polynomials p_s of the variables $n \cdot z_i$, $n' \cdot z_i$, $z_i \cdot z_j$ with weight l_i in each variable z_i ,

$$\mathcal{G}_{\lambda}(r, \eta, n \cdot z_i, n' \cdot z_i, z_i \cdot z_j) = \sum_s p_s(n \cdot z_i, n' \cdot z_i, z_i \cdot z_j) F_s(r, \eta) , \quad (3.29)$$

where the coefficient multiplying each polynomial is a function F_s of r , and η and s labels the independent tensor structures of the four-point function. The functions F_s depend on the representation λ but we omit this label to avoid cluttering the equations. For one external vector we have

$$\mathcal{G}_{\Delta, l} = (n \cdot z_1) F_1(r, \eta) + (n' \cdot z_1) F_2(r, \eta) . \quad (3.30)$$

On the other hand, just as for the scalar case (3.6), the conformal block on the cylinder can be written as

$$\mathcal{G}_{\lambda}(r, \eta, n \cdot z_i, n' \cdot z_i, z_i \cdot z_j) = \langle \mathcal{O}_4(n', z_4) \mathcal{O}_3(-n', z_3) | r^{H_{cyl}} \mathcal{P}_{\lambda} | \mathcal{O}_2(-n, z_2) \mathcal{O}_1(n, z_1) \rangle , \quad (3.31)$$

where $\mathcal{P}_{\lambda}$ is the projector into the conformal family with highest weight λ . It is natural to

rewrite the projector as a sum over a complete basis of states

$$\mathcal{G}_\lambda = \sum_{m=0}^{\infty} r^{\Delta+m} \sum_{\mathbf{Y}} \sum_a \langle \mathcal{O}_4(n', z_4) \mathcal{O}_3(-n', z_3) | m, \mathbf{Y}, a \rangle \langle m, \mathbf{Y}, a | \mathcal{O}_2(-n, z_2) \mathcal{O}_1(n, z_1) \rangle, \quad (3.32)$$

where we sum over all states at level m of the conformal family, organized in irreducible representations (irreps) $\mathbf{Y}$ of $SO(d)$. For example, for one external vector, the representation $\mathbf{Y}$ can only be symmetric and traceless, therefore

$$\langle m, [\mu_1 \mu_2 \dots \mu_j], a | \mathcal{O}_2(-n) \mathcal{O}_1(n, z_1) \rangle = [u_1(m, j, a) z_1 \cdot n + u_2(m, j, a) z_1 \cdot \nabla_n] n_{(\mu_1} \dots n_{\mu_j)}, \quad (3.33)$$

where

$$(\nabla_n)_\mu = \frac{\partial}{\partial n^\mu} - n_\mu n \cdot \frac{\partial}{\partial n} \quad (3.34)$$

computes the covariant derivative on the sphere S^{d-1} parametrized by n and the coefficients u_k are not fixed by rotational invariance. We can choose a basis for the OPE coefficients $c_{12\mathcal{O}}^{(p)}$ such that

$$\langle 0, [\mu_1 \mu_2 \dots \mu_j] | \mathcal{O}_2(-n) \mathcal{O}_1(n, z_1) \rangle = [c_{12\mathcal{O}}^{(1)} z_1 \cdot n + c_{12\mathcal{O}}^{(2)} z_1 \cdot \nabla_n] n_{(\mu_1} \dots n_{\mu_j)}. \quad (3.35)$$

In this formalism we can easily deal with more complicated irreps $\mathbf{Y}$ of $SO(d)$, which will appear in the next sections. For now, we continue the study of the case with one external vector, for which (3.32) becomes

$$\mathcal{G}_{\Delta, l} = \sum_{m=0}^{\infty} r^{\Delta+m} \sum_j \sum_a \tilde{u}(m, j, a) [u_1(m, j, a) z_1 \cdot n + u_2(m, j, a) z_1 \cdot \nabla_n] \mathcal{C}_j(n \cdot n'), \quad (3.36)$$

where $\tilde{u}(m, j, a)$ comes from the inner product of the scalars $\mathcal{O}_3$ and $\mathcal{O}_4$ with the exchanged states. It is convenient to introduce the following two functions of r and η ,

$$W_s(r, \eta) \equiv \sum_{m=0}^{\infty} r^{\Delta+m} \sum_{j=\max(0, l-m)}^{l+m} w_s(m, j) \mathcal{C}_j(\eta), \quad s = 1, 2, \quad (3.37)$$

where $w_s(m, j) \equiv \sum_a \tilde{u}(m, j, a) u_s(m, j, a)$. The choice (3.35) corresponds to the following initial condition

$$w_s(0, j) = c_{12\mathcal{O}}^{(s)} c_{34\mathcal{O}} \delta_{j, l}. \quad (3.38)$$

The functions W_s are the basic building blocks for the radial expansion. On the other hand the Casimir equation is naturally written as a differential equation for the conformal blocks $g_{\lambda, s}$ (defined in (2.74)) as we schematically showed in equation (2.80). In the following we explain how to relate $g_{\lambda, s}$ to W_s in a generic framework while exemplifying it for the case of one external vector operator. This process is trivial, however it may look involved since it utilizes the following

sequence of relations

$$g_s(r, \eta) \implies G(P_i, Z_i) \implies F_s(r, \eta) \implies W_s(r, \eta) . \quad (3.39)$$

As a first step we relate the functions W_s to the functions F_s defined in (3.29). For one external vector this can be done expanding (3.36) and collecting terms according to (3.30),

$$F_1(r, \eta) = W_1(r, \eta) - \eta \partial_\eta W_2(r, \eta) , \quad F_2(r, \eta) = \partial_\eta W_2(r, \eta) . \quad (3.40)$$

The second step is to relate the conformal partial waves G_λ to the functions F_s . This can be done by mapping the partial waves G_λ of (2.76) to the cylinder using the definitions (3.2) and (3.27), and collecting terms according to (3.29). In this way we obtain a relation between G_λ and F_s . Finally one can relate G_λ and $g_{\lambda,s}$ by using formula (2.76). Notice that in order to map the functions $g_{\lambda,s}$ to the functions F_s we just need a linear transformation, since these functions only correspond to different choices of four point function tensor structures.

For example, in the one vector case we only have two possible tensor structures in (2.76),

$$Q_1(P_i, Z_i) = V_{1,23} , \quad Q_2(P_i, Z_i) = V_{1,43} , \quad (3.41)$$

which can be evaluated on the cylinder using (3.2) and (3.27),

$$\begin{aligned} V_{1,23} &\rightarrow \frac{(-r^2 + 2\eta r + 1) n \cdot z_1 - 2r z_1 \cdot n'}{(r^2 - 2\eta r + 1)^{1/2} (r^2 + 2\eta r + 1)^{1/2}} , \\ V_{1,43} &\rightarrow \frac{2\eta r^2 n \cdot z_1 - (r^2 + 1) z_1 \cdot n'}{(r^2 - 2\eta r + 1)^{1/2} (r^2 + 2\eta r + 1)^{1/2}} . \end{aligned} \quad (3.42)$$

We then replace (3.42) and (3.17) in (2.76). Collecting terms according to (3.30) we finally get

$$\begin{aligned} \sum_{p=1}^2 c_{12\mathcal{O}}^{(p)} c_{34\mathcal{O}} g_1^{(p)}(r, \eta) &= \mathcal{A}(r, \eta) [2\eta r^2 F_2(r, \eta) + (r^2 + 1) F_1(r, \eta)] , \\ \sum_{p=1}^2 c_{12\mathcal{O}}^{(p)} c_{34\mathcal{O}} g_2^{(p)}(r, \eta) &= \mathcal{A}(r, \eta) [(r^2 - 2\eta r - 1) F_2(r, \eta) - 2r F_1(r, \eta)] , \end{aligned} \quad (3.43)$$

where we defined the function

$$\mathcal{A}(r, \eta) = -\frac{2\sum_i \Delta_i}{1 - r^2} \left(\frac{1 - 2\eta r + r^2}{1 + 2\eta r + r^2} \right)^{\frac{1}{2}(\Delta_{12} - \Delta_{34} + 1)} . \quad (3.44)$$

The last step is to find a recurrence relation for the coefficients $w_s(m, j)$ of the functions W_s . To do so we write the Casimir equation as in (2.80), then we replace g_s in terms of W_s as described above. Repeating the same procedure described in the scalar case, we can trade (see also appendix A) the differential equations for coupled recurrence relations on the coefficients $w_s(m, j)$. When we do this for the case of one external vector we obtain two coupled recurrence

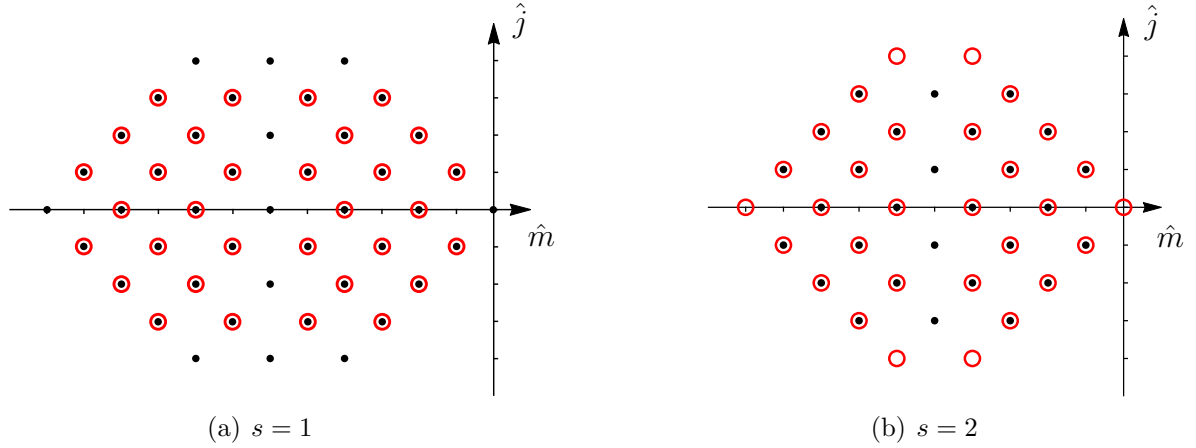


Figure 3.4: The sets $\mathcal{S}_{s'}$ appearing in the linear combination (3.45) for $s = 1, 2$. Red circles correspond to $s' = 2$ and black dots correspond to $s' = 1$.

relations of the kind (the precise formulae are defined in a `Mathematica` file)

$$\sum_{s'=1}^2 \sum_{(\hat{m}, \hat{j}) \in \mathcal{S}_{s'}^s} c_{s'}^s(\hat{m}, \hat{j}) w_{s'}(m + \hat{m}, j + \hat{j}) = 0, \quad s = 1, 2, \quad (3.45)$$

where the sets of point $\mathcal{S}_{s'}^s$ are represented in figures 3.4(a) and 3.4(b). The sets $\mathcal{S}_1^s$ are pictured by black dots while the set $\mathcal{S}_2^s$ by red circles. In particular the sets $\mathcal{S}_{s'}^1$ of the first recurrence relation are shown in figure 3.4(a) while the sets $\mathcal{S}_{s'}^2$ of the second one are in figure 3.4(b). We can then obtain the coefficients of the expansion iterating (3.45) with the initial condition (3.38). Equation (3.45) for $m = 0$ reduces to

$$(j - l)w_1(0, j) = 0, \quad j(j - l)w_2(0, j) = 0, \quad (3.46)$$

which constraints the possible initial conditions. In fact, (3.46) implies $w_1(0, j) = w_2(0, j) = 0$, for any $j \neq l$. On the other hand, $w_1(0, l)$ and $w_2(0, l)$ are not constrained by equation (3.45). This is consistent with the initial condition (3.38) that we derived from the structure of the conformal families. The two independent solutions of equation (3.45) are the two conformal blocks that we labeled with $p = 1, 2$. The last remark is that (3.45) does not constrain $w_2(m, 0)$. This actually make sense since in (3.40) the function W_2 only appears derived once in η and therefore the $j = 0$ term is annihilated. In fact, for practical purposes one could redefine W_2 avoiding to sum over the terms with $j = 0$.

As an example we present all the coefficients at level $m = 1$ of the expansion of the functions W_s , which build the conformal block $G_{\Delta, l}^{(p)}$. At level zero we set the initial condition $w_s(0, l) = \delta_{s, p}$.

Using (3.45) we therefore obtain

$$\begin{aligned}
w_1(1, l+1) &= -\frac{\Delta_{34}(2h+l-2)(\Delta_{12}\delta_{p,1} + l\delta_{p,2})}{(h+l-1)(\Delta+l)}, \\
w_2(1, l+1) &= -\frac{\Delta_{34}(2h+l-2)(\delta_{p,1} + \Delta_{12}l\delta_{p,2})}{(l+1)(h+l-1)(\Delta+l)}, \\
w_1(1, l-1) &= \frac{\Delta_{34}l(\Delta_{12}\delta_{p,1} - (2h+l-2)\delta_{p,2})}{(h+l-1)(-\Delta+2h+l-2)}, \\
w_2(1, l-1) &= -\frac{\Delta_{34}l(\delta_{p,1} - \Delta_{12}(2h+l-2)\delta_{p,2})}{(h+l-1)(2h+l-3)(-\Delta+2h+l-2)}.
\end{aligned} \tag{3.47}$$

Finally it is worth mentioning that the case $\Delta_{12} = 0 = \Delta_{34}$ drastically simplifies the recurrence relations. In fact in this case not only all the $c_{s'}^s(\hat{m}, \hat{j})$ with odd $\hat{m}$ are automatically zero, but also the two recurrence relations completely decouple. In fact $c_{s'}^s(\hat{m}, \hat{j}) = 0$ if $s' \neq s$. We stress that this result comes out by direct computation, but we do not have any insight of why this needs to happen. For more details about the simplification of the recurrence relations for special values of the parameters we encourage the reader to use the `Mathematica` files.

3.2.2 Two external vectors and two scalars

In this section we describe how to obtain a radial expansion for conformal blocks of two external vectors and two scalars. In particular, we consider the following conformal block decomposition

$$\begin{aligned}
\langle \mathcal{O}_1(P_1, Z_1) \mathcal{O}_2(P_2) \mathcal{O}_3(P_3, Z_3) \mathcal{O}_4(P_4) \rangle &= \sum_{\mathcal{O} \in \square\square\square\square} \sum_{p,q=1}^2 c_{12\mathcal{O}}^{(p)} c_{34\mathcal{O}}^{(q)} G_{\Delta,l}^{(p,q)}(P_i, Z_i) \\
&+ \sum_{\mathcal{O} \in \square\square\square\square} c_{12\mathcal{O}} c_{34\mathcal{O}} G_{\Delta,l,1}(P_i, Z_i).
\end{aligned} \tag{3.48}$$

There exist four conformal partial waves $G_{\Delta,l}^{(p,q)}$ (for $p, q = 1, 2$) when the exchanged multiplet has an highest weight $\mathcal{O}$ in the symmetric and traceless representation. Additionally, there is a single partial wave $G_{\Delta,l,1}$ when $\mathcal{O}$ is in the representation $(\Delta, l, 1)$. We shall see that the radial expansion allows us to find all the cases at once.

First, we consider the contribution from each conformal family on the cylinder

$$\sum_{p,q=1}^2 c_{12\mathcal{O}}^{(p)} c_{34\mathcal{O}}^{(q)} G_{\Delta,l}^{(p,q)}(P_i, Z_i) \rightarrow \mathcal{G}_{\Delta,l}, \quad c_{12\mathcal{O}} c_{34\mathcal{O}} G_{\Delta,l,1}(P_i, Z_i) \rightarrow \mathcal{G}_{\Delta,l,1}. \tag{3.49}$$

On the cylinder, both these functions can be expanded in the following tensor structures

$$\begin{aligned}
\mathcal{G}_\lambda &= z_1 \cdot n \, z_3 \cdot n' F_1(r, \eta) + z_1 \cdot n \, z_3 \cdot n F_2(r, \eta) + \\
&+ z_1 \cdot n' z_3 \cdot n' F_3(r, \eta) + z_1 \cdot n' z_3 \cdot n F_4(r, \eta) + z_1 \cdot z_3 F_5(r, \eta).
\end{aligned} \tag{3.50}$$

To implement the expansion (3.32) for two external vectors we need to consider the contribution from two types of states at each level m (the descendants of the two $SO(d)$ irreps that are exchanged in this case). The first is the same that we obtained in the one external vector case (3.33). The second one is new, with a coupling between one of the vectors and one of the scalars written as

$$\left\langle m, \begin{array}{|c|c|c|c|} \hline \mu_1 & \mu_2 & \dots & \mu_j \\ \hline \nu & & & \end{array}, a | \mathcal{O}_2(-n) \mathcal{O}_1(n, z_1) \right\rangle = N_{\mathcal{V}} u_0(m, j, a) \pi \left(\begin{array}{|c|c|c|c|} \hline \mu_1 & \mu_2 & \dots & \mu_j \\ \hline \nu & & & \end{array}, \begin{array}{|c|c|c|c|} \hline \alpha_1 & \alpha_2 & \dots & \alpha_j \\ \hline \beta & & & \end{array} \right) (z_1)_{\beta} n_{\alpha_1} \dots n_{\alpha_j} , \quad (3.51)$$

where the normalization constant $N_{\mathcal{V}}$ is introduced for later convenience. Notice that the value at $m = 0$ of $u_s(0, j)$ has to be different depending on what is the representation λ of the exchanged primary operator. In fact, if λ is the symmetric and traceless representation, then

$$u_0(0, l) = 0 , \quad u_q(0, l) = c_{12\mathcal{O}}^{(q)} , \quad q = 1, 2 , \quad (3.52)$$

and when $\lambda = (\Delta, l, 1)$, then

$$u_0(0, l) = c_{12\mathcal{O}} , \quad u_q(0, l) = 0 , \quad q = 1, 2 . \quad (3.53)$$

We are lead to the following expansion for the conformal block on the cylinder

$$\mathcal{G}_{\lambda} = \sum_{m=0}^{\infty} r^{\Delta+m} \sum_j \sum_a \left[N_{\mathcal{V}}^2 u_0(m, j, a) \tilde{u}_0(m, j, a) (z_1)_{\mu} (z_3)_{\nu} \mathcal{V}_j^{\mu\nu}(n, n') + \right. \\ \left. + (u_1(m, j, a) z_1 \cdot n + u_2(m, j, a) z_1 \cdot \nabla_n) (\tilde{u}_1(m, j, a) z_3 \cdot n' + \tilde{u}_2(m, j, a) z_3 \cdot \nabla_{n'}) \mathcal{C}_j(n \cdot n') \right] , \quad (3.54)$$

where

$$\mathcal{V}_j^{\mu\nu}(n, n') = n_{\alpha_1} \dots n_{\alpha_j} \pi \left(\begin{array}{|c|c|c|c|} \hline \alpha_1 & \alpha_2 & \dots & \alpha_j \\ \hline \mu & & & \end{array}, \begin{array}{|c|c|c|c|} \hline \beta_1 & \beta_2 & \dots & \beta_j \\ \hline \nu & & & \end{array} \right) n'_{\beta_1} \dots n'_{\beta_j} \quad (3.55)$$

is a (double) vector spherical harmonic ³ with index μ at the point n and index ν at the point n' . The double vector harmonic can be expressed in terms of Gegenbauer functions as follows [33]

$$N_{\mathcal{V}}^2 \mathcal{V}_l^{\mu\nu}(n, n') (z_1)_{\mu} (z_3)_{\nu} = \mathcal{C}_l''(\eta) (-\eta n \cdot z_1 z_3 \cdot n' + z_1 \cdot n' z_3 \cdot n' + n \cdot z_1 n \cdot z_3 - z_1 \cdot z_3) \\ - (n \cdot z_3 z_1 \cdot n' - \eta z_1 \cdot z_3) ((2h - 2) \mathcal{C}_l'(\eta) + \eta \mathcal{C}_l''(\eta)) , \quad (3.56)$$

where the normalization constant $N_{\mathcal{V}}^2$ is fixed by requiring that π is a projector

$$N_{\mathcal{V}}^2 \equiv \frac{2^l l(h)_{l-1}}{(2h-1)_{l-2}} . \quad (3.57)$$

It is then convenient to introduce five functions which have a nice expansion in radial coordinates

³See appendix F of [33] on the relation between projectors and spherical harmonics.

and which encode the information to generate the conformal blocks

$$W_s(r, \eta) = \sum_{j,m} w_s(j, m) r^{\Delta+m} \mathcal{C}_j(\eta) , \quad s = 1, 2, 3, 4, 5 , \quad (3.58)$$

with

$$\begin{aligned} w_1(m, j) &= \sum_a u_1(m, j, a) \tilde{u}_1(m, j, a) , \\ w_2(m, j) &= \sum_a u_1(m, j, a) \tilde{u}_2(m, j, a) , \\ w_3(m, j) &= \sum_a u_2(m, j, a) \tilde{u}_1(m, j, a) , \\ w_4(m, j) &= \sum_a u_2(m, j, a) \tilde{u}_2(m, j, a) , \\ w_5(m, j) &= \sum_a u_0(m, j, a) \tilde{u}_0(m, j, a) . \end{aligned} \quad (3.59)$$

Next we want to write the conformal partial waves in terms of the functions W_s . First we obtain the relation between F_s and W_s according to the definition (3.50),

$$\begin{aligned} F_1(r, \eta) &= W_1(r, \eta) + \eta \partial_\eta [\eta \partial_\eta W_4(r, \eta) - W_2(r, \eta) - W_3(r, \eta) - \partial_\eta W_5(r, \eta)] , \\ F_2(r, \eta) &= \partial_\eta [-\eta \partial_\eta W_4(r, \eta) + W_2(r, \eta) + \partial_\eta W_5(r, \eta)] , \\ F_3(r, \eta) &= \partial_\eta [-\eta \partial_\eta W_4(r, \eta) + W_3(r, \eta) + \partial_\eta W_5(r, \eta)] , \\ F_4(r, \eta) &= -[2(-1+h) + \eta \partial_\eta] \partial_\eta W_5(r, \eta) + \partial_\eta^2 W_4(r, \eta) , \\ F_5(r, \eta) &= [2(-1+h)\eta + (-1+\eta^2) \partial_\eta] \partial_\eta W_5(r, \eta) + \partial_\eta W_4(r, \eta) . \end{aligned} \quad (3.60)$$

Secondly we write the conformal partial wave in the embedding space in terms of functions of the cross ratios

$$\sum_{p,q} c_{12\mathcal{O}}^{(p)} c_{34\mathcal{O}}^{(q)} G_\lambda^{(p,q)}(\{P_i, Z_i\}) = \frac{\left(\frac{P_{24}}{P_{14}}\right)^{\frac{\Delta_1-\Delta_2}{2}} \left(\frac{P_{14}}{P_{13}}\right)^{\frac{\Delta_3-\Delta_4}{2}}}{(P_{12})^{\frac{\Delta_1+\Delta_2}{2}} (P_{34})^{\frac{\Delta_3+\Delta_4}{2}}} \sum_{s=1}^5 g_{\lambda,s}(r, \eta) Q_s(\{P_i, Z_i\}) , \quad (3.61)$$

where

$$\begin{aligned} Q_1 &= H_{1,3} , \\ Q_2 &= V_{1,23} V_{3,21} , \\ Q_3 &= V_{1,23} V_{3,41} , \\ Q_4 &= V_{1,43} V_{3,21} , \\ Q_5 &= V_{1,43} V_{3,41} . \end{aligned} \quad (3.62)$$

It is then straightforward to write the functions $g_{\lambda,s}$ in terms of the functions F_s ,

$$\begin{aligned} g_1(r, \eta) &= \mathcal{A} f_4 F_5 (r^2 - 1)^2 , \\ g_2(r, \eta) &= \mathcal{A} f_1 (-2r^2 (f_4 F_5 + 2F_3 \eta r) + f_2 (2F_4 \eta r^2 + F_2 (r^2 + 1)) - 2F_1 (r^3 + r)) , \\ g_3(r, \eta) &= \mathcal{A} f_1 (2f_4 F_5 r + 2\eta (2F_4 \eta r^2 + (r^2 + 1) (F_3 r^2 + F_2)) + F_1 (r^2 + 1)^2) , \\ g_4(r, \eta) &= \mathcal{A} f_1 (2r (f_3 F_3 + f_4 F_5) - f_2 (f_3 F_4 + 2F_2 r) + 4F_1 r^2) , \\ g_5(r, \eta) &= \mathcal{A} f_1 (-f_3 (2F_4 \eta + F_3 (r^2 + 1)) - 2f_4 F_5 - 2F_1 (r^3 + r) - 4F_2 \eta r) , \end{aligned} \quad (3.63)$$

where again we dropped the label λ and here the functions f_i are given by

$$\begin{aligned} f_1 &= r^2 - 2\eta r + 1, & f_2 &= -r^2 - 2\eta r + 1, \\ f_3 &= -r^2 + 2\eta r + 1, & f_4 &= r^2 + 2\eta r + 1, \end{aligned} \quad (3.64)$$

and

$$\mathcal{A}(r, \eta) = \frac{2^{\sum_i \Delta_i}}{(1-r^2)^2 (r^2 + 2\eta r + 1)} \left(\frac{r^2 - 2\eta r + 1}{r^2 + 2\eta r + 1} \right)^{\frac{1}{2}(\Delta_{12} - \Delta_{34})}. \quad (3.65)$$

Now we can already express the partial waves in terms of the function W_s . The last step is to obtain a recurrence relation for the coefficients w_s of the series expansion (3.58).

As we did in the one external vector case we use the Casimir equation to obtain five coupled differential equations involving the functions W_s . Then we rewrite them in the basis of the expansion in such a way to get a recurrence relation for the coefficients (see appendix A). The result is a set of five coupled recurrence relations

$$\sum_{s'=1}^5 \sum_{(\hat{m}, \hat{j}) \in \mathcal{S}_{s'}} c_{s'}^s(\hat{m}, \hat{j}) w_{s'}(m + \hat{m}, j + \hat{j}) = 0, \quad s = 1, \dots, 5. \quad (3.66)$$

As before we represent the sets $\mathcal{S}_{s'}$ in figure 3.5. The colored circles are related to the value of s' in (3.66), $s' = 1$ corresponds to the black dot, increasing values of s' correspond to larger circles. Each picture represents a different recurrence relation. The recurrence relations (3.66) for $m = 0$ reduce to

$$\begin{aligned} [c_{(\Delta, j)} - c_\lambda] w_1(0, j) &= 0, \\ j[c_{(\Delta, j)} - c_\lambda] w_2(0, j) &= 0, \\ j[c_{(\Delta, j)} - c_\lambda] w_3(0, j) &= 0, \\ j[c_{(\Delta, j)} - c_\lambda] w_4(0, j) &= 0, \\ j[c_{(\Delta, j, 1)} - c_\lambda] w_5(0, j) &= 0. \end{aligned} \quad (3.67)$$

The coefficients multiplying $w_s(0, j)$ depend on the representation λ of the exchanged primary operator. For the exchange of a traceless symmetric representation $\lambda = (\Delta, l)$, it is easy to see that $w_s(0, j) = 0$ for $j \neq l$. On the other hand, for $j = l$, equations (3.67) only impose $w_5(0, l) = 0$ leaving $w_s(0, l)$ for $s = 1, 2, 3, 4$ undetermined. These 4 independent solutions correspond precisely to the 4 independent conformal blocks that we parametrized by (p, q) with $p, q = 1, 2$. In the case of the representation $\lambda = (\Delta, l, 1)$ we also find $w_s(0, j) = 0$ for all $j \neq l$. For $j = l$, equation (3.67) implies $w_s(0, l) = 0$ for $s = 1, 2, 3, 4$, and leaves $w_5(0, l)$ as the only free parameter. This is consistent with the uniqueness of the conformal block $G_{\Delta, l, 1}$.

It is interesting to notice that equations (3.67) do not constrain $w_s(0, 0)$ for $s = 2, 3, 4, 5$. This should not come as a surprise since the functions W_s (with $s = 2, 3, 4, 5$) defined in (3.60) appear with a derivative in η which cancels the term $O(\eta^0)$. Therefore the only coefficients $w_s(j, m)$ (with $s = 2, 3, 4, 5$) that are physical are the ones with $j > 0$.

The coefficients of the expansion of the functions W_s have the same structure as shown in figure 3.2. As an example we want to show the coefficients at level $m = 1$ for $j = l + 1$. In the case of the conformal blocks $G_{\Delta, l}^{(p, q)}$, these can be found inputting the initial conditions $w_1(0, l) = \delta_{p, 1} \delta_{q, 1}$,

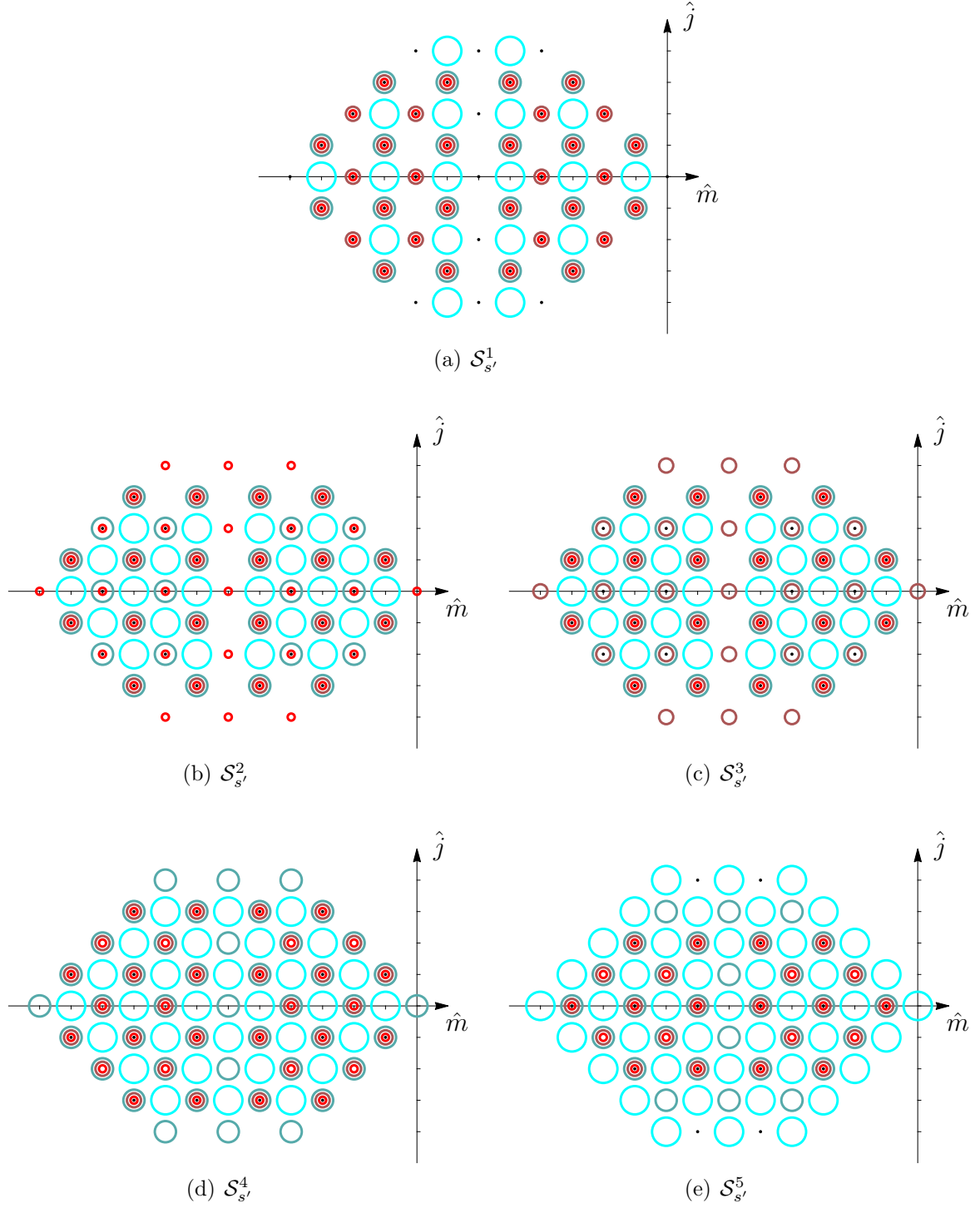


Figure 3.5: Pictorial representation of the sets $\mathcal{S}_{s'}^s$ in formula (3.66). Increasing values of s' correspond to increasing radius of the circles.

$w_2(0, l) = \delta_{p,1}\delta_{q,2}$, $w_3(0, l) = \delta_{p,2}\delta_{q,1}$, $w_4(0, l) = \delta_{p,2}\delta_{q,2}$, $w_5(0, l) = 0$ and using the recurrence relations (3.66). The result is

$$\begin{aligned}
w_1(1, l+1) &= \frac{(2h+l-2)(l\delta_{p,2} + \Delta_{12}\delta_{p,1})(l\delta_{q,2} - \Delta_{34}\delta_{q,1})}{(h+l-1)(\Delta+l)}, \\
w_2(1, l+1) &= \frac{(2h+l-2)(l\delta_{p,2} + \Delta_{12}\delta_{p,1})(\delta_{q,1} - \Delta_{34}l\delta_{q,2})}{(l+1)(h+l-1)(\Delta+l)}, \\
w_3(1, l+1) &= \frac{(2h+l-2)(\Delta_{12}l\delta_{p,2} + \delta_{p,1})(l\delta_{q,2} - \Delta_{34}\delta_{q,1})}{(l+1)(h+l-1)(\Delta+l)}, \\
w_4(1, l+1) &= \frac{(2h+l-2)(\Delta_{12}l\delta_{p,2} + \delta_{p,1})(\delta_{q,1} - \Delta_{34}l\delta_{q,2})}{(l+1)^2(h+l-1)(\Delta+l)},
\end{aligned} \tag{3.68}$$

and $w_5(1, l+1) = 0$. Notice that the $w_5(m, j)$ do not vanish in general. For example, $w_5(1, l) \neq 0$ because there is a descendant of the primary $\mathcal{O}_{\Delta, l}$ in the representation $(\Delta+1, l, 1)$ constructed by taking one derivative and antisymmetrizing it with an index of the operator. On the other hand, $w_5(n, l+n) = 0$ for any n , because the n -th symmetrized derivative constructs a descendant labeled by $(\Delta+n, l+n)$. These features are also evident from figure 3.5.

Similarly we can study the block $G_{\Delta, l, 1}$ by setting the initial conditions $w_5(0, l) = 1$ and $w_s(0, l) = 0$ for $s = 1, 2, 3, 4$. In this case we obtain

$$w_5(1, l+1) = -\frac{\Delta_{12}\Delta_{34}l(2h+l-3)}{(l+1)(h+l-1)(\Delta+l)}, \tag{3.69}$$

and $w_s(1, l+1) = 0$ for $s = 1, 2, 3, 4$. Again, by looking at figure 3.5 one it is clear that $w_s(1, l) \neq 0$ (for $s = 1, 2, 3, 4$). Therefore all the five functions W_s contribute to the construction of the conformal block at level $m > 0$.

In this case the condition $\Delta_{12} = 0 = \Delta_{34}$ gives a surprising result. In fact we obtain that many $c_s^s(\hat{m}, \hat{j})$ are zero in such a way that the recurrence relation for $s = 1, 4, 5$ are just coupled between them-self (and similarly we find two coupled relations for $s = 2, 3$). Another new feature is that the condition $\Delta_{12} = 0 = \Delta_{34}$ does not imply that the coefficients $c_s^s(\hat{m}, \hat{j})$ with odd $\hat{m}$ vanish. However they have to respect a specific parity rule: for each recurrence relation if a dot is at a $\hat{m}$ odd (even), then all the dots of the same color appear at $\hat{m}$ odd (even). Again these results come just by inspection but it would be interesting to have more analytical insights on why this should be the case (and on weather these are the minimal recurrence relations). Other special cases can be analysed by using the recurrence relations defined in the `Mathematica` files.

3.2.3 Two external spin 2 operators and two scalars

In this section we study all the conformal blocks appearing the decomposition of a four point function with two spin 2 tensors and two scalar operators,

$$\begin{aligned} \langle \mathcal{O}_1(P_1, Z_1) \mathcal{O}_2(P_2) \mathcal{O}_3(P_3, Z_3) \mathcal{O}_4(P_4) \rangle &= \sum_{\mathcal{O} \in \square\square\square\square} \sum_{p,q=3}^5 c_{12\mathcal{O}}^{(p)} c_{34\mathcal{O}}^{(q)} G_{\Delta,l}^{(p,q)}(P_i, Z_i) \\ &+ \sum_{\mathcal{O} \in \square\square\square\square} \sum_{p,q=1}^2 c_{12\mathcal{O}}^{(p)} c_{34\mathcal{O}}^{(q)} G_{\Delta,l,1}^{(p,q)}(P_i, Z_i) + \sum_{\mathcal{O} \in \square\square\square\square} c_{12\mathcal{O}} c_{34\mathcal{O}} G_{\Delta,l,2}(P_i, Z_i) . \end{aligned} \quad (3.70)$$

There are now nine conformal partial waves $G_{\Delta,l}^{(p,q)}$ (for $p, q = 3, 4, 5$) when $\mathcal{O}$ is in the symmetric and traceless representation (Δ, l) , four $G_{\Delta,l,1}^{(p,q)}$ (for $p, q = 1, 2$) if $\mathcal{O}$ is in the representation $(\Delta, l, 1)$ and a single one $G_{\Delta,l,2}$ when $\mathcal{O} \in (\Delta, l, 2)$. As in the two external vector case, we are going to study all the cases at once.

First we study the conformal block $\mathcal{G}_\lambda$ on the cylinder and expand it in the allowed polynomial structures p_s as described in (3.29). In particular, there can exist 14 possible structures

$$\begin{aligned} p_1 &= (n \cdot z_1)^2 (n \cdot z_3)^2, & p_8 &= (n \cdot z_1) (n' \cdot z_1) (n' \cdot z_3)^2, \\ p_2 &= (n \cdot z_1)^2 (n \cdot z_3) (n' \cdot z_3), & p_9 &= (n' \cdot z_1)^2 (n' \cdot z_3)^2, \\ p_3 &= (n \cdot z_1) (n \cdot z_3)^2 (n' \cdot z_1), & p_{10} &= (z_1 \cdot z_3) (n \cdot z_1) (n \cdot z_3), \\ p_4 &= (n \cdot z_1)^2 (n' \cdot z_3)^2, & p_{11} &= (z_1 \cdot z_3) (n' \cdot z_1) (n \cdot z_3), \\ p_5 &= (n \cdot z_1) (n' \cdot z_1) (n \cdot z_3) (n' \cdot z_3), & p_{12} &= (z_1 \cdot z_3) (n \cdot z_1) (n' \cdot z_3), \\ p_6 &= (n' \cdot z_1)^2 (n \cdot z_3)^2, & p_{13} &= (z_1 \cdot z_3) (n' \cdot z_1) (n' \cdot z_3), \\ p_7 &= (n' \cdot z_1)^2 (n' \cdot z_3) (n \cdot z_3), & p_{14} &= (z_1 \cdot z_3)^2, \end{aligned} \quad (3.71)$$

therefore

$$\mathcal{G}_\lambda = \sum_{s=1}^{14} F_s(r, \eta) p_s . \quad (3.72)$$

To write a nice expansion for the functions F_s , we study the decomposition of the four point function in eigenstates of the cylinder Hamiltonian. We obtain

$$\begin{aligned} \mathcal{G}_\lambda &= \sum_{m=0}^{\infty} r^{\Delta+m} \sum_j \sum_a \left[N_{\mathcal{T}}^2 u_0 \tilde{u}_0 (z_1)_{\mu_1} (z_1)_{\mu_2} (z_3)_{\nu_1} (z_3)_{\nu_2} \mathcal{T}_j^{\mu_1 \mu_2 \nu_1 \nu_2}(n, n') \right. \\ &\quad + N_{\mathcal{V}}^2 (u_1 z_1 \cdot n + u_2 z_1 \cdot \nabla_n) (\tilde{u}_1 z_3 \cdot n' + \tilde{u}_2 z_3 \cdot \nabla_{n'}) (z_1)_\mu (z_3)_\nu \mathcal{V}_j^{\mu\nu}(n, n') \\ &\quad + (u_3 (z_1 \cdot n)^2 + u_4 z_1 \cdot n z_1 \cdot \nabla_n + u_5 (z_1 \cdot \nabla_n)^2) \\ &\quad \left. \times (\tilde{u}_3 (z_3 \cdot n')^2 + \tilde{u}_4 z_3 \cdot n' z_3 \cdot \nabla_{n'} + \tilde{u}_5 (z_3 \cdot \nabla_{n'})^2) \mathcal{C}_j(n \cdot n') \right], \end{aligned} \quad (3.73)$$

where u_s and $\tilde{u}_s$ are coefficients dependent on j, m and a . As in the two external vectors case, we chose to factorize out the coefficients $N_{\mathcal{V}}^2$ and $N_{\mathcal{T}}^2$ in order to obtain simpler formulae. Again one has to fix their value at $m = 0$ depending on the possible primary representation λ that is

exchanged in the conformal block. The tensor harmonic

$$\mathcal{T}_j^{\mu_1\mu_2\nu_1\nu_2}(n, n') = n_{\alpha_1} \dots n_{\alpha_j} \pi \left(\begin{array}{c|c|c} \alpha_1 & \alpha_2 & \dots \alpha_j \\ \hline \mu_1 & \mu_2 & \end{array} \right), \begin{array}{c|c|c} \beta_1 & \beta_2 & \dots \beta_j \\ \hline \nu_1 & \nu_2 & \end{array} \right) n'_{\beta_1} \dots n'_{\beta_j} \quad (3.74)$$

can be written in terms of scalar harmonics as shown in [33]. From this representation it is straightforward to find a set of 14 functions that possess a nice radial expansion, which we can label as follows

$$W_{p,q} = \sum_{j,m} w_{p,q}(j, m) r^{\Delta+m} \mathcal{C}_j(\eta), \quad \begin{cases} p = q = 0 \\ p, q \in \{1, 2\} \\ p, q \in \{3, 4, 5\} \end{cases}, \quad (3.75)$$

where

$$w_{p,q}(j, m) = \sum_a u_p(m, j, a) \tilde{u}_q(m, j, a). \quad (3.76)$$

Next we want to relate the functions $W_{p,q}$ to the conformal partial waves. As usual the first step is to find a relation between $W_{p,q}$ and F_s according to (3.72). The result is, for example,⁴

$$F_1 = \partial_\eta^2 W_{3,5} - 2\partial_\eta^2 W_{4,5} + 6\partial_\eta^2 W_{5,5} + \partial_\eta^3 W_{1,2} - 3\partial_\eta^3 W_{2,2} + (3 + 4(-2 + h)h) \partial_\eta^4 W_{0,0} \\ + \eta (-\partial_\eta^3 W_{4,5} + 6\partial_\eta^3 W_{5,5} - \partial_\eta^4 W_{2,2} + \eta \partial_\eta^4 W_{5,5}). \quad (3.77)$$

We then write the conformal partial wave in terms of the conformal blocks g_s using (3.61), where now we have the following allowed structures

$$\begin{aligned} Q_1 &= H_{1,3}^2, & Q_8 &= H_{1,3} V_{1,4,3} V_{3,2,1}, \\ Q_2 &= V_{1,2,3}^2 V_{3,2,1}^2, & Q_9 &= H_{1,3} V_{1,4,3} V_{3,4,1}, \\ Q_3 &= V_{1,2,3}^2 V_{3,4,1}^2, & Q_{10} &= V_{1,2,3} V_{1,4,3} V_{3,2,1}^2, \\ Q_4 &= V_{1,4,3}^2 V_{3,2,1}^2, & Q_{11} &= V_{1,2,3} V_{1,4,3} V_{3,4,1}^2, \\ Q_5 &= V_{1,4,3}^2 V_{3,4,1}^2, & Q_{12} &= V_{1,2,3}^2 V_{3,2,1} V_{3,4,1}, \\ Q_6 &= H_{1,3} V_{1,2,3} V_{3,2,1}, & Q_{13} &= V_{1,4,3}^2 V_{3,2,1} V_{3,4,1}, \\ Q_7 &= H_{1,3} V_{1,2,3} V_{3,4,1}, & Q_{14} &= V_{1,2,3} V_{1,4,3} V_{3,2,1} V_{3,4,1}, \end{aligned} \quad (3.78)$$

Therefore we obtain linear relations between the F_s and g_s , for example,⁵

$$g_1 = 2^{\sum_i \Delta_i} \left(\frac{r^2 - 2\eta r + 1}{r^2 + 2\eta r + 1} \right)^{\frac{1}{2}(\Delta_{12} - \Delta_{34})} F_{14}. \quad (3.79)$$

The result of these redefinitions is that the Casimir equation can be written as a set of 14 coupled differential equations on the functions $W_{p,q}$, which we present in a **Mathematica** file. It is then a tedious exercise to rewrite them as fourteen algebraic relations for the coefficients $w_{p,q}$, as explained in the previous sections. We leave this exercise for the reader.

There are three allowed sets of initial conditions for the coefficients $w_{p,q}$. When the exchanged

⁴For the sake of brevity the remaining thirteen equations are appended in a **Mathematica** file.

⁵Again the full set of relations is presented in a **Mathematica** file.

representation is labeled by $\lambda = (\Delta, l)$ there exist nine conformal blocks $G_{\Delta, l}^{(p, q)}$ with $p, q \in \{3, 4, 5\}$. We can obtain each block $G_{\Delta, l}^{(p, q)}$ by setting as an initial condition the function $w_{p, q}(0, j)$ (with the same p and q) to $\delta_{j, l}$ and all the other $w_{p', q'}(0, j)$ to zero. In other words, for a given $p, q \in \{3, 4, 5\}$, the set of fourteen $w_{p', q'}(0, j)$ are defined by the initial condition

$$w_{p', q'}(0, j) = \delta_{p', p} \delta_{q', q} \delta_{j, l} . \quad (3.80)$$

Similarly, when the exchanged representation is $\lambda = (\Delta, l, 1)$, we have four blocks $G_{\Delta, l, 1}^{(p, q)}$ with $p, q \in \{1, 2\}$ which can be obtained by setting

$$w_{p', q'}(0, j) = \delta_{p', p} \delta_{q', q} \delta_{j, l} . \quad (3.81)$$

Finally, when we exchange $\lambda = (\Delta, l, 2)$, there exists a single block $G_{\Delta, l, 2}$ which is obtained by setting

$$w_{p', q'}(0, j) = \delta_{p', 0} \delta_{q', 0} \delta_{j, l} . \quad (3.82)$$

We checked that these choices of initial condition actually solve the corresponding Casimir equation at lowest order in r and therefore reproduce the correct leading OPE behavior of the four-point function.

Chapter 4

Recurrence Relation for Conformal Blocks

In this chapter we find all the possible singularities of the conformal blocks as functions of the scaling dimension Δ of the exchanged operator. In particular, we argue, using representation theory of parabolic Verma modules, that in odd spacetime dimension the singularities are only simple poles. We discuss how to use this information to write recursion relations that determine the conformal blocks. We first recover the recursion relation introduced in [15] for conformal blocks of external scalar operators. We then generalize this recursion relation for the conformal blocks associated to the four point function of three scalar and one vector operator. We further specialize to the case in which the vector operator is a conserved current. Finally we consider the case of four conserved currents in three dimensions.

4.1 Basic Idea

We start by summarizing the basic idea that gives rise to recursion relations for the conformal blocks.

It is convenient to express the conformal block as a sum over states in radial quantization

$$c_{12\mathcal{O}}c_{34\mathcal{O}}G_{\Delta,l}(\{x_i\}) = \sum_{\alpha \in \mathcal{H}_{\mathcal{O}}} \frac{\langle 0|\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)|\alpha\rangle\langle\alpha|\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)|0\rangle}{\langle\alpha|\alpha\rangle}, \quad (4.1)$$

where $\mathcal{H}_{\mathcal{O}}$ is the irreducible representation of the conformal group associated with the primary $\mathcal{O}$ (*i.e.*, it is $\mathcal{O}$ with all its descendants).

We will study $G_{\Delta,l}(\{x_i\})$ as a function of Δ . For some special values $\Delta = \Delta_A^*$, there exists a descendant state ¹

$$|\mathcal{O}_A\rangle \in \mathcal{H}_{\mathcal{O}} \quad \text{with} \quad \begin{cases} \Delta_A &= \Delta_A^* + n_A \\ l_A & \end{cases} \quad (4.2)$$

¹Again we suppress the indices of $\mathcal{O}_A$.

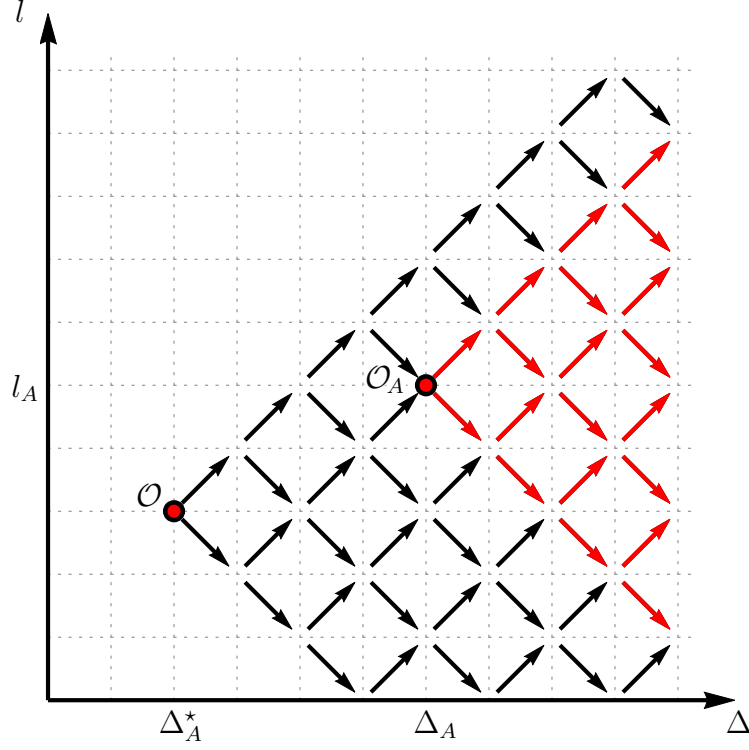


Figure 4.1: The conformal representation $\mathcal{H}_{\mathcal{O}}$ becomes reducible for some special values Δ_A^* of the conformal dimension of $\mathcal{O}$. The descendant state $\mathcal{O}_A$ becomes a primary and the representation $\mathcal{H}_{\mathcal{O}_A}$ only contains null states.

at the level n_A , that becomes a primary, as shown in picture 4.1. Namely

$$K_\nu |\mathcal{O}_A\rangle = 0 \quad (4.3)$$

for all special conformal generators K_ν . It is easy to show that any state that is both primary and descendant must have zero norm. Therefore, it follows that the denominator of (4.1) becomes zero when $\Delta = \Delta_A^*$,

$$G_{\Delta,l} \xrightarrow{\Delta \rightarrow \Delta_A^*} \infty. \quad (4.4)$$

If $|\mathcal{O}_A\rangle$ is a null primary state, its descendants are also null and together they form a conformal sub-representation. It then follows that $G_{\Delta,l}$ has a pole at $\Delta = \Delta_A^*$ and its residue is proportional to a conformal block,²

$$G_{\Delta,l}(\{x_i\}) = \frac{R_A}{\Delta - \Delta_A^*} G_{\Delta_A l_A}(\{x_i\}) + O((\Delta - \Delta_A^*)^0). \quad (4.5)$$

The coefficient R_A will be given by three contributions

$$R_A = M_A^{(L)} Q_A M_A^{(R)}, \quad (4.6)$$

²When d is even the singularity can be a higher order pole (as discussed in section 4.5). However these formulas have a well defined analytic continuation in d , which give the right result also in even dimensions.

where Q_A comes from the inverse of the norm $\langle \alpha | \alpha \rangle$ of the intermediate state α , while $M_A^{(L)}$ and $M_A^{(R)}$ come from the three point functions $\langle 0 | \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) | \alpha \rangle$ and $\langle \alpha | \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) | 0 \rangle$. Schematically we have

$$\langle \mathcal{O}_A | \mathcal{O}_A \rangle^{-1} = \frac{Q_A}{\Delta - \Delta_A^*} \langle \mathcal{O} | \mathcal{O} \rangle^{-1} + O((\Delta - \Delta_A^*)^0) , \quad (4.7)$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 | \mathcal{O}_A \rangle = M_A^{(L)} \langle \mathcal{O}_1 \mathcal{O}_2 | \mathcal{O} \rangle . \quad (4.8)$$

The idea is to find all the poles Δ_A^* in Δ of the conformal block and the associated primary descendant states $|\mathcal{O}_A\rangle$. Using this information we will compute the residue R_A of the conformal block at each pole Δ_A^* . In addition, we will study the behavior of the conformal block $g_{\Delta l}$ as $\Delta \rightarrow \infty$. To be precise $g_{\Delta l}$ has an essential singularity when $\Delta \rightarrow \infty$. However, one can define a regularized conformal block $h_{\Delta l}(r, \eta)$ such that [15]

$$g_{\Delta l}(r, \eta) = (4r)^\Delta h_{\Delta l}(r, \eta) , \quad (4.9)$$

$$h_{\Delta l}(r, \eta) = h_{\infty l}(r, \eta) + \sum_A \frac{R_A}{\Delta - \Delta_A^*} (4r)^{n_A} h_{\Delta_A l_A}(r, \eta) , \quad (4.10)$$

where $h_{\infty l}(r, \eta) = \lim_{\Delta \rightarrow \infty} h_{\Delta l}(r, \eta)$ is finite and can be computed explicitly (see below). Equation (4.10) can be used as a recursion relation to determine the conformal block. Notice that if we plug into the right hand side of (4.10) a series expansion for $h_{\Delta l}(r, \eta)$ correct up to $O(r^k)$, then the left hand side will be correct up to $O(r^{k+1})$ because $n_A \geq 1$. Furthermore, for this purpose, we only need to keep a finite number of terms in the sum over primary descendants A .

4.2 Conformal Blocks for External Scalar Operators

In this section we rederive the recursion relation for conformal blocks of external scalar operators presented in [15, 20]. As we discussed in section 4.1, three ingredients are needed to write such a formula: the knowledge of the data relative to the null primary states (namely Δ_A^* , Δ_A and l_A), the value of the residue R_A and the conformal block at large Δ . In the following three subsections we will study each one of these three ingredients and in section 4.2.4 we will put them together into a recursion relation.

4.2.1 Null States

In this section we list all the null states that can appear in the OPE between two scalars. A null state is a state $|\psi\rangle$ that is orthogonal to all other states. Notice that every null state is either a primary descendant, *i.e.* it is a descendant that is annihilated by the generators of special conformal transformations K_μ , or it is a descendant of a primary descendant. This follows from the fact that if $|\psi\rangle$ is null and $K_\mu |\psi\rangle \neq 0$ then $K_\mu |\psi\rangle$ is also null and it has lower dimension. We can continue this process until we reach a null state $|\psi_0\rangle$ that is also primary. The original null state $|\psi\rangle$ is then a descendant of $|\psi_0\rangle$. Therefore, it is sufficient to look for all primary

descendants that arise in a conformal family when the dimension Δ of the primary varies. Since any operator that appears in the OPE of two scalar operators has to belong to the traceless and symmetric representation of the rotation group $SO(d)$, it is sufficient to look for primary descendants in this representation.

Consider traceless and symmetric primary state with spin l

$$|\Delta, l; z\rangle \equiv z_{\mu_1} \dots z_{\mu_l} \mathcal{O}^{\mu_1 \dots \mu_l}(0)|0\rangle \equiv \mathcal{O}(z, 0)|0\rangle, \quad (4.11)$$

normalized as $\langle \mathcal{O}; z | \mathcal{O}; z' \rangle = (z \cdot z')^l$. We find (see section 4.5 for justification) that, in the OPE of two scalars, all primary descendants are of the following 3 types, which we call type I, type II and type III (see figure 4.2)³. For each type the null states are labeled by an integer n , which as we will see runs over all positive integers for type I and type III, and over a finite set for type II. In the following we will collectively denote the type (I, II, III) and the integer n by the label A ,

$$A \equiv \text{Type}, n, \quad \begin{cases} \text{Type} &= \text{I, II, III} \\ n &= 1, 2, \dots \end{cases}. \quad (4.12)$$

All the primary descendant states can be written as a differential operator $\mathcal{D}_A$ acting on the primary state:

$$|\Delta_A, l_A; z\rangle = \mathcal{D}_A |\Delta, l; z\rangle. \quad (4.13)$$

Type I) Type I states are the maximal spin descendant at level n

$$|\Delta + n, n + l; z\rangle = \mathcal{D}_{\text{I}, n} |\Delta, l; z\rangle \equiv (z \cdot P)^n |\Delta, l; z\rangle, \quad (4.14)$$

where P^μ is the generator of translations. In formula (B.6) in appendix B we find that the norm of the state (4.14) becomes null when

$$\Delta = \Delta_{\text{I}, n}^* \equiv 1 - l - n \quad (n = 1, 2, \dots). \quad (4.15)$$

Type II) Type II states are the minimal spin descendant at level $n \leq l$

$$|\Delta + n, l - n; z\rangle = \mathcal{D}_{\text{II}, n} |\Delta, l; z\rangle \equiv \frac{(D_z \cdot P)^n}{(2 - h - l)_n (-l)_n} |\Delta, l; z\rangle, \quad (4.16)$$

where D_z is the operator introduced in (2.13) and we recall that $h = d/2$. In formula (B.13) in appendix B we compute its norm and we show that the state becomes null when

$$\Delta = \Delta_{\text{II}, n}^* \equiv l + d - 1 - n \quad (n = 1, 2, \dots, l). \quad (4.17)$$

Type III) Type III primary descendant states have the same spin as the original primary,

$$|\Delta + 2n, l; z\rangle = \mathcal{D}_{\text{III}, n} |\Delta, l; z\rangle \equiv \mathcal{V}_0 \cdot \mathcal{V}_1 \dots \mathcal{V}_{n-1} |\Delta, l; z\rangle, \quad (4.18)$$

³Type II and type III in our paper is type III and type II in [15], respectively.

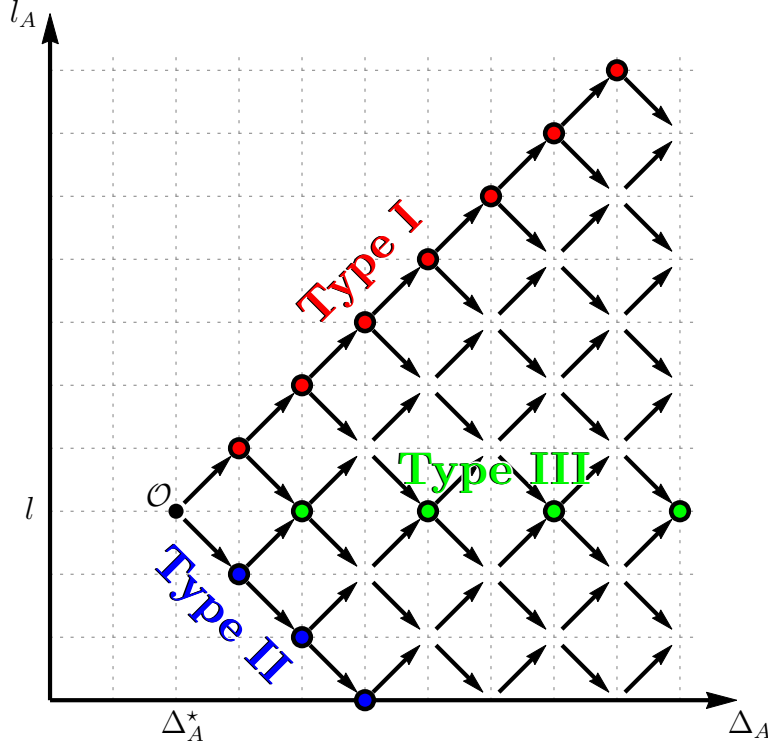


Figure 4.2: The picture shows the traceless and symmetric part of $\mathcal{H}_\mathcal{O}$. Each sequence of arrows creates a descendants of $\mathcal{O}$. When the conformal dimension of $\mathcal{O}$ takes a value Δ_A^* , one descendant placed at a colored dot becomes a primary state. There are three types of primary descendants labeled by an integer n that counts the dots from the left to the right.

where

$$\mathcal{V}_j \equiv P^2 - 2 \frac{(P \cdot z)(P \cdot D_z)}{(h + l + j - 1)(h + l - j - 2)} . \quad (4.19)$$

In appendix B we compute the norm of (4.18) in various cases and we conjecture that this descendant state becomes primary when

$$\Delta = \Delta_{\text{III},n}^* = \frac{d}{2} - n \quad (n = 1, 2, \dots) . \quad (4.20)$$

In appendix B we computed the norm

$$\langle \mathcal{O}_A; z | \mathcal{O}_A; z' \rangle \equiv N_A (z \cdot z')^{l_A} , \quad (4.21)$$

of the three states and we found that N_A has a zero at $\Delta = \Delta_A^*$. In what follows, we will only need the residue Q_A of the inverse of the norm at the pole $\Delta = \Delta_A^*$,

$$\frac{1}{N_A} = \frac{Q_A}{\Delta - \Delta_A^*} + O((\Delta - \Delta_A^*)^0) . \quad (4.22)$$

We found

$$\begin{aligned}
Q_{\text{I},n} &= -\frac{n}{2^n(n!)^2} , \\
Q_{\text{II},n} &= -\frac{n(-l)_n}{(-2)^n(n!)^2(2h+l-n-2)_n} \frac{(h+l-n-1)}{(h+l-1)} , \\
Q_{\text{III},n} &= -\frac{n}{(-16)^n(n!)^2(h-n-1)_{2n}} \frac{(h+l-n-1)}{(h+l+n-1)} .
\end{aligned} \tag{4.23}$$

Notice that if we consider even spacetime dimensions, $Q_{\text{III},n}$ diverges for large enough n due to the Pochhammer symbol $(h-n-1)_{2n}$ in the denominator. In fact in even d the conformal blocks can have higher-order poles in Δ . This phenomenon is related to the structure of the conformal families as we will explain in section 4.5. The fact that $Q_{\text{III},n}$ diverges implies that we cannot use the recursion relation to obtain CBs when d is even. However the conformal blocks are defined for non-integer dimension by analytic continuation in d and they are actually finite and well defined in the limit of d even.

4.2.2 Residues R_A

The coefficient R_A in formula (4.10) is equal to the product $M_A^{(L)} Q_A M_A^{(R)}$ as shown in (4.6). The term Q_A was already obtained in the previous section by computing the norm of the states $|\mathcal{O}_A\rangle$. In this section we explain how to find the coefficient M_A , which arises from the different normalization of the three point function involving the primary descendant $\mathcal{O}_A$.

In order to compute M_A we first define the normalization of a three point function through its leading OPE (2.55). In particular it is convenient to write the leading OPE in the unconventional channel $\mathcal{O} \times \mathcal{O}_1 \sim \mathcal{O}_2$ as follows

$$\mathcal{O}(x, z) \mathcal{O}_1(0) \sim c_{12\mathcal{O}} \frac{(-x \cdot z)^l}{(x^2)^{\frac{\Delta+\Delta_{12}+l}{2}}} \mathcal{O}_2(0) . \tag{4.24}$$

A similar leading OPE can be written for the primary descendant operator $\mathcal{O}_A$. This should also take the form of (4.24), up to a possible normalization factor,

$$\mathcal{O}_A(x, z) \mathcal{O}_1(0, z_1) \sim c_{12\mathcal{O}} M_A^{(L)} \frac{(-x \cdot z)^{l_A}}{(x^2)^{\frac{\Delta+n_A+\Delta_{12}+l_A}{2}}} \mathcal{O}_2(0) , \tag{4.25}$$

(recall that $\Delta + n_A$ and l_A are the conformal dimension and spin of $\mathcal{O}_A$). On the other hand one can also obtain $\mathcal{O}_A$ by acting with the differential operator $\mathcal{D}_A$ on $\mathcal{O}$ as shown in (4.13). In fact it is easy to compute the constant $M_A^{(L)}$ by acting with $\mathcal{D}_A$ on the leading OPE (4.24). Therefore the constants $M_A^{(L)}$ can be simply defined by

$$\mathcal{D}_A \frac{(-x \cdot z)^l}{(x^2)^{\frac{\Delta+\Delta_{12}+l}{2}}} = M_A^{(L)} \frac{(-x \cdot z)^{l_A}}{(x^2)^{\frac{\Delta+n_A+\Delta_{12}+l_A}{2}}} . \tag{4.26}$$

Computing (4.26) for the three types we find (see appendix C.1)

$$\begin{aligned}
M_{\text{I},n}^{(L)} &= (2i)^n \left(\frac{\Delta + \Delta_{12} + l}{2} \right)_n , \\
M_{\text{II},n}^{(L)} &= i^n \frac{(d+l-n-2)_n}{(h+l-n-1)_n} \left(\frac{\Delta + \Delta_{12} + 2 - d - l}{2} \right)_n , \\
M_{\text{III},n}^{(L)} &= (-4)^n \frac{(h-n-1)_l}{(h+n-1)_l} \left(\frac{\Delta + \Delta_{12} + 2 - d - l}{2} \right)_n \left(\frac{l + \Delta + \Delta_{12}}{2} \right)_n .
\end{aligned} \tag{4.27}$$

With this information we can determine the residue R_A by formula (4.6) (of course $M_A^{(L)}$ should be evaluated at $\Delta = \Delta_A^*$). This result is consistent with the result of [15, 20]⁴.

4.2.3 Conformal Block at Large Δ

To compute the large Δ behavior of the conformal block we need to study the conformal Casimir equation (2.80), which in the scalar case look like

$$\mathcal{D}(r, \eta) g_{\Delta l}(r, \eta) = c_{\Delta l} g_{\Delta l}(r, \eta) , \tag{4.28}$$

where $\mathcal{D}(r, \eta)$ is a second order differential operator in r and η . The leading term in Δ of equation (4.28) can be used to find the large Δ behavior of the conformal block. In fact when Δ is large equation (2.80) simplifies drastically: it reduces to a first order differential equation that can be easily solved up to an integration constant. Finally we can fix the integration constant computing the OPE limit ($x_{12}, x_{34} \rightarrow 0$) of the conformal block.

To find $h_{\infty l}(r, \eta)$ we first substitute the definition $g_{\Delta l}(r, \eta) \equiv (4r)^\Delta h_{\Delta l}(r, \eta)$ of (4.9) in equation (4.28) in order to obtain a differential equation for $h_{\Delta l}(r, \eta)$. The resulting equation is naively a polynomial of order two in Δ . However it is easy to see that the terms proportional to Δ^2 exactly vanish from the two sides of the equation (one comes from $c_{\mathcal{O}}$, while the other from $\partial_r^2 (4r)^\Delta$). The leading term in Δ is therefore just $O(\Delta^1)$. It is also easy to see that this term just leads to a first order differential equation in r ,⁵

$$\begin{cases} \partial_r h_{\infty l}(r, \eta) = \mathcal{M}(r, \eta) h_{\infty l}(r, \eta) \\ \mathcal{M}(r, \eta) \equiv 4 \frac{\eta (\Delta_{12} - \Delta_{34}) (r^2 - 1)^2 - \eta^2 r ((1 - 2h)r^2 + 1) - \frac{r^3 + r}{2} (h(r^2 + 1) - 2)}{(r^2 - 1)(r^2 - 2\eta r + 1)(r^2 + 2\eta r + 1)} \end{cases} , \tag{4.29}$$

⁴We use the conventions of [40, 41]. Our convention is related to those of [36] and [15, 20] as follows:

$$g_{\Delta, l}^{[36]}(u, v) = 4^\Delta g_{\Delta, l}^{[15, 20]}(u, v) = \frac{(-2)^l \left(\frac{d-2}{2} \right)_l}{(d-2)_l} g_{\Delta, l}^{\text{here}}(u, v) .$$

⁵This reasoning is generic. In fact we will see that, for any conformal block, the leading terms in Δ of the Casimir equation leads always to a first order differential equation in the variable r .

which can be solved as

$$h_{\infty l}(r, \eta) = \frac{(r^2 - 2\eta r + 1)^{\frac{\Delta_{12} - \Delta_{34} - 1}{2}}}{(r^2 + 2\eta r + 1)^{\frac{\Delta_{12} - \Delta_{34} + 1}{2}}} (1 - r^2)^{1-h} f_l(\eta) . \quad (4.30)$$

Here $f_l(\eta) \equiv h_{\infty l}(0, \eta)$ is an integration constant.

In order to determine $f_l(\eta)$, we need to fix the behavior of the conformal block as $r \rightarrow 0$, or equivalently $x_{12}^2, x_{34}^2 \rightarrow 0$. In practice we take the leading OPE (2.55) of the the four point function in both the channels $\mathcal{O}_1 \times \mathcal{O}_2 \sim \mathcal{O}$ and $\mathcal{O}_3 \times \mathcal{O}_4 \sim \mathcal{O}$. We then use (2.47) to express the remaining two point function of $\langle \mathcal{O}\mathcal{O} \rangle$. The result of this operations can be written in a compact form using the expression (2.15) for the projector into the traceless and symmetric representation,

$$G_{\Delta, l}(x_i) \underset{x_{12}, x_{34} \rightarrow 0}{\sim} \frac{1}{l!^2 (h-1)_l} \frac{(x_{12} \cdot D_z)^l}{(x_{12}^2)^{\frac{\Delta_1 + \Delta_2 - \Delta + l}{2}}} \frac{(x_{34} \cdot \partial_{z'})^l}{(x_{34}^2)^{\frac{\Delta_3 + \Delta_4 - \Delta + l}{2}}} \frac{(z \cdot I(x_{24}) \cdot z')^l}{(x_{24}^2)^\Delta} . \quad (4.31)$$

In order to obtain the conformal blocks we need to remove the usual kinematical prefactor. Finally we rewrite the result in radial coordinates, which in this limit are simply $(4r)^2 \approx x_{12}^2 x_{34}^2 / (x_{24}^2)^2$ and $\eta \approx -\hat{x}_{12} \cdot I(x_{24}) \cdot \hat{x}_{34}$. Thus we obtain the leading OPE limit $h_{\Delta l}(0, \eta)$ of the conformal blocks,

$$f_l(\eta) = \frac{l!}{(-2)^l (h-1)_l} C_l^{h-1}(\eta) , \quad (4.32)$$

where we used the closed formula (2.16) for the project into the traceless and symmetric representation.

4.2.4 Recursion Relation

We now have all the ingredients that we need to build the scalar conformal blocks using the recursion relation (4.10). For the sake of clarity in this section we write down explicitly the recursion relation with all the data that we computed in the previous subsections. The recursion relation is

$$g_{\Delta l}(r, \eta) = (4r)^\Delta h_{\Delta l}(r, \eta) , \quad (4.33)$$

$$h_{\Delta l}(r, \eta) = h_{\infty l}(r, \eta) + \sum_A \frac{R_A}{\Delta - \Delta_A^*} (4r)^{n_A} h_{\Delta_A l_A}(r, \eta) , \quad (4.34)$$

where the sum over A is a sum over the three types and over n with

A	Δ_A^*	n_A	l_A
Type I: $n = 1, 2, \dots \infty$	$1 - l - n$	n	$l + n$
Type II: $n = 1, 2, \dots, l$	$l + 2h - 1 - n$	n	$l - n$
Type III: $n = 1, 2, \dots \infty$	$h - n$	$2n$	l

(4.35)

and $\Delta_A = \Delta_A^* + n_A$. All the labels in table (4.35) were obtained in section 4.2.1 from an educated guess. However we will rigorously obtain them in section 4.5 from a general analysis of conformal families. The residues R_A are computed using formula (4.6) with Q_A defined in (4.23) and M_A defined in (4.27),

$$\left\{ \begin{array}{l} R_{\text{I},n} = \frac{-n(-2)^n}{(n!)^2} \left(\frac{\Delta_{12}+1-n}{2} \right)_n \left(\frac{\Delta_{34}+1-n}{2} \right)_n , \\ R_{\text{II},n} = \frac{-n(-2)^{-n}}{(n!)^2} \frac{l!}{(l-n)!} \frac{(2h+l-n-2)_n}{(h+l-n)_n (h+l-n-1)_n} \left(\frac{\Delta_{12}+1-n}{2} \right)_n \left(\frac{\Delta_{34}+1-n}{2} \right)_n , \\ R_{\text{III},n} = \frac{-n(-1)^n}{(n!)^2} \frac{(h-n-1)_{2n}}{(h+l-n-1)_{2n} (h+l-n)_{2n}} \prod_{\delta=\Delta_{12}, \Delta_{34}} \left(\frac{\delta-h-l-n+2}{2} \right)_n \left(\frac{\delta+h+l-n}{2} \right)_n . \end{array} \right. \quad (4.36)$$

The regular part in Δ of the conformal block is computed by replacing (4.32) in (4.30),

$$h_{\infty l}(r, \eta) = \frac{l!}{(-2)^l (h-1)_l} \frac{(1-r^2)^{1-h}}{(r^2-2\eta r+1)^{\frac{\Delta_{12}-\Delta_{34}-1}{2}}} \frac{(r^2+2\eta r+1)^{\frac{\Delta_{12}-\Delta_{34}+1}{2}}}{C_l^{h-1}(\eta)} . \quad (4.37)$$

Formula (4.34) reproduces the result of [20]. One can further check the validity of (4.34) by proving that it solves the Casimir equation to several order in the r series expansion.

4.3 The general recursion relation

In this section we generalize the recurrence relation in order to obtain any conformal block in any dimension.

As we did in the scalar case, we can write the conformal partial wave in radial quantization as follows

$$\sum_{p,q} c_{12\mathcal{O}}^{(p)} c_{34\mathcal{O}}^{(q)} G_{\mathcal{O}}^{(p,q)} = \sum_{|\alpha\rangle \in \mathcal{H}_{\mathcal{O}}} \frac{\langle 0 | \mathcal{O}_1 \mathcal{O}_2 | \alpha \rangle \langle \alpha | \mathcal{O}_3 \mathcal{O}_4 | 0 \rangle}{\langle \alpha | \alpha \rangle} , \quad (4.38)$$

where $\mathcal{H}_{\mathcal{O}}$ is the conformal multiplet associated to the primary operator $\mathcal{O}$ and the operators $\mathcal{O}_i$ are primaries with conformal dimension Δ_i and any $SO(d)$ spin. Again it is clear that the conformal partial waves have poles for some tuned values of Δ , when a descendant $\mathcal{O}_A$ of $\mathcal{O}$ becomes primary. When the external operators have spin, however, there can be more poles than in the scalar case. In section 4.5 we obtain the full set of poles Δ_A^* of a bosonic conformal block

$G_\lambda^{(p,q)}$ for the exchange an operator $\mathcal{O}$ in the most generic representation $\lambda = (\Delta, l_1, \dots, l_{[h]})$. We will see that in odd dimensions the poles are always simple. The residue at the poles will be written in terms of conformal blocks correspondent to the exchange of the sub representation $\mathcal{H}_{\mathcal{O}_A} \subset \mathcal{H}_{\mathcal{O}}$. In section 4.5 we also determine all the possible weights $\lambda_A = (\Delta_A, l_{1A}, \dots, l_{[h]A})$ of the primary descendant operators $\mathcal{O}_A$. For concreteness we write the full set of poles Δ_A^* and associated primary descendants λ_A in the following table (which will be obtained in section 4.5)

A	Δ_A^*	n_A	l_{kA}
I $_k$, $n : n \in [1, l_{k-1} - l_k]$	$k - l_k - n$	n	$l_k + n$
II $_k$, $n : n \in [1, l_k - l_{k+1}]$	$2h + l_k - k - n$	n	$l_k - n$
III, $n : n \in [1, \infty)$	$h - n$	$2n$	l_k
IV, $n : n \in [1, l_{[h]}]$	$h + \frac{1}{2} - n$	$2n - 1$	l_k

(4.39)

where $k = 1, \dots, [h]$ and n is an integer. To obtain λ_A from the data of the table it is enough to know that $\Delta_A = \Delta_A^* + n_A$, and that all the l_{iA} which are not represented in the table are left unchanged ($l_{iA} = l_i$ for $i \neq k$).

Since the conformal partial waves $G_\lambda^{(p,q)}$ carry the indices (p, q) , the residue at each pole Δ_A^* can be written as a linear combination of $G_{\lambda_A}^{(p',q')}$ as follows

$$G_\lambda^{(p,q)} \sim \frac{1}{\Delta - \Delta_A^*} \sum_{p',q'} (R_A)_{pp'qq'} G_{\lambda_A}^{(p',q')} . \quad (4.40)$$

where again the coefficients $(R_A)_{pp'qq'} \equiv (M_A^{(L)})_{pp'} Q_A (M_A^{(R)})_{qq'}$ can be computed from a direct computation. In particular the term Q_A is obtained as in the scalar case, while the coefficients M_A are now matrices since the three point functions can have many tensor structures, schematically,

$$\langle \mathcal{O}_A | \mathcal{O}_A \rangle^{-1} = \frac{Q_A}{\Delta - \Delta_A^*} \langle \mathcal{O} | \mathcal{O} \rangle^{-1} + O((\Delta - \Delta_A^*)^0) , \quad (4.41)$$

$$\langle \mathcal{O}_1 \mathcal{O}_2 | \mathcal{O}_A \rangle^{(p)} = \sum_{p'} (M_A)_{pp'} \langle \mathcal{O}_1 \mathcal{O}_2 | \mathcal{O} \rangle^{(p')} . \quad (4.42)$$

For clarity we will be focusing only on external traceless and symmetric operators. In this case the conformal partial wave can be written in terms of conformal blocks using (2.76). We can then reconstruct the full conformal blocks summing over all the poles in Δ and over the regular part,

$$h_{\lambda,s}^{(p,q)}(r, \eta) = h_{\infty,\lambda,s}^{(p,q)}(r, \eta) + \sum_A \sum_{p',q'} (4r)^{n_A} \frac{(R_A)_{pp'qq'}}{\Delta - \Delta_A^*} h_{\lambda_A,s}^{(p',q')}(r, \eta) , \quad (4.43)$$

where, as in the scalar case, we defined

$$h_{\lambda,s}^{(p,q)}(r, \eta) \equiv (4r)^{-\Delta} g_{\lambda,s}^{(p,q)}(r, \eta) \xrightarrow{\Delta \rightarrow \infty} h_{\infty,\lambda,s}^{(p,q)}(r, \eta) . \quad (4.44)$$

The regular part $h_{\infty,\lambda,s}^{(p,q)}(r,\eta)$ can be computed as in the scalar case by solving the Casimir equation at leading order in large Δ .

Given the knowledge of the poles Δ_A^* , the coefficients R_A and the conformal block at infinity h_∞ , one can use (4.43) to obtain a series expansion (in r) for the conformal block. This is usually a very efficient method to compute conformal blocks. In the following section we will give some examples for how to obtain the recurrence relations computing R_A and h_∞ . In section 4.5 we will derive the table (4.94) which is needed in order to classify the poles and the residue of the conformal blocks.

4.4 Examples

In this section we compute all the ingredients which are needed to apply the recurrence relation in some cases. We will compute all the CBs

$$\begin{array}{ccc} \mathcal{O}_1 & & \mathcal{O}_3 \\ & \swarrow \quad \searrow & \\ & \text{p} \quad \text{q} & \\ & \nwarrow \quad \nearrow & \\ \mathcal{O}_2 & & \mathcal{O}_4 \end{array}, \quad (4.45)$$

where the external operators are as follows

- $\mathcal{O}_1$ vector, $\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4$ scalars,
- $\mathcal{O}_1$ conserved current, $\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4$ scalars,
- $\mathcal{O}_1, \mathcal{O}_3$ vectors, $\mathcal{O}_2, \mathcal{O}_4$ scalars,
- $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4$ vectors in $d = 3$.

4.4.1 One External Vector Operator

We now consider the case of a four point function of a vector operator $\mathcal{O}_1(x_1, z_1) = (z_1)_\mu \mathcal{O}_1^\mu(x_1)$ and three scalars $\mathcal{O}_2(x_2), \mathcal{O}_3(x_3), \mathcal{O}_4(x_4)$.

It is convenient to fix the normalization of the three point functions by defining the leading OPE structures in the channel $\mathcal{O} \times \mathcal{O}_1 \sim \mathcal{O}_2$

$$\mathcal{O}(x, z) \mathcal{O}_1(0, z_1) \sim \frac{\mathcal{O}_2(0)}{(x^2)^\alpha} \sum_{q=1}^2 c_{12\mathcal{O}}^{(q)} t_l^{(q)}(x, z, z_1), \quad (4.46)$$

where $\mathcal{O}$ is a spin l operator with dimension Δ and $\alpha \equiv \frac{\Delta + \Delta_1 - \Delta_2 + l + 1}{2}$. There are only two allowed tensor structures $t_l^{(q)}(x, z, z_1)$ (see section 2.2.4)

$$\begin{aligned} t_l^{(1)}(x, z, z_1) &\equiv (x \cdot z)^l (z_1 \cdot x), \\ t_l^{(2)}(x, z, z_1) &\equiv (x \cdot z)^{l-1} (z \cdot z_1) x^2. \end{aligned} \quad (4.47)$$

Notice that for $l = 0$ only $t_l^{(1)}$ exists.

Let us now consider the conformal block decomposition of a four point function of the four point function $\mathcal{F}_4(\{x_i\}, z_1) = \langle \mathcal{O}_1(x_1, z_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \mathcal{O}_4(x_4) \rangle$ of the operators $\mathcal{O}_i$ defined above. We define (see section 2.2.6)

$$\mathcal{F}_4(\{x_i\}, z_1) = \sum_{\mathcal{O}} \sum_{q=1}^2 c_{12\mathcal{O}}^{(q)} c_{34\mathcal{O}} G_{\Delta l}^{(q)}(\{x_i\}, z_1) \quad (4.48)$$

$$= \sum_{\mathcal{O}} \sum_{q=1}^2 \begin{array}{c} \mathcal{O}_1 \\ \diagdown \\ \textcircled{q} \\ \diagup \\ \mathcal{O}_2 \end{array} \begin{array}{c} \mathcal{O} \\ \text{---} \\ \textcircled{} \\ \text{---} \end{array} \begin{array}{c} \mathcal{O}_3 \\ \diagup \\ \textcircled{} \\ \diagdown \\ \mathcal{O}_4 \end{array}, \quad (4.49)$$

where the exchanged operator $\mathcal{O}$ belongs to the symmetric traceless representation of $SO(d)$, since all the other representations do not couple to the OPE $\mathcal{O}_3 \times \mathcal{O}_4$.

As explained in section 2.2.5 we can use conformal symmetry to write the four point function as a function of just two cross ratios. In particular we will define the four point function tensor structures Q_s by

$$\begin{cases} Q_1(\{P_i\}, Z_1) = V_{1,23} \\ Q_2(\{P_i\}, Z_1) = V_{1,34} \end{cases}. \quad (4.50)$$

as already exemplified in (2.71).

In the following sections we will explain how to find a recursion relation which determines the conformal blocks $G_{\Delta l}^{(q)}$.

4.4.1.1 Null States

The null states that can propagate are exactly of the same three types that we listed in the scalar case, since they have to belong to traceless symmetric representations.

4.4.1.2 Residues R_A

In the scalar case we found R_A applying formula (4.6). Since Q_A and $M_A^{(R)}$ did not change we now need to compute only $M_A^{(L)}$.

The basic strategy to find $M_A^{(L)}$ is rather similar to the scalar case and it basically boils down to acting with the operators $\mathcal{D}^A$ on the OPE as in section 4.2.2. Concretely we need to act with $\mathcal{D}^A$ on the OPE (4.46). Since $\mathcal{D}_A \mathcal{O}(x, z)$ is also a primary operator that belongs to the symmetric and traceless representation, we expect to find as result an OPE of the kind (4.46),

$$\mathcal{D}_A \mathcal{O}(x, z) \mathcal{O}_1(0, z_1) = \frac{\mathcal{O}_2(0)}{(x^2)^{\alpha_A}} \sum_{q'=1}^2 c_{12\mathcal{O}_A}^{(q')} t_{l_A}^{(q')}(x, z, z_1), \quad (4.51)$$

where $\alpha_A \equiv \frac{\Delta + n_A + \Delta_1 - \Delta_2 + l_A + 1}{2}$, $c_{12\mathcal{O}_A}^{(q)}$ are new OPE constants and $\Delta + n_A, l_A$ are respectively the conformal dimension and the spin of the primary descendant $\mathcal{O}_A \equiv \mathcal{D}_A \mathcal{O}$. Moreover we can think of (4.51) as the action of $\mathcal{D}^A$ on the OPE (4.46), thus we can fix the OPE constants $c_{12\mathcal{O}_A}^{(q)}$

in terms of the $c_{12\mathcal{O}}^{(q)}$ of (4.46). In the scalar case this operation was simple since there was just one OPE coefficient, while in this case the computation is slightly more involved since the two coefficients can mix. In fact the action of $\mathcal{D}^A$ on each structure of the OPE (4.46) can generate the other one, therefore we expect to find a 2×2 matrix $M_A^{(L)}$ such that

$$c_{12\mathcal{O}_A}^{(q')} = \sum_{q=1}^2 c_{12\mathcal{O}}^{(q)} \left(M_A^{(L)} \right)_{qq'} . \quad (4.52)$$

In practice we can find $M_A^{(L)}$ performing the following computation

$$\mathcal{D}_A \frac{t_l^{(q)}(x, z, z_1)}{(x^2)^\alpha} = \sum_{q'=1}^2 \left(M_A^{(L)} \right)_{qq'} \frac{t_{l_A}^{(q')}(x, z, z_1)}{(x^2)^{\alpha_A}} . \quad (4.53)$$

As an example, here we show the result for type I,

$$M_{I,n}^{(L)} = (-2i)^n (\alpha)_{n-1} \begin{pmatrix} n + \alpha - 1 & -\frac{n}{2} \\ 0 & \alpha - 1 \end{pmatrix} . \quad (4.54)$$

The result for the other types is presented in formula (C.12) in appendix C.2).

4.4.1.3 Conformal Block at Large Δ

The goal of this section is to find the leading behavior in Δ of $g_{\Delta,l,s}^{(q)}(r, \eta)$. To achieve this goal we want to apply the same procedure we used in the scalar case. In particular we want to solve the leading term in Δ of the Casimir equation (2.80) with the appropriate leading behavior of $G_{\Delta l}^{(q)}$ when $x_{12}, x_{34} \rightarrow 0$.

As we explained in the section 4.2.3 the leading term in Δ of the Casimir equation (2.80) leads to a first order differential equation in the variable r . However now the differential equation is slightly more involved since we have two tensor structures in the four point functions and therefore we generally get system of two coupled differential equations

$$\partial_r h_{\infty l, s}^{(p, q)}(r, \eta) = \sum_{s'=1}^2 \mathbf{m}_{s s'}(r, \eta) h_{\infty l, s'}^{(p, q)}(r, \eta) , \quad (4.55)$$

where $\mathbf{m}$ is a 2×2 matrix that depends on r and η . We consider the following ansatz for the leading term in Δ of the conformal blocks

$$h_{\infty l, s}^{(q)}(r, \eta) = \mathcal{A}^{\Delta_{12}, \Delta_{34}}(r, \eta) \sum_{t=1}^2 F_s^t(r, \eta) f_{l, t}^{(q)}(\eta) , \quad (4.56)$$

$$\mathcal{A}^{\Delta_{12}, \Delta_{34}}(r, \eta) = \frac{(1 + r^2 - 2r\eta)^{\frac{\Delta_{12} - \Delta_{34}}{2}}}{(1 + r^2 + 2r\eta)^{1 + \frac{\Delta_{12} - \Delta_{34}}{2}}} (1 - r^2)^{-h} , \quad (4.57)$$

where $F_s^t(0, \eta) = \delta_{s,t}$ so that $h_{\infty, l, s}^{(q)}(0, \eta) = f_{l, s}^{(q)}(\eta)$. The overall function $\mathcal{A}^{\Delta_{12}, \Delta_{34}}(r, \eta)$ depends on the external data, and the expression in (4.57) is inspired by its scalar counterpart (4.30), and is chosen so that the differential equation simplifies. The Casimir equation of course will fix only the matrix F_s^t , while to fix $f_{l, s}^{(q)}(\eta)$ we need to specify the initial conditions as $r \rightarrow 0$. Using the ansatz (4.56), the two coupled differential equations (4.55) take the simple form

$$\partial_r F_s^t(r, \eta) = \sum_{s'=1}^2 \mathcal{M}_{s s'}(r, \eta) F_{s'}^t(r, \eta) , \quad (4.58)$$

$$\mathcal{M}(r, \eta) = \frac{2}{(r^2 - 1)(r^2 + 2\eta r + 1)} \begin{pmatrix} r(r^2 + 2\eta r - 1) & 2r\eta \\ r^2 + 1 & r^3 + \eta r^2 + r - \eta \end{pmatrix} . \quad (4.59)$$

The label $t = 1, 2$ of F_s^t parametrize the two independent solutions of the two coupled linear differential equations in (4.58). The result is

$$F(r, \eta) = \begin{pmatrix} r^2 + 1 & -2r^2\eta \\ -2r & -r^2 + 2\eta r + 1 \end{pmatrix} . \quad (4.60)$$

To find the functions $f_{l, s}^{(p)}(\eta) = h_{\infty, l, s}^{(p)}(0, \eta)$ we need to study the leading behavior of $G_{\Delta l}^{(p, q)}$ for $r \rightarrow 0$. This limit corresponds to the leading contribution in the left and the right OPE in the four point function. In particular one can completely fix the two functions $f_{l, s}^{(q)}(\eta)$ using

$$\frac{t_l^{(q)}(\hat{x}_{12}, I(x_{24}) \cdot D_z, I(x_{12}) \cdot z_1)(-\hat{x}_{34} \cdot z)^l}{l!(h-1)_l} \approx \sum_s f_{l, s}^{(q)}(\eta) Q_s(\{x_i, z_i\}) , \quad (4.61)$$

where the left hand side arises from the leading OPE limit of the four point function and the right hand side comes from the $r \rightarrow 0$ limit of (2.67). More details about this formula can be found in appendix H.1.1. The result is

$$\begin{aligned} f_{l, 1}^{(1)}(\eta) &= \frac{l! C_l^{(h-1)}(\eta)}{-2^l(h-1)_l} , & f_{l, 2}^{(1)}(\eta) &= 0 , \\ f_{l, 1}^{(2)}(\eta) &= \frac{(l-1)!(2\eta(h-1)C_{l-1}^{(h)}(\eta) + lC_l^{(h-1)}(\eta))}{-2^l(h-1)_l} , & f_{l, 2}^{(2)}(\eta) &= \frac{(l-1)!C_{l-1}^{(h)}(\eta)}{-2^{l-1}(h)_{l-1}} . \end{aligned} \quad (4.62)$$

4.4.1.4 Recursion Relation

We now have all the ingredients to write the recursion relation for all the conformal blocks of one vector operators and three scalars in any dimensions:

$$\begin{aligned} g_{\Delta l, s}^{(q)}(r, \eta) &= (4r)^\Delta h_{\Delta l, s}^{(q)}(r, \eta) , \\ h_{\Delta l, s}^{(q)}(r, \eta) &= h_{\infty, l, s}^{(q)}(r, \eta) + \sum_{q'=1}^2 \sum_A \frac{(R_A)_{qq'} (4r)^{n_A}}{\Delta - \Delta_A^*} h_{\Delta_A l_A, s}^{(q')}(r, \eta) , \end{aligned} \quad (4.63)$$

where the coefficient R_A is computed by

$$(R_A)_{qq'} = (M_A^{(L)})_{qq'} Q_A M_A^{(R)} , \quad (4.64)$$

where $(M_A^{(L)})_{qq'}$ is defined in section 4.4.1.2, Q_A in equation (4.23) and M_A in (4.27). The regular part $h_{\infty l, s}^{(q)}$ is defined in equation (4.57) after substituting (4.60) and (4.62). The labels A , Δ_A^* , Δ_A , l_A and n_A are the same as in the scalar case (4.35).

4.4.2 One External Conserved Current

Let us next consider the case where the external vector operator is a conserved current. The OPE between a conserved current and a scalar is specified by a single tensor structure

$$\mathcal{O}(x, z) \mathcal{O}_1(0, z_1) \sim \frac{\mathcal{O}_2(0)}{(x^2)^\alpha} c_{12\mathcal{O}} \tau_{\Delta l}(x, z, z_1) , \quad (4.65)$$

where $\alpha = \frac{\Delta - \Delta_2 + l + d}{2}$ because the operator $\mathcal{O}_1$ is a conserved current with $\Delta_1 = d - 1$. In fact, imposing conservation $\partial_{z_1} \cdot \partial_{x_1} \mathcal{O}_1(x_1, z_1) = 0$ on the vector OPE (4.46), we find that $\tau_{\Delta l}$ is a specific linear combination of the two structures,

$$\tau_{\Delta l} = \sum_{q=1}^2 t_l^{(q)} (\omega_{\Delta l})_q \quad (l > 0) , \quad (4.66)$$

$$\omega_{\alpha l} = (2(\alpha - 1), -2\alpha + 2h + l) . \quad (4.67)$$

When $l = 0$ the second structure does not exist and when we impose conservation we find that there is no allowed $\tau_{\Delta 0}$ unless $\Delta = \Delta_2$. We will see that the recursion relation for $l > 0$ does not couple to the case $l = 0$ and $\Delta = \Delta_2$ so we can compute it separately.

The conformal block for conserved current will be then defined as

$$\tilde{g}_{\Delta l, s} \equiv \sum_{q=1}^2 (\omega_{\Delta l})_q g_{\Delta l, s}^{(q)} , \quad (4.68)$$

where $g_{\Delta l, s}^{(q)}$ are the conformal blocks computed in section 4.4.1. Therefore, the conformal blocks for a conserved current are given by a linear combination of the conformal blocks for a vector operator. Nevertheless in the next section, we explain how to obtain a recursion relation, which directly gives $\tilde{h}_{\Delta l, s}(r, \eta)$

$$\tilde{h}_{\Delta l, s}(r, \eta) \equiv (4r)^{-\Delta} \tilde{g}_{\Delta l, s}(r, \eta) . \quad (4.69)$$

4.4.2.1 Residues R_A

The action of $\mathcal{D}_A$ on the OPE (4.65) gives back structures of the form (4.65), namely

$$\mathcal{D}_A \mathcal{O}(x, z) \mathcal{O}_1(0, z_1) = \frac{\mathcal{O}_2(0, \partial_{z_2})}{(x^2)^{\alpha_A}} c_{12\mathcal{O}} \mu_A^L \tau_{\Delta_A l_A}(x, z, z_1, z_2) , \quad (4.70)$$

where $\alpha_A \equiv \frac{\Delta - \Delta_2 + l_A + d}{2}$. Now, since there exists just one allowed structure, μ_A^L is just a scalar function. Moreover it can be easily computed knowing M_A :

$$\mu_A (\omega_{\Delta_A l_A})_q = \sum_{q'=1}^2 (\omega_{\Delta l})_{q'} (M_A)_{q'q} , \quad q = 1, 2 , \quad (4.71)$$

where $\omega_{\Delta l}$ are defined in (4.67). Notice that the two equations in (4.71) define the same value μ_A . This is a non trivial consistency condition for the matrices M_A given in section 4.4.1.2. We obtain for the three types

$$\begin{aligned} \mu_{\text{I},n}^L &= (-2i)^n \left(\frac{\Delta_{12} - n}{2} \right)_n , \\ \mu_{\text{II},n}^L &= (-i)^n \frac{(l-n)}{l} \frac{(-2h-l+2)_n}{(-h-l+2)_n} \left(\frac{\Delta_{12} - n}{2} \right)_n , \\ \mu_{\text{III},n}^L &= 4^n \frac{(h-n)_{2n}}{(h+l-n-1)_{2n}} \left(\frac{h+l-n+\Delta_{12}-1}{2} \right)_n \left(\frac{h+l-n-\Delta_{12}+1}{2} \right)_n , \end{aligned}$$

where $\Delta_{12} = d - 1 - \Delta_2$.

4.4.2.2 Conformal Block at Large Δ

We next want to find the large Δ behaviour of $\tilde{h}_{\Delta,l,s}$ defined in (4.69). The function $\tilde{h}_{\Delta,l,s}$ is a linear combination of the functions $h_{\Delta,l,s}^{(q)}$ that are regular at $\Delta = \infty$ but the coefficients $(\omega_{\Delta l})_q$ grow linearly in Δ . This means that we need to compute the sub leading behaviour of $h_{\Delta,l,s}^{(q)}$ at large Δ in order to determine the terms of order Δ and Δ^0 in $\tilde{h}_{\Delta,l,s}$. To do so we change the ansatz (4.56) for the conformal block at large Δ to

$$h_{\Delta,l,s}^{(q)}(r, \eta) = \mathcal{A}^{\Delta_{12}, \Delta_{34}}(r, \eta) \sum_{t=1}^2 F_s^t(r, \eta) \left[f_{l,t}^{(q)}(\eta) + \frac{1}{\Delta} \hat{f}_{l,t}^{(q)}(r, \eta) + O\left(\frac{1}{\Delta^2}\right) \right] , \quad (4.72)$$

where $\mathcal{A}$, F and f are defined in section E.2 so that they solve the Casimir equation at leading order in the large Δ expansion. Plugging the ansatz (4.72) in the Casimir equation, we find an equation of the form $\partial_r \hat{f}_{l,t}^{(q)}(r, \eta)$ equal to an explicit function of r . This can be trivially integrated and the integration constant can be fixed considering that $\hat{f}_{l,t}^{(q)}(0, \eta) = 0$ for consistency with the leading OPE. The final result $\tilde{h}_{\Delta,l,s}^\infty(r, \eta)$ is then achieved keeping the terms of order Δ and Δ^0 in $\sum_{q=1}^2 (\omega_{\Delta l})_q h_{\Delta,l,s}^{(q)}(r, \eta)$. The result is written in formulas (D.1) and (D.2) in appendix D.

4.4.2.3 Recursion Relation

Collecting the results above, we can give a simplified version of (4.69) that computes directly the block of a conserved current without passing through the vector case:

$$\tilde{g}_{\Delta l, s}(r, \eta) = (4r)^\Delta \tilde{h}_{\Delta l, s}(r, \eta) , \quad (4.73)$$

$$\tilde{h}_{\Delta l, s}(r, \eta) = \tilde{h}_{\Delta l, s}^\infty(r, \eta) + \sum_A \frac{(\rho_A) (4r)^{n_A}}{\Delta - \Delta_A^*} \tilde{h}_{\Delta_A, l_A, s}(r, \eta) \quad (l > 0) , \quad (4.74)$$

where

$$\rho_A = \mu_A^L Q_A M_A^R \quad (4.75)$$

and $\tilde{h}_{\Delta l, s}^\infty(r, \eta)$ contains terms linear and constant in Δ and is given explicitly in appendix D. Notice that when we consider $l > 0$, the formula never couples to any conformal block with $l = 0$. In fact the only way to do so would be through a descendant of type II at the level l , but $\mu_{\Pi, l}^L = 0$. One can check that (4.74) agrees with (4.69) at the first orders in the r expansion. For $l = 0$ it does not exist the conformal block $\tilde{g}_{\Delta l, s}$ unless $\Delta = \Delta_2$. To study $\tilde{g}_{\Delta_2 0, s}$ one can use formula (4.69).

The blocks computed in (4.74) are not completely independent, and in fact they have to satisfy the conservation condition

$$\partial_{x_1} \cdot \partial_{z_1} \tilde{G}_{\Delta l}(\{x_i\}, z_1) = 0 . \quad (4.76)$$

This gives a constraint for the functions $\tilde{h}_{\Delta l, 1}(r, \eta)$ and $\tilde{h}_{\Delta l, 2}(r, \eta)$ that we can schematically write as

$$\mathfrak{D}_\Delta \left[\tilde{h}_{\Delta l, s}(r, \eta) \right] = 0 , \quad (4.77)$$

where $\mathfrak{D}_\Delta$ is explicitly defined in equation (D.3) in appendix D. Using the identity

$$\mathfrak{D}_\Delta \left[r^{n_A} \tilde{h}_{\Delta_A l_A, s}(r, \eta) \right] = r^{n_A} \mathfrak{D}_{\Delta + n_A} \left[\tilde{h}_{\Delta_A l_A, s}(r, \eta) \right] \xrightarrow{\Delta \rightarrow \Delta_A^*} 0 , \quad (4.78)$$

we conclude that applying $\mathfrak{D}_\Delta$ to (4.74) removes all the poles in Δ . One can also check that

$$\mathfrak{D}_\Delta \left[\tilde{h}_{\Delta l, s}^\infty(r, \eta) \right] = O(\Delta^0) . \quad (4.79)$$

This shows that the recursion relation (4.74) respects conservation up to a term independent of Δ . This general strategy is not sufficient to show that this term is actually zero but we have checked this up to $O(r^7)$ in the series expansion in r .

4.4.3 Two External Vectors and Two Scalars

We now focus on all the conformal blocks appearing in the four point function of two external vectors. We will use the same notation and normalization used in section 3.2.2.

Let us start by studying the conformal block $g_{\lambda, s}^{(p, q)}$ for the exchange of a symmetric and traceless representation $\lambda = (\Delta, l)$. The poles Δ_A^* and the residue $g_{\lambda_A, s}^{(p, q)}$ which can couple to the external

states are characterized by the following table

A	Δ_A^*	n_A	l_{1A}	l_{2A}
I ₁ , $n : n \in [1, \infty)$	$1 - l - n$	n	$l + n$	0
II ₁ , $n : n \in [1, l]$	$l + 2h - 1 - n$	n	$l - n$	0
III, $n : n \in [1, \infty)$	$h - n$	$2n$	l	0
I ₂ , $n : n = 1$	1	1	l	1

(4.80)

Therefore we can write the following recurrence relation

$$\begin{aligned}
h_{(\Delta, l), s}^{(p, q)}(r, \eta) &= h_{(\infty, l), s}^{(p, q)}(r, \eta) + \sum_A \sum_{p', q'=1}^2 (4r)^{n_A} \frac{(R_A)_{pp'qq'}}{\Delta - \Delta_A^*} h_{(\Delta_A, l_A), s}^{(p', q')}(r, \eta) \\
&+ \frac{(4r)}{\Delta - 1} (R_{I_{2,1}})_{pq} h_{(2, l, 1), s}(r, \eta),
\end{aligned}
\tag{4.81}$$

where the sum in A runs over (I₁, n), (II₁, n) and (III, n). The coefficients $(R_A)_{pp'qq'}$ are given in appendix E.1 using the same the results of section 4.4.1. In appendix E.1 we also explain how to find $(R_{I_{2,1}})_{pq}$, which was not previously computed,⁶

$$(R_{I_{2,1}})_{pq} = \frac{1}{2(l+1)(2h+l-3)} \begin{pmatrix} 1, -\Delta_{12} \end{pmatrix}_p \begin{pmatrix} 1, \Delta_{34} \end{pmatrix}_q, \quad p, q = 1, 2. \tag{4.82}$$

In appendix E.2 we also compute $h_{(\infty, l), s}^{(p, q)}(r, \eta)$ by solving the Casimir equation at the leading term in large Δ .

When we exchange a representation $\lambda = (\Delta, l, 1)$ we have the following set of poles

A	Δ_A^*	n_A	l_{1A}	l_{2A}
I ₁ , $n : n \in [1, \infty)$	$1 - l - n$	n	$l + n$	1
II ₁ , $n : n \in [1, l - 1]$	$l + 2h - 1 - n$	n	$l - n$	1
III, $n : n \in [1, \infty)$	$h - n$	$2n$	l	1
II ₂ , $n : n = 1$	$2h - 2$	1	l	0

(4.83)

We can write the following recurrence relation to determine the block $h_{(\Delta, l, 1)}$

$$\begin{aligned}
h_{(\Delta, l, 1), s}(r, \eta) &= h_{(\infty, l, 1), s}(r, \eta) + \sum_A (4r)^{n_A} \frac{R_A}{\Delta - \Delta_A^*} h_{(\Delta_A, l_A, 1), s}(r, \eta) \\
&+ (4r) \sum_{p, q=1}^2 \frac{(R_{II_{2,1}})_{pq}}{\Delta - (d-2)} h_{(d-1, l), s}^{(p, q)}(r, \eta),
\end{aligned}
\tag{4.84}$$

⁶Here the notation $(a, b)_p$ means that we pick the component p of the vector (a, b) .

where again $\sum_A$ runs over (I_1, n) , (II_1, n) and (III, n) . Since R_A are just numbers (not matrices), they can be easily obtained matching the coefficients from the expansion in r . We present them in appendix E.1. Moreover in appendix E.1 we also explain how to compute $R_{II_2,1}$ from first principles,

$$(R_{II_2,1})_{pq} = \frac{l(h-1)}{2h+l-2} \left(2h+l-2, \frac{\Delta_{12}}{l} \right)_p \left(2h+l-2, -\frac{\Delta_{34}}{l} \right)_q, \quad p, q = 1, 2. \quad (4.85)$$

Finally in appendix (E.2) we also find $h_{(\infty, l, 1), s}(r, \eta)$.

We would like to comment on an important feature of the recursion relation (4.43). At first sight, the reader might be puzzled by the fact that for most of the poles Δ_A^* of formula (4.43) the residue looks divergent because the conformal blocks $G_{\lambda_A}^{(p', q')}$ have a pole precisely at this value of Δ , for every A which is not of the type (II_1, n) and (III, n) . In the example above, this phenomena happens in (4.84) for the residue of the block $h_{(\Delta, l, 1), s}$ at the pole $\Delta = (d-2)$. In fact, the term

$$\sum_{p, q=1}^2 (R_{II_2,1})_{pq} h_{(d-1, l), s}^{(p, q)}(r, \eta), \quad (4.86)$$

in the second line of (4.84), looks divergent because equation (4.81) tells us that $h_{(\Delta, l), s}^{(p, q)}(r, \eta)$ has a pole at $\Delta = d-1$. However, one can easily check that $R_{II_2,1}$, as defined in (4.85), is such that the combination (4.86) is actually finite. We expect this to happen in general so that, in odd spacetime dimension, the only singularities of the conformal blocks in the Δ -plane are simple poles. Moreover, this analytic structure should extend to non-integer spacetime dimension by analytic continuation.⁷

In this section, we presented the recursion relations (4.81) and (4.84) which allow us to determine all the conformal blocks that can appear in the conformal block decomposition (3.48) of the four-point function with two vector and two scalar operators. In [41], it was shown that the conformal blocks associated to the exchange of symmetric traceless primary operators could be written as differential operators acting on scalar conformal blocks. This gives an alternative way to determine $h_{(\Delta, l), s}^{(p, q)}(r, \eta)$ (see appendix F). On the other hand, $h_{(\Delta, l, 1), s}(r, \eta)$ must be determined using (4.84). Notice that in this context, the last term in (4.84) is a source term that can be written in terms of (derivatives of) the scalar conformal blocks. This is a specific realization of the strategy proposed in [41, 42]. They suggested that one should start by computing *seed* conformal blocks, which have the simplest external operators that can exchange a primary in a given $SO(d)$ irreducible representation. For example, the *seed* blocks for symmetric traceless operators are scalar CBs. Then, one can create external spin by acting with appropriate differential operators [28, 41, 42]. The *seed* conformal blocks have the advantage that they are unique, in the sense that there is only one block for the chosen external operators. This means that the recursion relations (4.43) are simpler for *seed* CBs. This is clear in the particular example of equation

⁷In even spacetime dimension there are higher order poles in the Δ -plane.

(4.84). This is just one equation, where the poles are proportional to the same CB with the exception of one pole that can be determined from other *seed* CBs.

4.4.4 Four Vectors in $3d$

We now discuss the case of four external vectors J_i with dimensions Δ_i in three dimensions. The partial wave decomposition of the four point function is

$$\langle J_1 J_2 J_3 J_4 \rangle = \sum_{\mathcal{O}} \sum_{p,q=1}^5 c_{12\mathcal{O}}^{(p)} c_{34\mathcal{O}}^{(q)} G_{\Delta l+}^{(p,q)} + \sum_{\mathcal{O}^-} \sum_{p,q=1}^4 c_{12\mathcal{O}^-}^{(p)} c_{34\mathcal{O}^-}^{(q)} G_{\Delta l-}^{(p,q)} \quad (4.87)$$

$$= \sum_{\mathcal{O}^+} \sum_{p,q=1}^5 \begin{array}{c} J_1 \\ \diagdown \\ \textcircled{p} \end{array} \begin{array}{c} \textcircled{q} \\ \diagup \\ J_3 \end{array} \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \begin{array}{c} J_2 \\ \diagup \\ \textcircled{p} \end{array} \begin{array}{c} \textcircled{q} \\ \diagdown \\ J_4 \end{array} + \sum_{\mathcal{O}^-} \sum_{p,q=1}^4 \begin{array}{c} J_1 \\ \diagdown \\ \textcircled{p} \end{array} \begin{array}{c} \textcircled{q} \\ \diagup \\ J_3 \end{array} \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \end{array} \begin{array}{c} J_2 \\ \diagup \\ \textcircled{p} \end{array} \begin{array}{c} \textcircled{q} \\ \diagdown \\ J_4 \end{array}, \quad (4.88)$$

where $\mathcal{O}^\pm$ are operators with spin l dimension Δ and parity $\pm$. The precise definition for the normalization of the three point functions is given in appendix G.1. Using formula (2.76) we can decomposed each partial wave $G_{\Delta l\pm}^{(p,q)}$ in 43 conformal blocks⁸ dependent on just two cross ratios. To do so, we will use the basis presented in table J.1. In the following we thus study the conformal blocks

$$h_{\Delta l+,s}^{(p,q)}(r, \eta) \quad p, q = 1 \dots 5, \quad (4.89)$$

$$h_{\Delta l-,s}^{(p,q)}(r, \eta) \quad p, q = 1 \dots 4, \quad (4.90)$$

where $s = 1, \dots, 43$. As usual we obtain a set of recurrence relations for the conformal blocks which are diagonal in the label s but which couple the labels p, q ,

$$\begin{aligned} h_{\Delta l+,s}^{(p,q)}(r, \eta) &= h_{\infty l+,s}^{(p,q)}(r, \eta) + \sum_{p',q'=1}^5 \sum_{A \in \mathcal{A}_+} \frac{(R_{+A})_{pq,p'q'} (4r)^{n_A}}{\Delta - \Delta_A^*} h_{\Delta_A l_{A+},s}^{(p',q')}(r, \eta) \\ &\quad + \sum_{p',q'=1}^4 \sum_{A \in \mathcal{A}_-} \frac{(R_{+A})_{pq,p'q'} (4r)^{n_A}}{\Delta - \Delta_A^*} h_{\Delta_A l_{A-},s}^{(p',q')}(r, \eta) \end{aligned} \quad (4.91)$$

$$\begin{aligned} h_{\Delta l-,s}^{(p,q)}(r, \eta) &= h_{\infty l-,s}^{(p,q)}(r, \eta) + \sum_{p',q'=1}^4 \sum_{A \in \mathcal{A}_+} \frac{(R_{-A})_{pq,p'q'} (4r)^{n_A}}{\Delta - \Delta_A^*} h_{\Delta_A l_{A-},s}^{(p',q')}(r, \eta) \\ &\quad + \sum_{p',q'=1}^5 \sum_{A \in \mathcal{A}_-} \frac{(R_{-A})_{pq,p'q'} (4r)^{n_A}}{\Delta - \Delta_A^*} h_{\Delta_A l_{A+},s}^{(p',q')}(r, \eta), \end{aligned} \quad (4.92)$$

where the label A stands for (T, n) where T are is one of the four types I, II, III, IV and n is an integer belonging to the set $\mathcal{S}_T$ which can be finite or infinite depending on the type T . In

⁸The actual independent structures are 41, but we find it more convenient to work in the 43-dimensional space and project out the final result into the 41-dimensional space.

particular we have

$$\sum_{A \in \mathcal{A}_+} \equiv \sum_{T=I,II,III} \sum_{n \in \mathcal{S}_T}, \quad \sum_{A \in \mathcal{A}_-} \equiv \sum_{T=IV} \sum_{n \in \mathcal{S}_T}. \quad (4.93)$$

We present here a table which specifies the labels A , $\mathcal{S}_T$, Δ_A^* , n_A , l_A .

A	$\mathcal{S}_T$	Δ_A^*	n_A	l_A
I, n	$[1, \infty)$	$1 - l - n$	n	$l + n$
II, n	$[1, l]$	$2 + l - n$	n	$l - n$
III, n	$[1, \infty)$	$\frac{3}{2} - n$	$2n$	l
IV, n	$[1, \min(2, l)]$	$2 - n$	$2n - 1$	l

(4.94)

Comments on the appearance of these four families can be found in the appendix G. The conformal blocks at large delta h_∞ are computed exactly by solving the Casimir equation at the leading term in the large Δ expansion as explained in the appendix G.4. In appendix G.3 we detail how to obtain the coefficients R .

For further details and comments on the recurrence relations we send the reader to the appendix G.

4.5 Structure of Conformal Families

In this section we discuss the general structure of conformal families and a precise mathematical criterion for the absence of null states.

4.5.1 Parabolic Verma Modules

The mathematical concept of parabolic Verma module applies precisely to the conformal families relevant for CFT. Understanding this connection and using a theorem of Jantzen [43], we will be able to find the general conditions for the absence of primary descendants. We shall follow the notation of the book [44].

To define a parabolic Verma module, we start with a Lie algebra $\mathfrak{g}$ with its Cartan subalgebra $\mathfrak{h}$. We denote its root system by Φ and the set of positive and negative roots by Φ^+ and Φ^- , respectively. We then choose a subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing $\mathfrak{h}$ and denote its root system by $\Phi_{\mathfrak{p}} \subset \Phi$. Introduce a highest weight state $|\lambda\rangle$ of $\mathfrak{g}$,

$$H^i |\lambda\rangle = \lambda^i |\lambda\rangle, \quad \forall H^i \in \mathfrak{h}, \quad E_\alpha |\lambda\rangle = 0, \quad \forall \alpha \in \Phi^+. \quad (4.95)$$

Since $|\lambda\rangle$ is also an highest weight state of $\mathfrak{p}$, we can use it to construct a finite dimensional irreducible representation $L_{\mathfrak{p}}(\lambda)$ of $\mathfrak{p}$. Finally, the parabolic Verma module $M_{\mathfrak{p}}(\lambda)$ is constructed

by acting freely on $L_{\mathfrak{p}}(\lambda)$ with the generators $E_{-\alpha}$ for all $\alpha \in \Psi^+ \equiv \Phi^+/\Phi_{\mathfrak{p}}^+$. The case of conformal field theories in d dimensions corresponds to

$$\mathfrak{g} = \mathfrak{so}(d+2) , \quad \mathfrak{p} = \mathfrak{so}(2) \bigoplus \mathfrak{so}(d) . \quad (4.96)$$

In fact a primary state $|\mathcal{O}_\lambda\rangle$ can be labeled by $\lambda = (-\Delta, l_1, \dots, l_{[\frac{d}{2}]})$, where $-\Delta$ is the eigenvalue of $\mathfrak{so}(2)$ and the set $l_1, \dots, l_{[\frac{d}{2}]}$ defines an irreducible representation of $\mathfrak{so}(d)$. In other words, the primary state transforms in an irreducible representation $L_{\mathfrak{p}}(\lambda)$ of the algebra $\mathfrak{p}$. The set Ψ^+ contains the positive roots associated with the generators K_μ . The parabolic Verma module is therefore obtained by acting with the negative roots P_μ on the primary state, which gives the usual conformal multiplet

$$M_{\mathfrak{p}}(\lambda) = \{|\mathcal{O}_\lambda\rangle, P_\mu|\mathcal{O}_\lambda\rangle, P_\mu P_\nu|\mathcal{O}_\lambda\rangle, \dots\} . \quad (4.97)$$

A parabolic Verma module is said to be simple if it does not contain any submodule. Simplicity is equivalent to the absence of primary descendants in the conformal case. In order to state the conditions for simplicity of a parabolic Verma module we need to introduce some important concepts. Firstly, we introduce the formal character of the parabolic modules $M_{\mathfrak{p}}(\lambda)$ in terms of standard Verma modules $M(\lambda)$ and of the irreducible module $L_{\mathfrak{p}}(\lambda)$

$$\text{ch } M_{\mathfrak{p}}(\lambda) = \sum_{w \in W_{\mathfrak{p}}} (-1)^{\ell(w)} \text{ch } M(w \cdot \lambda) \quad (4.98)$$

$$= \frac{\text{ch } L_{\mathfrak{p}}(\lambda)}{\prod_{\alpha \in \Psi^+} (1 - e^{-\alpha})} , \quad (4.99)$$

where $W_{\mathfrak{p}}$ is the Weyl group of (the semisimple part of) $\mathfrak{p}$, $\ell(w)$ is the length of the Weyl reflection w and the dot action of a Weyl reflection $w = s_\beta$ is defined by

$$s_\beta \cdot \lambda \equiv \lambda - \frac{2\langle \lambda + \rho, \beta \rangle}{\langle \beta, \beta \rangle} \beta , \quad \rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha . \quad (4.100)$$

We also define

$$\Psi_\lambda^+ \equiv \left\{ \beta \in \Psi^+ : n_\beta \equiv \frac{2\langle \lambda + \rho, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z}_{>0} \right\} \quad (4.101)$$

as the subset of Ψ^+ such that the $s_\beta \cdot \lambda = \lambda - n_\beta \beta$ for a non-negative integer n_β . We are now ready to state Jantzen's simplicity criterion [43]: $M_{\mathfrak{p}}(\lambda)$ is simple if and only if

$$\sum_{\beta \in \Psi_\lambda^+} \text{ch } M_{\mathfrak{p}}(s_\beta \cdot \lambda) = 0 . \quad (4.102)$$

Notice that when Ψ_λ^+ is empty, then the Jantzen's criterion is trivially satisfied. Moreover, when Ψ_λ^+ is not empty, there are very simple geometrical ways to see if (4.102) is satisfied. To do so, it is very convenient to introduce the notion of a wall between Weyl chambers. The wall Ω_γ is

the hyperplane perpendicular to the root $\gamma \in \Phi$ that contains the point $-\rho$,

$$\Omega_\gamma = \{\lambda : \langle \lambda + \rho, \gamma \rangle = 0\} . \quad (4.103)$$

We can already dramatically simplify the search of simple modules using the following statement: when Ψ_λ^+ is not empty $M_{\mathfrak{p}}(\lambda)$ can be simple only if λ lives in a wall of a Weyl chamber. This means that we have to check condition (4.102) only when $\lambda \in \Omega_\gamma$ for some γ .

We now want to give some intuition on the reason why the sum in (4.102) can be zero. The main idea is that if two weights λ and $\tilde{\lambda}$ are related by a Weyl transformation $w \in W_{\mathfrak{p}}$ with odd length, then the sum of their characters is zero,

$$w \cdot \lambda = \tilde{\lambda} , \quad w \in W_{\mathfrak{p}} , \quad (-1)^{\ell(w)} = -1 \quad \implies \quad \text{ch } M_{\mathfrak{p}}(\lambda) + \text{ch } M_{\mathfrak{p}}(\tilde{\lambda}) = 0 . \quad (4.104)$$

Furthermore, if $w \cdot \lambda = \lambda$ then $\text{ch } M_{\mathfrak{p}}(\lambda) = 0$.

When the parabolic module $M_{\mathfrak{p}}(\lambda)$ is not simple, it will contain at least one primary descendant. Notice that the weight vectors $s_\beta \cdot \lambda = \lambda - n_\beta \beta$ with $\beta \in \Psi_\lambda^+$ are good candidates to be primary descendants because their quadratic Casimir is equal to the one of λ ,

$$\mathcal{C}|s_\beta \cdot \lambda\rangle = (s_\beta \cdot \lambda, s_\beta \cdot \lambda + 2\rho) = (\lambda, \lambda + 2\rho) = \mathcal{C}|\lambda\rangle . \quad (4.105)$$

The general decomposition of the module $M_{\mathfrak{p}}(\lambda)$ into irreducible modules is rather complicated, however this question can be approached using the *Kazhdan-Lusztig theory* [45].⁹

The Kazhdan-Lusztig conjecture for parabolic Verma modules (*relative Kazhdan-Lusztig conjecture*) [47, 48] tells us how to decompose a parabolic Verma module $M_{\mathfrak{p}}(\lambda)$ into irreducible modules $L(\lambda)$.

For this purpose, it is convenient to parametrize the weights λ and μ by elements of the Weyl group. It is known that we can parametrize the weight of independent irreducible modules by an element of the coset: $w \in W_{\mathfrak{p}} \backslash W$. Let us write the corresponding weight by $\lambda(w)$ ¹⁰. Then, the characters of the irreducible modules are given by appropriate linear combinations of the characters of parabolic Verma modules,

$$\text{ch } L(\lambda(x)) = \sum_{x \leq w} m_{\lambda(x), \lambda(w)} \text{ch } M_{\mathfrak{p}}(\lambda(w)) , \quad (4.106)$$

where $\leq$ in the sum denotes the Bruhat ordering. The relative Kazhdan-Lusztig (KL) conjecture states that the multiplicity $m_{\lambda(w), \lambda(x)}$ is a special value of the Kazhdan-Lusztig polynomial $P_{x,w}(q)$:

$$m_{\lambda(w), \lambda(x)} = (-1)^{\ell(w) - \ell(x)} P_{x,w}(1) . \quad (4.107)$$

In appendix I.1 we present explicit expressions for the KL polynomial.

In the following we apply both the Jantzen's criterion and *relative Kazhdan-Lusztig conjecture* to conformal field theories. We will first study the example of a CFT₃ and then we will generalize

⁹See appendix C of [46] (and references therein) for a more physics oriented summary.

¹⁰See e.g. [48] for an explicit description of $\lambda(w)$.

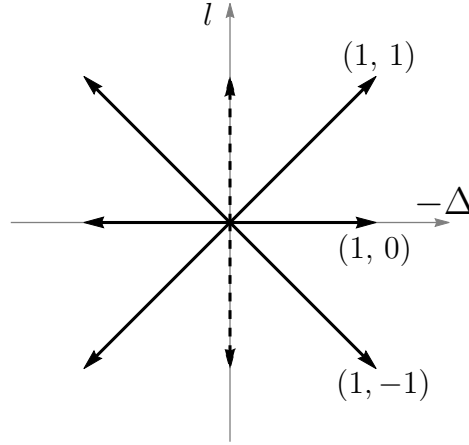


Figure 4.3: The root system of $\mathfrak{so}(5)$. The dashed roots belong to $\mathfrak{so}(3) \subset \mathfrak{so}(5)$.

it to CFT in generic odd and even dimensions. A summary of the results is presented in section 4.5.5, where we will completely classify the poles of conformal blocks and their residues.

4.5.2 Example: $\mathfrak{so}(5)$

It is instructive to apply the general statement of the previous section to the case of 3 dimensional CFTs. In this case, $\mathfrak{p} = \mathfrak{so}(2) \oplus \mathfrak{so}(3)$, thus we can label the highest weight by $\lambda = (-\Delta, l)$. The finite dimensional representation $L_{\mathfrak{p}}(\lambda)$ are the spin l representations of $\mathfrak{so}(3)$ with eigenvalue $-\Delta$ under $\mathfrak{so}(2)$,

$$\text{ch } L_{\mathfrak{p}}(\lambda) = \frac{e^{(-\Delta, l)} - e^{(-\Delta, -l-1)}}{1 - e^{-(0, 1)}} = \sum_{s=-l}^l e^{(-\Delta, s)} . \quad (4.108)$$

The root system of $\mathfrak{so}(5)$, depicted in figure 4.3, leads to

$$\Psi^+ = \{(1, 1), (1, 0), (1, -1)\} . \quad (4.109)$$

From the numbers

$$n_{(1,1)} = l + 2 - \Delta , \quad n_{(1,0)} = 3 - 2\Delta , \quad n_{(1,-1)} = 1 - l - \Delta , \quad (4.110)$$

we conclude that for $\Delta > l + 1$ the set Ψ_{λ}^+ is empty and therefore the module $M_{\mathfrak{p}}(\lambda)$ is simple. However, if any of the numbers in (4.110) takes a positive integer value, then we have to check condition (4.102).

First, we notice that n_{β} in (4.110) is an integer only when Δ is either an integer or a semi-integer. When Δ is an integer there exist three special cases for which Ψ_{λ}^+ is not empty and the module

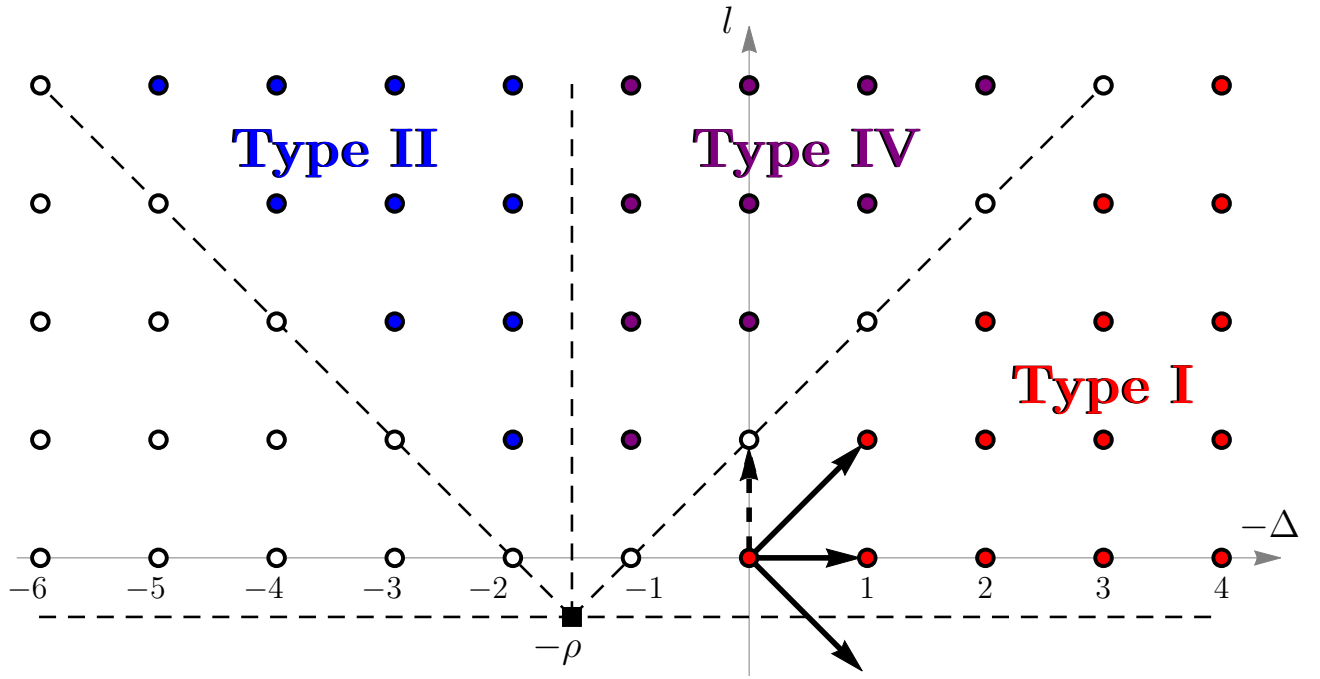


Figure 4.4: Integral weights of $\mathfrak{so}(5)$.

is not simple: ¹¹

$$\begin{array}{llll}
 \text{Type I.} & \Delta = 1 - l - k, \quad (k = 1, 2, \dots) & \implies & \Psi_{\lambda}^{+} = \{(1, 1), (1, 0), (1, -1)\} , \\
 \text{Type II.} & \Delta = l + 2 - k, \quad (k = 1, 2, \dots, l) & \implies & \Psi_{\lambda}^{+} = \{(1, 1)\} , \\
 \text{Type IV.} & \Delta = 2 - k, \quad (k = 1, 2, \dots, l) & \implies & \Psi_{\lambda}^{+} = \{(1, 1), (1, 0)\} .
 \end{array}$$

This can be understood geometrically from figure 4.4. The three special regions I, II and IV are just the colored dots in the Weyl chambers. The dot action $s_{\beta} \cdot \lambda$ corresponds to reflections (towards the left) on the dashed lines. Moreover, the set Ψ_{λ}^{+} is simply the set of allowed dot action reflections. It is then trivial to see why in case II there is only one element in Ψ_{λ}^{+} , while in the cases IV and I there are respectively 2 and 3.

There exists one special case in which Ψ_{λ}^{+} is not empty but the module is simple. This happens when λ belongs to the wall $\Omega_{(1,-1)}$ defined by $\Delta = 1 - l$. Notice that the wall $\Omega_{(1,-1)}$ is given by the line perpendicular to $(1, -1)$ that passes trough $-\rho$, which corresponds to the dashed line between the regions IV and I in figure 4.4. It easy to check that $\Psi_{\lambda}^{+} = \{(1, 1), (1, 0)\}$ and

$$\text{ch } M_{\mathfrak{p}}(s_{(1,1)} \cdot \lambda) + \text{ch } M_{\mathfrak{p}}(s_{(1,0)} \cdot \lambda) = 0 , \quad (4.111)$$

thus the condition (4.102) holds and the module is simple.

When the module is not simple we can understand the structure of submodules using Kazhdan-Luzstig theory. In this case, all Kazhdan-Luzstig polynomials are equal to 1 and using (4.98)

¹¹The denomination of the types is made to match section 4.2.1. We will comment later on the type IV.

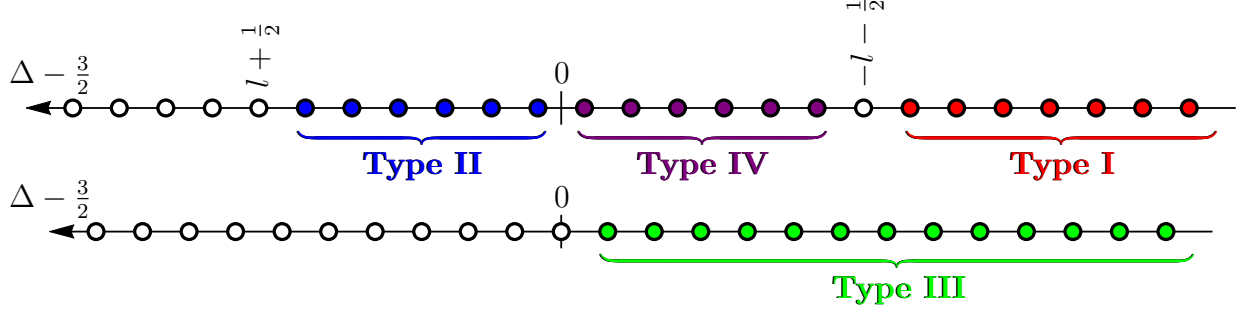


Figure 4.5: Summary of the weights of $\mathfrak{so}(5)$, for Δ integer (above) and Δ half-integer (below).

and (4.106) we obtain (see appendix I.1)

$$\text{Type I.} \quad \text{ch } M_{\mathfrak{p}}(\lambda) = \text{ch } L(\lambda) + \text{ch } L(s_{(1,-1)} \cdot \lambda) , \quad (4.112)$$

$$\text{Type II.} \quad \text{ch } M_{\mathfrak{p}}(\lambda) = \text{ch } L(\lambda) + \text{ch } M_{\mathfrak{p}}(s_{(1,1)} \cdot \lambda) , \quad (4.113)$$

$$\text{Type IV.} \quad \text{ch } M_{\mathfrak{p}}(\lambda) = \text{ch } L(\lambda) + \text{ch } L(s_{(1,0)} \cdot \lambda) . \quad (4.114)$$

In the case II the module $M_{\mathfrak{p}}(\lambda)$ contains the simple parabolic submodule $M_{\mathfrak{p}}(s_{\beta} \cdot \lambda)$, where β is the unique element in Ψ_{λ}^{+} . Geometrically we can check that when λ is a blue dot in figure 4.4, $s_{(1,1)} \cdot \lambda$ is a white dot which correspond to a simple parabolic module. In the cases IV and I, things are more interesting and we find that the module $M_{\mathfrak{p}}(\lambda)$ does not contain any parabolic submodule, instead it decomposes into the two irreducible modules $L(\lambda)$ and $L(s_{\beta} \cdot \lambda)$ where β is a root in Ψ_{λ}^{+} . Notice that the actual root β that we found in (4.112) and (4.114) is the one corresponding to the lowest $n_{\beta} > 0$. Geometrically this means that s_{β} is the Weyl transformation that maps λ to the closest Weyl chamber on the left.

If Δ is a half-integer there is only a single special case when Ψ_{λ}^{+} is not empty:

$$\text{Type III.} \quad \Delta = \frac{3}{2} - k \quad (k = 1, 2, \dots) \quad \implies \quad \Psi_{\lambda}^{+} = \{(1, 0)\} . \quad (4.115)$$

In the case III, as in the case II, there exists only one element in Ψ_{λ}^{+} . Therefore the module $M_{\mathfrak{p}}(\lambda)$ contains only the simple parabolic submodule $M_{\mathfrak{p}}(s_{(1,0)} \cdot \lambda)$

$$\text{Type III.} \quad \text{ch } M_{\mathfrak{p}}(\lambda) = \text{ch } L(\lambda) + \text{ch } M_{\mathfrak{p}}(s_{(1,0)} \cdot \lambda) . \quad (4.116)$$

The full set of weights is summarized in figure 4.5.

The structure of submodules has important implications for the analytic structure of conformal blocks as functions of Δ . The absence of second generation primary descendants leads to absence of higher order poles. Therefore, we conclude that in 3D the conformal blocks only have single poles. Moreover, the residues of the poles are given by the blocks associated to the submodules that become null.

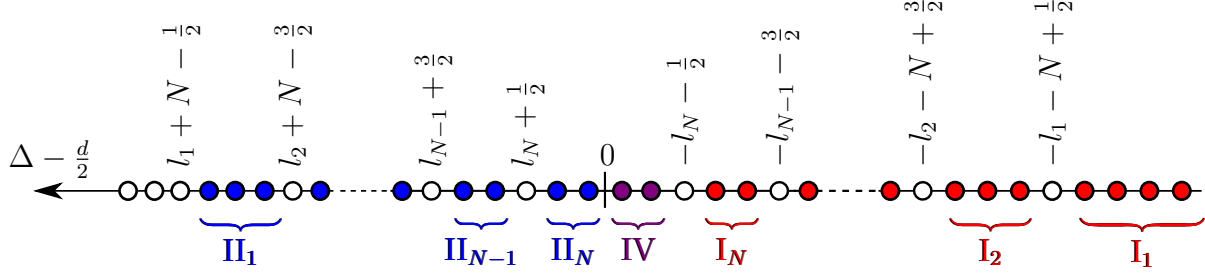


Figure 4.6: The diagram represents all integer values of Δ when $d = 2N + 1$. The white dots mark values of Δ for which the module $M_{\mathfrak{p}}(\lambda) = M_{\mathfrak{p}}(-\Delta, l_1, \dots, l_N)$ is simple. The coloured dots mark the values for which $M_{\mathfrak{p}}(\lambda)$ is not simple. In the set Π_k , there is a submodule with smaller l_k and all other l 's unchanged. In the set IV , there is a submodule with exactly the same l 's. In the set I_k there is a submodule with larger l_k and all other l 's unchanged.

4.5.3 Odd Spacetime Dimension

We now consider the general case in odd spacetime dimension $d = 2N + 1$. The complexified conformal algebra is $\mathfrak{so}(2N + 3)$. The set of positive roots is

$$\Phi^+ = \{\alpha_{jk}^{+-}, \alpha_{jk}^{++}, \alpha_j^+\}, \quad (4.117)$$

defined in (2.23), and the vector ρ is given by

$$\rho = \left(N + \frac{1}{2}, N - \frac{1}{2}, \dots, \frac{3}{2}, \frac{1}{2} \right). \quad (4.118)$$

We consider the highest weight $\lambda = (-\Delta, l_1, \dots, l_N)$ and the parabolic Verma modules based on the subalgebra $\mathfrak{p} = \mathfrak{so}(2) \oplus \mathfrak{so}(2N + 1) \subset \mathfrak{so}(2N + 3)$. We have

$$\Psi^+ = \{\alpha_0^+, \alpha_{01}^{++}, \alpha_{02}^{++}, \dots, \alpha_{0N}^{++}, \alpha_{01}^{+-}, \alpha_{02}^{+-}, \dots, \alpha_{0N}^{+-}\}, \quad (4.119)$$

with

$$n_{\alpha_0^+} = 2N + 1 - 2\Delta, \quad (4.120)$$

$$n_{\alpha_{0j}^{++}} = 2N + 1 - \Delta + l_j - j \quad (j = 1, 2, \dots, N), \quad (4.121)$$

$$n_{\alpha_{0j}^{+-}} = -\Delta - l_j + j \quad (j = 1, 2, \dots, N). \quad (4.122)$$

We conclude that for generic real values of Δ the module $M_{\mathfrak{p}}(\lambda)$ is simple. For integer values of Δ the question is much more non-trivial. It is clear that for $\Delta \geq 2N + l_1$ the module $M_{\mathfrak{p}}(\lambda)$ is simple. For smaller integer values of Δ there is always some n from (4.120-4.122) which takes a positive integer value. In these cases we have to check condition (4.102). The results for integers roots are summarized in figure 4.6. The semi integer case is similar to the case III in $\mathfrak{so}(5)$.

To explain figure 4.6 we start by studying the special integer values of Δ for which $M_{\mathfrak{p}}(\lambda)$ is

simple. These organize in two sequences. The first is when λ belongs to the wall of the Weyl chamber $\Omega_{\alpha_{0k}^{++}}$, given by

$$\Delta = 2N + 1 + l_k - k \quad (k = 1, \dots, N) . \quad (4.123)$$

The second is when λ belongs to the wall $\Omega_{\alpha_{0k}^{-+}}$,

$$\Delta = k - l_k \quad (k = 1, 2, \dots, N) . \quad (4.124)$$

In both cases, Jantzen's criterion implies that the module $M_{\mathfrak{p}}(\lambda)$ is simple. We explain this in detail in appendix I.

Let us now describe the integer values of Δ for which $M_{\mathfrak{p}}(\lambda)$ is not simple. These cases are naturally divided into four types

$$\lambda_{I_k, n}^* \equiv \{\lambda : \Delta = k - l_k - n\} \quad (n = 1, 2, \dots, l_{k-1} - l_k) , \quad (4.125)$$

$$\lambda_{II_k, n}^* \equiv \{\lambda : \Delta = 2N + 1 + l_k - k - n\} \quad (n = 1, 2, \dots, l_k - l_{k+1}) , \quad (4.126)$$

$$\lambda_{III, n}^* \equiv \{\lambda : \Delta = \frac{1}{2} + N - n\} \quad (n = 1, 2, \dots) , \quad (4.127)$$

$$\lambda_{IV, n}^* \equiv \{\lambda : \Delta = N + 1 - n\} \quad (n = 1, 2, \dots, l_k) , \quad (4.128)$$

where $k = 1, 2, \dots, N$ and we defined $l_{N+1} \equiv 0$ and $l_0 \equiv \infty$. Each of these cases has the following set of roots Ψ_{λ}^+

$$\Psi_{\lambda}^+ = \begin{cases} \{\alpha_{01}^{++}, \dots, \alpha_{0N}^{++}, \alpha_0^+, \alpha_{0N}^{+-}, \dots, \alpha_{0k}^{+-}\} & (\lambda = \lambda_{I_k, n}^*) , \\ \{\alpha_{01}^{++}, \dots, \alpha_{0k}^{++}\} & (\lambda = \lambda_{II_k, n}^*) , \\ \{\alpha_0^+\} & (\lambda = \lambda_{III, n}^*) , \\ \{\alpha_{01}^{++}, \dots, \alpha_{0N}^{++}, \alpha_0^+\} & (\lambda = \lambda_{IV, n}^*) . \end{cases}$$

In the cases II_1 and III the set Ψ_{λ}^+ contains only one element and the decomposition in submodules is straightforward. In all the other cases, one can work out the detailed decomposition of the parabolic modules using Kazhdan-Lusztig theory as we show in appendix I.1. The result is as follows

$$\begin{aligned} \text{ch } M_{\mathfrak{p}}(\lambda_A^*) &= \text{ch } L(\lambda_A^*) + \text{ch } M_{\mathfrak{p}}(\lambda_A) , & A = (II_1, n), (III, n) , \\ \text{ch } M_{\mathfrak{p}}(\lambda_A^*) &= \text{ch } L(\lambda_A^*) + \text{ch } L(\lambda_A) , & \text{otherwise} , \end{aligned} \quad (4.129)$$

where the irreducible submodules are labeled by the following weights λ_A

$$\begin{aligned} \lambda_{I_k, n} &\equiv s_{\alpha_{0k}^{+-}} \cdot \lambda_{I_k, n}^* &= (l_k - k, l_1, \dots, l_k + n, \dots, l_N), \\ \lambda_{II_k, n} &\equiv s_{\alpha_{0k}^{++}} \cdot \lambda_{II_k, n}^* &= (-2N - 1 - l_k + k, l_1, \dots, l_k - n, \dots, l_N) , \\ \lambda_{III, n} &\equiv s_{\alpha_0^+} \cdot \lambda_{III, n}^* &= (\frac{1}{2} + N + q, l_1, \dots, l_N) , \\ \lambda_{IV, n} &\equiv s_{\alpha_0^+} \cdot \lambda_{IV, n}^* &= (1 - N + q, l_1, \dots, l_N) . \end{aligned} \quad (4.130)$$

Notice that in type II_k the first submodule has smaller l_k , in type IV and III the first submodule

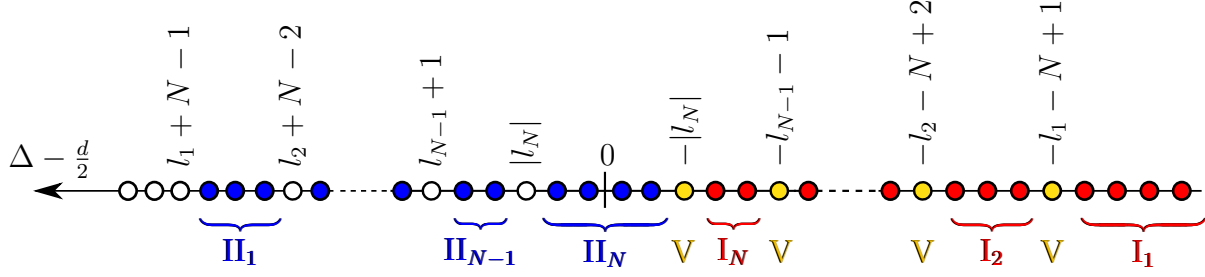


Figure 4.7: The diagram represents all integer values of Δ when $d = 2N$. The white circles mark values of Δ for which the module $M_{\mathfrak{p}}(\lambda) = M_{\mathfrak{p}}(-\Delta, l_1, \dots, l_N)$ is simple. The coloured dots mark the values for which $M_{\mathfrak{p}}(\lambda)$ is not simple. In the set Π_k , there is a submodule with smaller $|l_k|$ and all other l 's unchanged. In the set I_k there is a submodule with larger $|l_k|$ and all other l 's unchanged. In case V (yellow circles), there is a submodule with the opposite sign of l_N and all other l 's unchanged.

has the same l_k 's, and in type I_k the first submodule has larger l_k .

4.5.4 Even Spacetime Dimension

The case $d = 2N$ is similar. The set of positive roots is

$$\Phi^+ = \{\alpha_{jk}^{+-}, \alpha_{jk}^{++}\} , \quad (4.131)$$

and the vector ρ is given by

$$\rho = (N, N-1, \dots, 1, 0) . \quad (4.132)$$

We have

$$\Psi^+ = \{\alpha_{01}^{++}, \alpha_{02}^{++}, \dots, \alpha_{0N}^{++}, \alpha_{01}^{+-}, \alpha_{02}^{+-}, \dots, \alpha_{0N}^{+-}\} \quad (4.133)$$

with

$$n_{\alpha_{0j}^{++}} = 2N - \Delta + l_j - j \quad (j = 1, 2, \dots, N) , \quad (4.134)$$

$$n_{\alpha_{0j}^{+-}} = -\Delta - l_j + j \quad (j = 1, 2, \dots, N) . \quad (4.135)$$

We conclude that for generic values of Δ the module $M_{\mathfrak{p}}(\lambda)$ is simple. For integer values of Δ the answer is summarized in figure 4.7. For $\Delta \geq 2N + l_1 - 1$ the set Ψ_{λ}^+ is empty and the module is simple. For the special values

$$\Delta = 2N + |l_k| - k \quad (k = 1, 2, \dots, N) , \quad (4.136)$$

the set Ψ_{λ}^+ is not empty but one can show that the modules are still simple (see appendix I). For all other integer values of Δ the module $M_{\mathfrak{p}}(\lambda)$ is not simple. We divide these cases into three groups depending on the first submodule that appears in the decomposition of $M_{\mathfrak{p}}(\lambda)$. Cases Π_k and I_k are very similar to what happens in odd dimensions, in the sense that they correspond

to decreasing and increasing $|l_k|$. In case V, the first submodule flips the sign of l_N and keeps all other l 's unchanged.

In appendix I.1, we use Kazhdan-Lusztig theory to find the precise decomposition of the parabolic modules into irreducible modules. We show the non-trivial decompositions that arise in even spacetime dimension which implies a more complicated pole structure of the conformal blocks.

4.5.5 Comments on Conformal Representation Theory

It is useful to summarize the results that we obtained from the study of conformal representation theory stressing the implications for the study of conformal blocks and unitary CFTs.

First we saw that the odd dimensional case is easier than the even dimensional one. Thus the analytic continuation of the conformal blocks in the spacetime dimension d should be easier starting from odd dimension. In this case, we know the full set of poles of any conformal block:

$$\begin{aligned}\Delta_{\text{I}_k,n}^* &= k - l_k - n & (n = 1, 2, \dots, l_{k-1} - l_k) , \\ \Delta_{\text{II}_k,n}^* &= d + l_k - k - n & (n = 1, 2, \dots, l_k - l_{k+1}) , \\ \Delta_{\text{III},n}^* &= \frac{d}{2} - n & (n = 1, 2, \dots) , \\ \Delta_{\text{IV},n}^* &= \frac{d}{2} + \frac{1}{2} - n & (n = 1, 2, \dots, l_k) .\end{aligned}\tag{4.137}$$

We conjecture that a conformal block G_λ associated to the exchange of the operator $\mathcal{O}_\lambda$, labeled by an highest weight $\lambda = (-\Delta, l_1, \dots, l_N)$ of $\mathfrak{so}(d+2)$, has the following behavior at the pole

$$G_\lambda^{(p,q)} = \sum_{p',q'} \frac{(R_A)_{pp'qq'}}{\Delta - \Delta_A^*} G_{\lambda_A}^{(p',q')} + O((\Delta - \Delta_A)^0) ,\tag{4.138}$$

where $A = (\text{I}_k, n), (\text{II}_k, n), (\text{III}, n), (\text{IV}, n)$, λ_A are the ones defined in (4.130) and the coefficients $(R_A)_{pp'qq'}$ can be computed using the techniques introduced in the previous sections. As we showed in the previous section, also the regular part in Δ of the conformal blocks can be also obtained from a direct computation. Combining the knowledge of the pole structure (4.138) and the regular part one can in principle build any conformal block.

At a first sight formula (4.138) may look divergent since $G_{\lambda_A}^{(p',q')}$ has a pole precisely at the values of Δ specified by λ_A (for all A which are not of the type (II_1, n) and (III, n)). However in the examples of sections 4.4.3 and 4.4.4 we showed that the specific linear combination provided by the coefficients R_A is such that the divergence is canceled. This matches the analysis of section 4.5.3, where we found that in odd spacetime dimensions all the null submodules are simple, which in turns implies that there cannot exist higher order poles in the conformal blocks. We therefore conjecture that (4.138) is correct and always well defined.

From representation theory we also obtain a very sharp way to constrain the spectrum of unitary theories. Since unitary theories can only contain states $|\mathcal{O}_\lambda\rangle$ with positive norms, then all the respective modules $M_{\mathfrak{p}}(\lambda)$ have to be simple. Watching figure 4.6 it is easy to understand that

this requires that the value of the conformal dimension Δ of any operator has to satisfy

$$\Delta \geq \Delta_A^* \quad \text{for any } A . \quad (4.139)$$

Actually (4.139) gives many redundant conditions, for example it is clear that the bounds are optimized when n is as low as possible. In fact when the operator is a scalar we obtain that the bound (4.139) reduces to the usual

$$\Delta \geq \frac{d-2}{2} . \quad (4.140)$$

Moreover for a generic operator with $l_1 > l_2 > 0$ we have

$$\Delta \geq l_1 + d - 2 , \quad (4.141)$$

which gives the usual unitarity bound for tensor operators. When the first k lines of the Young tableau are of the same length, the bound becomes

$$\Delta \geq d - 1 - k + l \quad (l_1, \dots, l_k = l > 0) . \quad (4.142)$$

This analysis matches the results known in the literature [49–51].

Chapter 5

Charting the space of 3D CFTs with a continuous global symmetry

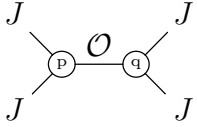
5.1 Introduction

Any local CFT with a continuous global symmetry contains a conserved current J_μ , whose flux through the boundary of a region B measures the total charge inside this region.¹ This property is encoded in the Ward identity,

$$\int_{\partial B} dx n^\mu \langle J_\mu(x) \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = -\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle \sum_{x_i \in B} q_i, \quad (5.1)$$

where n^μ is the unit normal to the boundary of the region B and q_i are the charges of the local operators $\mathcal{O}_i$. We shall study the four-point function of J_μ which is an observable that exists in any CFT with a continuous global symmetry. This will allow us to constrain the spectrum of operators that appear in the Operator Product Expansion (OPE) of two currents. In three spacetime dimensions, these neutral operators can be classified by their scaling dimension Δ , $SO(3)$ spin ℓ and parity.² More precisely, we will study the conformal block decomposition

$$\langle J_{\mu_1}(x_1) \dots J_{\mu_4}(x_4) \rangle = \sum_{\mathcal{O}} \sum_{p,q} \lambda_{JJ\mathcal{O}}^{(p)} \lambda_{JJ\mathcal{O}}^{(q)} \text{Diagram}, \quad (5.2)$$

where $\lambda_{JJ\mathcal{O}}^{(p)}$ are the coefficients of the operator $\mathcal{O}$ in the OPE of two currents. The index p, q run over a finite range, which depends on the spin and parity of the operator $\mathcal{O}$. The symbol  stands for the conformal blocks that are labeled by p, q and the quantum numbers Δ, ℓ and parity of $\mathcal{O}$. This is described in detail in section 5.2. Following the usual bootstrap

¹The existence of a conserved current follows from the Noether theorem in any Lagrangian CFT. However, we do not know of a more general (bootstrap) proof of this statement.

²We shall restrict our analysis to parity invariant CFTs. It would be interesting to relax this condition since there are many examples of parity breaking 3D CFTs involving Chern-Simons gauge fields.

strategy, we then impose crossing symmetry of this four-point function. However, due to current conservation, not all crossing equations are linearly independent. In section 5.2, we explain how to select a minimal set of independent crossing equations to be imposed numerically. With these ingredients and assuming unitarity, we applied the usual bootstrap semi-definite programming method (SDPB) to constrain the spectrum of neutral operators and some OPE coefficients $\lambda_{JJ\mathcal{O}}^{(p)}$. In figure 5.1, we show our result for the excluded region of the plane (Δ_0^+, Δ_0^-) , where $\Delta_\ell^\pm$ denotes the scaling dimension of the lightest parity even/odd neutral spin- ℓ operator. This curve was calculated using up to $\Lambda = 23$ derivatives of the crossing equations at the crossing symmetric point (451 components). The parameter Λ is defined in eq. (5.47). In this plot, we represented several known theories to verify that they all fall inside the allowed region. On one hand, the theories of a free Dirac fermion and of a free complex scalar field lie well within the allowed region. On the other hand, the critical $O(2)$ -model and the generalized free theory (GFVF) of a current seem to play an important role in determining the boundary of the allowed region. Our results suggest that these theories sit at kinks of the optimal boundary corresponding to $\Lambda = \infty$.

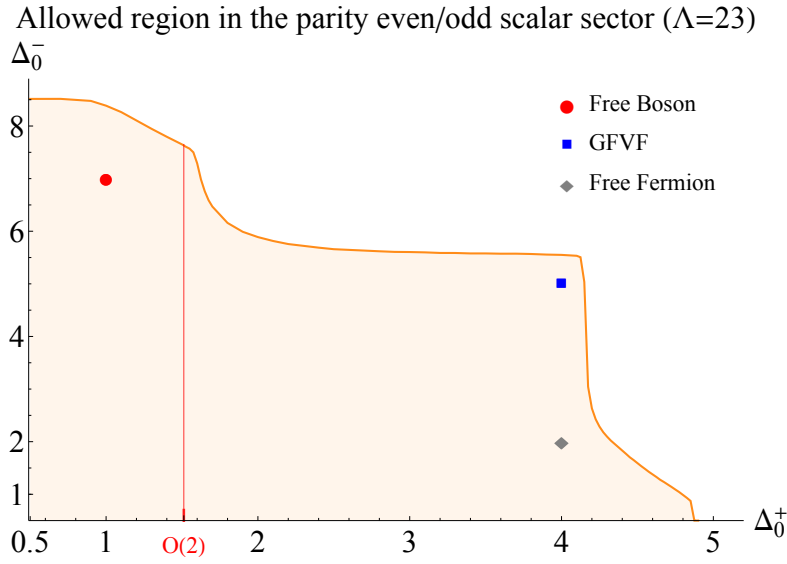


Figure 5.1: Exclusion plot in the plane (Δ_0^+, Δ_0^-) corresponding to the lightest parity even and parity odd scalars appearing in the OPE of two equal conserved currents. The shaded region is allowed.

The stress-energy tensor appears in the OPE of two currents,

$$J_\mu(x)J_\nu(0) = C_J \frac{\delta_{\mu\nu} - 2\hat{x}_\mu\hat{x}_\nu}{x^4} + \frac{3C_J}{32\pi|x|} [t_{\mu\nu}^{\alpha\beta}(\hat{x}) + 12\gamma\tilde{t}_{\mu\nu}^{\alpha\beta}(\hat{x})] T_{\alpha\beta}(0) + \dots \quad (5.3)$$

where $\hat{x}^\mu = \frac{x^\mu}{|x|}$ and the dots represent the contributions from all other operators besides the

identity and the stress tensor $T_{\alpha\beta}$. There are two independent tensor structures³ compatible with conservation and permutation symmetry. The conformal Ward identities relate the overall coefficient to the OPE coefficient of the identity operator (C_J) but the relative coefficient γ is an independent parameter that characterizes the CFT. In particular, it controls the high frequency/low temperature behaviour of the conductivity [52]. In the holographic context, γ corresponds to a higher derivative coupling between two photons and a graviton in the bulk. In particular, γ vanishes for Einstein-Maxwell theory. The conformal collider analysis of [53] gives rise to the bounds $-1 \leq 12\gamma \leq 1$ (see also [54–56]). This bound was recently proven using only unitarity and convergence of the OPE expansion [57] (also see [58] for an alternative approach). The bound is saturated by free complex bosons ($12\gamma = -1$) and free fermions ($12\gamma = 1$).

In figure 5.2, we plot the minimal value of the central charge C_T as a function of γ and for several values of Λ (number of derivatives of the crossing equations imposed). In dashed lines we plot the conformal collider bounds and the value of the central charge C_T of the minimal theories that saturate them: a free complex scalar and a free Dirac fermion. It is encouraging to notice that the lower bound on C_T grows very rapidly outside the region $-1 \leq 12\gamma \leq 1$. We suspect it diverges when $\Lambda \rightarrow \infty$. On the other hand, for $-1 < 12\gamma < 1$ the lower bound on C_T seems to be converging to a finite curve as we increase Λ .

Figures 5.1 and 5.2 are just an appetiser for the results presented in section 5.3. To facilitate the interpretation of our results we listed in appendix J.1 some known 3D CFTs with a continuous global symmetry. In section 5.2, we summarize the steps involved in setting up the numerical conformal bootstrap approach to the four point function of a conserved current, leaving many details to appendices J.2, J.3, J.4, G and J.5.

5.2 Setup

In this section we define our notation for three and four point correlation functions of spin 1 currents. We will often work in general spacetime dimension d and specialize to $d = 3$ at the end. Through this section we will work in embedding space introduced in section 2.2.

5.2.1 Three point functions $\langle JJO_{\Delta,\ell}^\pm \rangle$

In order to decompose the four-point function $\langle JJJJ \rangle$ in conformal blocks, we need to understand the structure of the OPE of two currents $J \times J$. This is equivalent to classifying all the conformal invariant three-point functions $\langle JJO_{\Delta,\ell}^\pm \rangle$. Since we are assuming the CFT is parity preserving, the three-point functions $\langle JJO_{\Delta,\ell}^+ \rangle$ will not involve the ϵ -tensor while the three-point functions $\langle JJO_{\Delta,\ell}^- \rangle$ will do.

³Their explicit form is:

$$\begin{aligned} t_{\mu\nu}^{\alpha\beta}(\hat{x}) &= 6\hat{x}_{(\mu}\delta_{\nu)}^{\alpha}\hat{x}^{\beta} + 2\delta_{\mu}^{\alpha}\delta_{\nu}^{\beta} + 3\hat{x}_{\mu}\hat{x}_{\nu}\hat{x}^{\alpha}\hat{x}^{\beta} - 5\delta_{\mu\nu}\hat{x}^{\alpha}\hat{x}^{\beta} , \\ \tilde{t}_{\mu\nu}^{\alpha\beta}(\hat{x}) &= 2\hat{x}_{(\mu}\delta_{\nu)}^{\alpha}\hat{x}^{\beta} - 2\delta_{\mu}^{\alpha}\delta_{\nu}^{\beta} - 3\hat{x}_{\mu}\hat{x}_{\nu}\hat{x}^{\alpha}\hat{x}^{\beta} - 3\delta_{\mu\nu}\hat{x}^{\alpha}\hat{x}^{\beta} . \end{aligned}$$

Lower bound on C_T ($\Delta=9,11,\dots,21,23$)

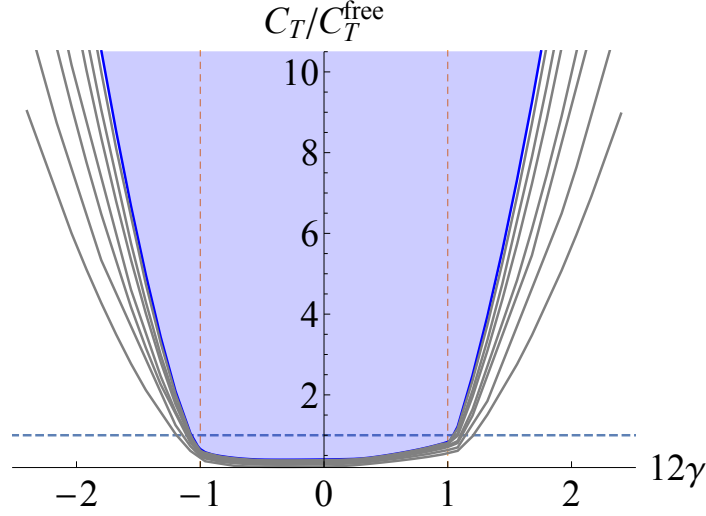


Figure 5.2: Lower bound on the central charge normalized to the one of a free complex scalar as a function of γ . The vertical dashed lines corresponds to the conformal collider bound $-1 \leq 12\gamma \leq 1$. We impose that the first spin-2 operator after $T_{\mu\nu}$ has dimension larger than 3.5 (see section 5.3 for explanation).

Let us start by writing the most general form of the three-point function between two equal vector operators (of dimension $d-1$) and a parity even operator,

$$\begin{aligned} & \langle J(P_1, Z_1) J(P_2, Z_2) \mathcal{O}_{\Delta, \ell}^+(P_3, Z_3) \rangle (P_{12})^{d-1-\frac{\Delta}{2}} (P_{13})^{\frac{\Delta}{2}} (P_{23})^{\frac{\Delta}{2}} = \\ & = \begin{cases} \lambda_{JJ\mathcal{O}^+}^{(1)} V_1 V_2 V_3^\ell + \lambda_{JJ\mathcal{O}^+}^{(2)} H_{12} V_3^\ell & \text{if } \ell = 0 \\ \lambda_{JJ\mathcal{O}^+}^{(3)} (H_{13} V_2 - H_{23} V_1) V_3^{\ell-1} & \text{if } \ell \geq 1, \text{ odd} \\ \lambda_{JJ\mathcal{O}^+}^{(1)} V_1 V_2 V_3^\ell + \lambda_{JJ\mathcal{O}^+}^{(2)} H_{12} V_3^\ell + \\ \quad + \lambda_{JJ\mathcal{O}^+}^{(3)} (H_{13} V_2 + H_{23} V_1) V_3^{\ell-1} + \lambda_{JJ\mathcal{O}^+}^{(4)} H_{13} H_{23} V_3^{\ell-2} & \text{if } \ell \geq 2, \text{ even} \end{cases} \end{aligned} \quad (5.4)$$

where $\lambda_{JJ\mathcal{O}^+}^{(i)}$ are undetermined constants and we used the notation

$$V_1 = V_{1,23}, \quad V_2 = V_{2,31}, \quad V_3 = V_{3,12}. \quad (5.5)$$

In this expression, we only imposed conformal and permutation symmetry. To impose conservation of $J(P_1, Z_1)$, it is enough to demand that the embedding space differential operator $\frac{\partial}{\partial P_1} \cdot \frac{\partial}{\partial Z_1}$ annihilates the three-point function. This further reduces the number of independent constants. In the case of a scalar operator, conservation implies $\lambda_{JJ\mathcal{O}^+}^{(2)} = \frac{\Delta-d+1}{\Delta} \lambda_{JJ\mathcal{O}^+}^{(1)}$ leaving only one free constant. In the case of odd spin, conservation implies $\lambda_{JJ\mathcal{O}^+}^{(3)} = 0$, which means that a parity even, spin odd operator cannot appear in the OPE $J \times J$. Finally, in the case of even spin $\ell \geq 2$

we find

$$\lambda_{JJ\mathcal{O}^+}^{(4)} = \frac{(d - \Delta - 2)(d - \Delta - 1)\lambda_{JJ\mathcal{O}^+}^{(1)} + (\Delta(d - \Delta - 2) + \ell(3d - 2\Delta - 4) + \ell^2)\lambda_{JJ\mathcal{O}^+}^{(2)}}{(2d - \Delta + \ell - 4)(2d - \Delta + \ell - 2)},$$

$$\lambda_{JJ\mathcal{O}^+}^{(3)} = \frac{(d - \Delta - 1)\lambda_{JJ\mathcal{O}^+}^{(1)} + (\Delta + \ell)\lambda_{JJ\mathcal{O}^+}^{(2)}}{2d - \Delta + \ell - 2},$$

which reduces the number of independent structures down to 2.

Let us now turn our attention to the three point function of two conserved currents and one parity odd operator $\mathcal{O}_{\Delta,\ell}^-$. In $d \leq 4$ one can use the ϵ -tensor to make parity odd conformally invariant three point functions. Indeed, any parity odd structure can be obtained by multiplying parity even structures by ϵ -tensors. In $d = 3$, there are three parity odd building blocks:

$$\epsilon_{ij} \equiv \epsilon(Z_i, Z_j, P_1, P_2, P_3) \quad (i, j = 1, 2, 3). \quad (5.6)$$

Conformal invariance and permutation symmetry restricts the tensor structures to be:⁴

$$\begin{aligned} & \langle J(P_1, Z_1)J(P_2, Z_2)\mathcal{O}_{\Delta,\ell}^-(P_3, Z_3) \rangle (P_{12})^{d-\frac{\Delta+1}{2}}(P_{13})^{\frac{\Delta+1}{2}}(P_{23})^{\frac{\Delta+1}{2}} = \\ & = \begin{cases} \lambda_{JJ\mathcal{O}^-}^{(1)}\epsilon_{12}V_3^\ell & \text{if } \ell = 0 \\ \lambda_{JJ\mathcal{O}^-}^{(2)}(\epsilon_{13}V_2 + \epsilon_{23}V_1)V_3^{\ell-1} & \text{if } \ell = 1 \\ \lambda_{JJ\mathcal{O}^-}^{(2)}(\epsilon_{13}V_2 - \epsilon_{23}V_1)V_3^{\ell-1} + \lambda_{JJ\mathcal{O}^-}^{(3)}(\epsilon_{13}H_{23} - \epsilon_{23}H_{13})V_3^{\ell-2} & \text{if } \ell \geq 2, \text{ even} \\ \lambda_{JJ\mathcal{O}^-}^{(2)}(\epsilon_{13}V_2 + \epsilon_{23}V_1)V_3^{\ell-1} + \lambda_{JJ\mathcal{O}^-}^{(3)}(\epsilon_{13}H_{23} + \epsilon_{23}H_{13})V_3^{\ell-2} & \text{if } \ell \geq 3, \text{ odd} \end{cases} \end{aligned} \quad (5.7)$$

where $\lambda_{JJ\mathcal{O}^-}^{(i)}$ are undetermined constants. Current conservation then fixes

$$\lambda_{JJ\mathcal{O}^-}^{(3)} = \frac{\Delta - 2 - (-1)^\ell}{\Delta - \ell - 3}\lambda_{JJ\mathcal{O}^-}^{(2)}, \quad (5.8)$$

for $\ell \geq 2$. In the case $\ell = 0$, conservation is automatic and $\lambda_{JJ\mathcal{O}^-}^{(1)}$ is a free parameter. In the case $\ell = 1$, conservation implies that $\lambda_{JJ\mathcal{O}^-}^{(2)} = 0$. In other words, no spin 1 operator that can appear in the OPE of two equal currents.

In summary, the number of independent constants in the three-point function $\langle JJ\mathcal{O}_{\Delta,\ell}^\pm \rangle$ is given in the following table:

		Parity	
Spin	Dimension	+	-
$\ell = 0$	$\Delta \geq 1/2$	1	1
$\ell = 1$	$\Delta \geq 2$	0	0
$\ell \geq 2$ even	$\Delta \geq \ell + 1$	2	1
$\ell \geq 3$ odd	$\Delta \geq \ell + 1$	0	1

⁴As explained in appendix J.2, the structure $\epsilon_{12}V_3^\ell$ is not independent of the ones we used for $\ell \geq 2$. Structures involving an ϵ tensor contracted with three polarization vectors can also be expressed in terms of ϵ_{ij} .

Finally, let us comment on the special cases when $\mathcal{O}$ saturates the unitarity bound. For $\ell \geq 1$ this happens when $\mathcal{O}$ is a conserved current with $\Delta = \ell + d - 2$. It is easy to check that the three-point function (5.4) of $\mathcal{O}^+$ are automatically conserved at P_3 if we set $\Delta = \ell + d - 2$. On the other hand, conservation of $\mathcal{O}^-(P_3)$ implies that the three-point functions (5.7) vanish for $\ell > 2$. For $\ell = 2$ conservation follows from $\Delta = 3$. This is consistent with the fact that it is possible to couple the currents to the stress tensor with a parity odd three-point function in theories that violate parity. For $\ell = 0$, one should impose $\partial^2 \mathcal{O} = 0$ when $\Delta = \frac{1}{2}$. This implies that the three-point function $\langle JJ\mathcal{O} \rangle$ must vanish for both $\pm$ parity.

5.2.1.1 Special case: $\langle JJT_{\mu\nu} \rangle$

Let us study in detail the three point function of two identical conserved currents and the energy momentum tensor. As discussed in the previous section there are only two independent structures. The three-point function is given by (5.4) with $\ell = 2$, $\Delta = d$ and

$$\lambda_{JJT}^{(1)} = (2 - 3d - 4d^3\gamma)C, \quad \lambda_{JJT}^{(2)} = (1 - 2d - 4d^2\gamma)C, \quad (5.9)$$

$$\lambda_{JJT}^{(3)} = -2d(1 + 4\gamma)C, \quad \lambda_{JJT}^{(4)} = 2\left(\frac{1}{d-2} - 4\gamma d\right)C, \quad (5.10)$$

where

$$C = \frac{d(d-2)}{2(d-1)^2 S_d} C_J, \quad (5.11)$$

is related via the conformal Ward identity to the current two-point function

$$\langle J(P_1, Z_1)J(P_2, Z_2) \rangle = C_J \frac{H_{12}}{(P_{12})^{d-1}}. \quad (5.12)$$

The symbol $S_d = 2\pi^{\frac{d}{2}}/\Gamma(\frac{d}{2})$ is the volume of a $(d-1)$ -dimensional sphere and γ is an independent parameter that appears in the OPE (5.3). The parameter γ controls the anisotropy of the energy correlator of a state created by the current [53, 55, 56]. Positivity of this energy correlator implies the bounds

$$-\frac{1}{4d} \leq \gamma \leq \frac{1}{4d(d-2)}, \quad (5.13)$$

which are saturated by free scalars and free fermions, respectively. This bound was recently proven relying only on unitarity and OPE convergence [57]. The parameter γ also has a nice physical meaning from the perspective of the dual AdS description. The current three-point function can be computed from the bulk action

$$S_{AdS} = C_J \int d^{d+1}x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \gamma L^2 W^{\mu\nu\tau\rho} F_{\mu\nu} F_{\tau\rho} \right] \quad (5.14)$$

where L is the AdS radius, W is the Weyl tensor and F is the field strength of the bulk gauge field dual to the current. In this form, it is clear that γ does not contribute to the two-point function of the current in the vacuum.

In the conformal bootstrap analysis of the four-point function $\langle JJJJ \rangle$ we normalize all operators to have unit two-point function. Recall that the stress tensor has a natural normalization due to the Ward identities,

$$\langle T(P_1, Z_1) T(P_2, Z_2) \rangle = C_T \frac{H_{12}^2}{(P_{12})^d} . \quad (5.15)$$

That means that we should multiply the OPE coefficients (5.10) by

$$\frac{1}{C_J \sqrt{C_T}} . \quad (5.16)$$

This shows that C_J is not accessible in the bootstrap analysis of $\langle JJJJ \rangle$. On the other hand, C_T does affect the OPE coefficients of normalized operators. For comparison, we recall the values of C_T for free theories [59]. Each real scalar field contributes

$$C_T = \frac{1}{S_d^2} \frac{d}{d-1} . \quad (5.17)$$

Each Dirac field contributes

$$C_T = \frac{1}{S_d^2} \frac{d}{2} 2^{\lfloor \frac{d}{2} \rfloor} , \quad (5.18)$$

where $2^{\lfloor \frac{d}{2} \rfloor}$ is the dimension of the Dirac γ -matrices in d spacetime dimensions. Notice that in $d = 3$, a complex scalar contributes the same as a Dirac fermion

$$C_T^{\text{free}} = \frac{3}{S_3^2} = \frac{3}{16\pi^2} . \quad (5.19)$$

This is the minimal matter content of free theories with a $U(1)$ global symmetry.

5.2.2 Four point function $\langle JJJJ \rangle$

The general structure of the four point function is⁵

$$\langle J(P_1, Z_1) \dots J(P_4, Z_4) \rangle = \frac{v^{1-d}}{(P_{12} P_{34})^{d-1}} \sum_s f_s(u, v) Q_s(\{P_i; Z_i\}) , \quad (5.20)$$

where

$$u = \frac{P_{12} P_{34}}{P_{13} P_{24}} , \quad v = \frac{P_{14} P_{23}}{P_{13} P_{24}} , \quad (5.21)$$

are the usual conformal invariant cross ratios and Q_s encode tensor structures in the embedding formalism. In Table J.1 we list all the parity even structures Q_s contributing to the four point function. In general dimension they are 43. As explained below, when $d = 3$ they reduce to 41. In addition, since we are considering equal conserved currents, there are two permutations which leave unchanged the conformal invariants u, v : $1234 \rightarrow 2143$ and $1234 \rightarrow 3412$. The action of these permutations simply sends one structure into another. The final effect is to

⁵The factor of v^{1-d} is convenient to make the crossing equations simpler.

reduce the number of independent functions $f_s(u, v)$ that appear in (5.20) to 19 (17 for $d = 3$). The transformation properties of each tensor structure, together with a list of the independent ones is reported in Table J.1.

5.2.2.1 Crossing Symmetry

The crossing symmetry $1234 \rightarrow 2134$ sends the cross ratios (u, v) into $(\frac{u}{v}, \frac{1}{v})$. As usual in the conformal bootstrap analysis, this crossing symmetry follows automatically from the conformal block expansion in the (12)(34) channel associated to the three-point functions studied in section 5.2.1.

On the other hand, the crossing symmetry $1234 \rightarrow 3214$ is not satisfied by the conformal blocks in the (12)(34) channel and gives rise to non-trivial constraints on the operator spectrum and OPE coefficients. The crossing symmetry $1234 \rightarrow 3214$ leads to ⁶

$$f_s(u, v) = \sum_{s'} [\mathbf{C}_{13}]_{ss'} f_{s'}(v, u), \quad (5.22)$$

where the matrix $[\mathbf{C}_{13}]$ is a permutation, which can be decomposed as follows

$$[\mathbf{C}_{13}] \equiv [\mathbf{M}]^{-1} [\mathbf{P}] [\mathbf{M}] \quad (5.23)$$

where $[\mathbf{P}]$ is a diagonal matrix with diagonal entries equal to ± 1 . This leads to a simpler form of the crossing equations. Introducing new functions $\tilde{f}_s(u, v) = \sum_{s'} [\mathbf{M}]_{ss'} f_{s'}(u, v)$ (see appendix J.4 for the precise definitions), the crossing equations simplify to

$$\begin{aligned} \tilde{f}_s(u, v) &= -\tilde{f}_s(v, u), & s = 1, 2, \dots, 7 \text{ and } s = 19, \\ \tilde{f}_s(u, v) &= \tilde{f}_s(v, u), & s = 8, 9, \dots, 18. \end{aligned} \quad (5.24)$$

In other words, we have 8 odd and 11 even functions under the crossing symmetry $u \leftrightarrow v$. We will see that the functions $\tilde{f}_{18}$ and $\tilde{f}_{19}$ will disappear in 3 dimensions, hence the choice to put them at the end of the list.

5.2.2.2 Conservation

In the numerical conformal bootstrap approach one writes the four point function as a sum of conformal blocks and imposes (a truncated version) of the 19 crossing equations (5.24). Fortunately, we can use conservation of the external currents to reduce this large number of crossing equations. Imposing conservation directly on the four point function produces a set of differential constraints that the functions $\tilde{f}_s(u, v)$ must satisfy. The four point function of three vectors and one scalar operator contains 14 independent tensor structures (in any dimension). As a consequence, each conservation condition will produce 14 first order differential equations

⁶These equations are derived for $\sqrt{u} + \sqrt{v} \neq 1$ where all the 43 tensor structures are linearly independent. By continuity, the equations also hold for any u and v . This is indeed the case for the free theory examples discussed in appendix J.1.

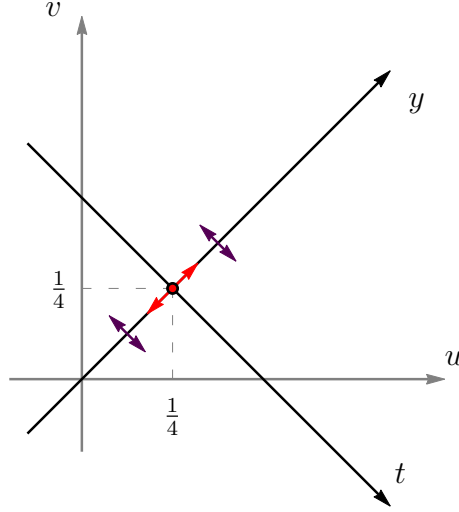


Figure 5.3: Change of coordinates from the usual cross ratio u and v to the new ones $y = u + v - 1/2$ and $t = u - v$. The conservation equation can be used to evolve 12 functions (6 even and 6 odd under the action of $[P]$) from the line $t = 0$ to the full plane. We can further evolve 2 functions (1 even and 1 odd) from the point $(t, y) = (0, 0)$, to the line $t = 0$.

of the form

$$\sum_{s=1}^{19} \left[[K(u, v)]_{is} \tilde{f}_s(u, v) + [K^u(u, v)]_{is} \partial_u \tilde{f}_s(u, v) + [K^v(u, v)]_{is} \partial_v \tilde{f}_s(u, v) \right] = 0, \quad (5.25)$$

where $i = 1, \dots, 14$. The first important observation is that the conformal block decomposition⁷ automatically satisfies these equations.⁸ The second observation is that the equations (5.25) are crossing symmetric. In other words, applying the crossing symmetry $u \leftrightarrow v$ to (5.25) and using (5.24) we obtain an equivalent set of differential equations. This means that if we use these differential equations to determine the functions $\tilde{f}_s$ evolving from a crossing symmetric "initial condition", then crossing symmetry is guaranteed everywhere. Therefore, if we start from a conformal block decomposition, it is sufficient to impose crossing symmetry on a minimal set of data about the functions $\tilde{f}_s$ that determines these functions everywhere via the differential equations (5.25).

To make this idea more precise it is convenient to introduce new coordinates

$$t = u - v, \quad y = u + v - \frac{1}{2}, \quad (5.26)$$

which are represented in figure 5.3. We will think of the t as time and y as space. Crossing symmetry (5.24) means that 8 functions $\tilde{f}$ are odd under time-reversal, while the remaining 11 functions are even. The conservation equations (5.25) become the following first order time

⁷See for instance (5.40) in the next section.

⁸In fact, we used this to cross check the computation of the conformal blocks.

evolution equations

$$\sum_{s=1}^{19} \left[[K]_{is} \tilde{f}_s + [K^y]_{is} \partial_y \tilde{f}_s + [K^t]_{is} \partial_t \tilde{f}_s \right] = 0, \quad i = 1, \dots, 14, \quad (5.27)$$

where $[K^t] = [K^u] - [K^v]$ and $[K^y] = [K^u] + [K^v]$ are 14×19 matrices.

One can check that the matrix $[K^t]$ has rank 12.⁹ That means that we can evolve 12 functions $\tilde{f}_s$ starting from an initial time slice, which we choose to be $t = 0$. Since the functions $\tilde{f}_s$ are either even or odd under $t \rightarrow -t$, crossing symmetric boundary conditions are obtained by simply imposing the odd ones to vanish on the line $t = 0$, while the even ones are left unconstrained. One can explicitly check that the (7 dimensional) Kernel of $[K^t]$ decomposes in two orthogonal subspaces (of dimension 5 and 2) associated to the eigenvalues ± 1 of the crossing symmetry matrix $[P]$ defined in (5.23). This means that we can evolve $8 - 2 = 6$ odd functions and $11 - 5 = 6$ even functions. One possible choice is $\tilde{f}_1, \dots, \tilde{f}_6$ and $\tilde{f}_8, \tilde{f}_9, \tilde{f}_{10}, \tilde{f}_{11}, \tilde{f}_{12}, \tilde{f}_{14}$. Hence, by using 12 out of the 14 conservation equations we reduced to the set of crossing symmetry conditions:

$$\begin{aligned} \tilde{f}_s(u, u) &= 0 & \text{for } s=1, 2, 3, 4, 5, 6 \\ \tilde{f}_s(u, v) &= -\tilde{f}_s(v, u) & \text{for } s=7, 19 \\ \tilde{f}_s(u, v) &= \tilde{f}_s(v, u) & \text{for } s=13, 15, 16, 17, 18 \end{aligned} \quad (5.28)$$

Note that the boundary condition on the line doesn't constrain the even functions: any initial condition $\tilde{f}_s$, $s = 8, 9, 10, 11, 12, 14$ will be automatically evolved into a crossing symmetric function.

In fact, this is still not the minimal set of data where we can impose crossing symmetry. We will use the two remaining conservation equations to reduce further the set of crossing symmetry equations. The remaining conservation equations are not evolution equations. They are two constraint equations on the initial data at $t = 0$. One can check that at $t = 0$, the first constraint equation only involves odd functions and the second only involves even functions. More precisely, the first constraint equation can be written as

$$\partial_y \tilde{f}_3(0, y) = \sum_{s \neq 3} A_s(y) \partial_y \tilde{f}_s(0, y) + \sum_s B_s(y) \tilde{f}_s(0, y), \quad (5.29)$$

where the sum runs over the odd functions ($s = 1, 2, \dots, 7$ and $s = 19$) and the coefficients $A_s(y)$ and $B_s(y)$ are regular at the crossing symmetric point $y = 0$. This means that it is sufficient to impose $\tilde{f}_3(t = 0, y = 0) = 0$ because this equation will ensure that $\tilde{f}_3(t = 0, y) = 0$ for any y . Since the second constraint equation only involves even functions, which are unconstrained at the initial surface $t = 0$ it is not useful to further reduce the crossing symmetry constraints.

⁹In fact, this is true for a generic choice of time coordinate around the point $u = v = 1/4$. The exception being the coordinate y . In this special case, the rank of $[K^y]$ is 10.

In the end the minimal set of crossing symmetry conditions is:

$$\tilde{f}_3(1/4, 1/4) = 0 \quad (5.30)$$

$$\tilde{f}_s(u, u) = 0 \quad \text{for } s = 1, 2, 4, 5, 6 \quad (5.31)$$

$$\tilde{f}_s(u, v) = -\tilde{f}_s(v, u) \quad \text{for } s = 7, 19 \quad (5.32)$$

$$\tilde{f}_s(u, v) = \tilde{f}_s(v, u) \quad \text{for } s = 13, 15, 16, 17, 18 \quad (5.33)$$

where we went back to the original coordinates u and v . In agreement with [60] in general d there are 7 equations in the “bulk”; additionally there are five constraints on the line and one at a crossing-symmetric point. We remark that our analysis of the conservation equations is valid only in a local neighbourhood of the crossing symmetric point $u = v = 1/4$. However, this is sufficient for the numerical bootstrap algorithm where we only consider a finite number of derivatives of the crossing equations at $u = v = 1/4$.

5.2.2.3 Three dimensions

In three dimensions not all 43 tensor structures of the four point function are linearly independent. As we described in section 2.2.5 we can consider the embedding space tensor

$$W_{ij}^{A_1 \dots A_6} = Z_i^{A_1} Z_j^{A_2} P_1^{A_3} P_2^{A_4} P_3^{A_5} P_4^{A_6}, \quad (5.34)$$

which vanishes identically in $\mathbb{R}^{d+1,1}$ for $d = 3$. The contractions of two tensors W_{ij} and W_{kl} can be written as a linear combination of the 43 tensor structures Q_s that form a basis for four-point functions of vector primary operators. As explained in section 2.2.5 we therefore obtain two independent linear relation between the 43 tensor structures Q_s . These constraints can be found in appendix J.5. We use these to express the structures Q_{31} and Q_{40} in terms of the other Q_s . According to the definitions in appendix J.4, this corresponds to the functions $\tilde{f}_{18}$ and $\tilde{f}_{19}$. The entire argument about the conservation equations proceeds in the same way just dropping these two functions.

In the end the minimal set of crossing symmetry conditions in $d = 3$ is as follows. It includes five equations in the “bulk” [60], five constraints on a line, and one at a point:

$$\tilde{f}_7(u, v) = -\tilde{f}_7(v, u) \quad (5.35)$$

$$\tilde{f}_s(u, v) = \tilde{f}_s(v, u) \quad \text{for } s = 13, 15, 16, 17 \quad (5.36)$$

$$\tilde{f}_s(u, u) = 0 \quad \text{for } s = 1, 2, 4, 5, 6 \quad (5.37)$$

$$\tilde{f}_3(1/4, 1/4) = 0 \quad (5.38)$$

5.2.3 Conformal blocks

In order to apply the technology of the numerical bootstrap we need to expand the functions $\tilde{f}_s$ in conformal blocks. To do so, we can use the conformal blocks computed in section 4.4.4.

Notice that with formulas (4.91-4.92) one can obtain all the blocks correspondent to four generic

external vector operators. In this work however we only need the blocks for conserved and equal currents. To obtain them we contract the labels p, q of the blocks with some matrices $m_{\pm}$ which come from the conservation of the 3-point function of $\langle JJ\mathcal{O}^{\pm}\rangle$ explained in section 5.2.1. We further contract the index s with a matrix $[\mathbf{M}]$ which simplifies the crossing equation of 4 equal vector operators as explained in section 5.2.2,

$$\tilde{g}_{\Delta\ell\pm,\tilde{s}}^{(\tilde{p},\tilde{q})}(r,\eta) \equiv v^{1-d} \sum_s [\mathbf{M}]_{\tilde{s}s} \sum_{pq} (m_{\pm})_{\tilde{p}p} (m_{\pm})_{\tilde{q}q} g_{\Delta\ell\pm,s}^{(p,q)}(r,\eta) . \quad (5.39)$$

Here the matrix m_+ is 2×5 while the matrix m_- is 1×4 , therefore $\tilde{p}, \tilde{q} = 1, 2$ for the parity even case and $\tilde{p}, \tilde{q} = 1$ for the parity odd case. In appendix G.5 we give the precise form of such matrices. The matrix $[\mathbf{M}]$ is 19×43 and it is defined in appendix J.4. It is worth to stress that since the equations (4.91-4.92) are diagonal in s , it is possible to compute only some structures, without having to compute the others. In the following sections we will drop the tilde symbol above the labels p, q, s .

Using the OPE channel (12)(34), one obtains the following conformal block expansion¹⁰

$$\tilde{f}_s(u, v) = \sum_{\mathcal{O}} \sum_{p,q} \tilde{\lambda}_{JJ\mathcal{O}}^{(p)} \tilde{\lambda}_{JJ\mathcal{O}}^{(q)} \tilde{g}_{\mathcal{O},s}^{(p,q)}(u, v) , \quad (5.40)$$

where the functions $\tilde{f}$ were defined in section 5.2.2.2. For further details we refer the reader to appendix G.

5.2.4 Bootstrap equations

Plugging the conformal blocks decomposition (5.40) into the three dimensional crossing equations (5.35) we explicitly obtain 11 conditions which can be nicely written in vector notation as

$$0 = \sum_{\substack{\mathcal{O}^+ \\ \ell \geq 2, \text{even}}} \begin{pmatrix} \tilde{\lambda}_{JJ\mathcal{O}^+}^{(1)} & \tilde{\lambda}_{JJ\mathcal{O}^+}^{(2)} \end{pmatrix} \vec{V}_{\Delta\ell+} \begin{pmatrix} \tilde{\lambda}_{JJ\mathcal{O}^+}^{(1)} \\ \tilde{\lambda}_{JJ\mathcal{O}^+}^{(2)} \end{pmatrix} + \sum_{\mathcal{O}^+, \ell=0} \left(\tilde{\lambda}_{JJ\mathcal{O}^+}^{(1)} \right)^2 \vec{V}_{\Delta+} + \sum_{\mathcal{O}^-, \ell} \left(\tilde{\lambda}_{JJ\mathcal{O}^-} \right)^2 \vec{V}_{\Delta\ell-} . \quad (5.41)$$

Here $\tilde{\lambda}_{JJ\mathcal{O}^+}^{(i)}, i = 1, 2$ are the OPE coefficients defined in Sec. G, while $\tilde{\lambda}_{JJ\mathcal{O}^-} = \tilde{\lambda}_{JJ\mathcal{O}^-}^{(1)}$ for scalars and $\tilde{\lambda}_{JJ\mathcal{O}^-} = \tilde{\lambda}_{JJ\mathcal{O}^-}^{(2)}$ for higher ℓ . In particular, for the stress energy tensor we have:

$$\tilde{\lambda}_{JJT}^{(1)} = \sqrt{\frac{3C_T^{\text{free}}}{16C_T}}(1 - 12\gamma) \quad \tilde{\lambda}_{JJT}^{(2)} = \sqrt{\frac{3C_T^{\text{free}}}{16C_T}}(5 - 12\gamma) \quad (5.42)$$

¹⁰In appendix G, we compute the conformal blocks in a three-point function basis which is different from the one of section 5.2.1, therefore the coefficients $\tilde{\lambda}_{JJ\mathcal{O}}^{(p)}$ are just a linear transformation of the coefficients $\lambda_{JJ\mathcal{O}}^{(p)}$.

Finally, $\vec{V}_{\Delta+}$, $\vec{V}_{\Delta\ell-}$ are 11-dimensional vectors and $\vec{V}_{\Delta\ell+}$ is a 11-vector of 2×2 matrices. Introducing the (anti)symmetric combination of conformal blocks defined in (5.39),

$$\begin{aligned}\tilde{F}_{\Delta\ell\pm,s}^{(p,q)}(u,v) &= \tilde{g}_{\Delta\ell\pm,s}^{(p,q)}(u,v) - \tilde{g}_{\Delta\ell\pm,s}^{(p,q)}(v,u), \\ \tilde{H}_{\Delta\ell\pm,s}^{(p,q)}(u,v) &= \tilde{g}_{\Delta\ell\pm,s}^{(p,q)}(u,v) + \tilde{g}_{\Delta\ell\pm,s}^{(p,q)}(v,u).\end{aligned}\tag{5.43}$$

we have

$$\vec{V}_{\Delta+} \equiv \begin{pmatrix} \tilde{H}_{\Delta 0+,3}(\frac{1}{4}, \frac{1}{4}) \\ \tilde{H}_{\Delta 0+,1}(u,u) \\ \tilde{H}_{\Delta 0+,2}(u,u) \\ \tilde{H}_{\Delta 0+,4}(u,u) \\ \tilde{H}_{\Delta 0+,5}(u,u) \\ \tilde{H}_{\Delta 0+,6}(u,u) \\ \tilde{H}_{\Delta 0+,7}(u,v) \\ \tilde{F}_{\Delta 0+,13}(u,v) \\ \tilde{F}_{\Delta 0+,15}(u,v) \\ \tilde{F}_{\Delta 0+,16}(u,v) \\ \tilde{F}_{\Delta 0+,17}(u,v) \end{pmatrix}, \quad \vec{V}_{\Delta\ell-} \equiv \begin{pmatrix} \tilde{H}_{\Delta\ell-,3}(\frac{1}{4}, \frac{1}{4}) \\ \tilde{H}_{\Delta\ell-,1}(u,u) \\ \tilde{H}_{\Delta\ell-,2}(u,u) \\ \tilde{H}_{\Delta\ell-,4}(u,u) \\ \tilde{H}_{\Delta\ell-,5}(u,u) \\ \tilde{H}_{\Delta\ell-,6}(u,u) \\ \tilde{H}_{\Delta\ell-,7}(u,v) \\ \tilde{F}_{\Delta\ell-,13}(u,v) \\ \tilde{F}_{\Delta\ell-,15}(u,v) \\ \tilde{F}_{\Delta\ell-,16}(u,v) \\ \tilde{F}_{\Delta\ell-,17}(u,v) \end{pmatrix}, \quad \vec{V}_{\Delta\ell+} \equiv \begin{pmatrix} \mathbf{s}[\tilde{H}_{\Delta\ell+,3}(\frac{1}{4}, \frac{1}{4})] \\ \mathbf{s}[\tilde{H}_{\Delta\ell+,1}(u,u)] \\ \mathbf{s}[\tilde{H}_{\Delta\ell+,2}(u,u)] \\ \mathbf{s}[\tilde{H}_{\Delta\ell+,4}(u,u)] \\ \mathbf{s}[\tilde{H}_{\Delta\ell+,5}(u,u)] \\ \mathbf{s}[\tilde{H}_{\Delta\ell+,6}(u,u)] \\ \mathbf{s}[\tilde{H}_{\Delta\ell+,7}(u,v)] \\ \mathbf{s}[\tilde{F}_{\Delta\ell+,13}(u,v)] \\ \mathbf{s}[\tilde{F}_{\Delta\ell+,15}(u,v)] \\ \mathbf{s}[\tilde{F}_{\Delta\ell+,16}(u,v)] \\ \mathbf{s}[\tilde{F}_{\Delta\ell+,17}(u,v)] \end{pmatrix}. \tag{5.44}$$

For any 2×2 matrix M , $\mathbf{s}[M] \equiv (M + M^T)/2$ selects its symmetric part. In the above expression we omitted the (p, q) upper indices when only one conserved conformal block is allowed, namely for parity even scalars and parity odd operators.

5.2.5 Setting up the Semi-Definite Programming

The feasibility of the above set of equations can be constrained using semidefinite programming (SDP). We refer to [61] for details. To rule out a hypothetical CFT spectrum, we must find a linear functional α such that

$$\begin{aligned}\vec{\alpha}[V_{0,+}] &\geq 0, \quad \text{for the identity operator,} \\ \vec{\alpha}[\vec{V}_{\Delta+}] &\geq 0, \quad \text{for all scalar operators,} \\ \vec{\alpha}[\vec{V}_{\Delta\ell+}] &\succeq 0, \quad \text{for all parity even operators in the spectrum with } \ell \text{ even,} \\ \vec{\alpha}[\vec{V}_{\Delta\ell-}] &\geq 0, \quad \text{for all parity odd operators in the spectrum with any } \ell \neq 1.\end{aligned}\tag{5.45}$$

Here, the notation “ $\succeq 0$ ” means “is positive semidefinite”. Since the 11 crossing equations have a different dependence on the conformal invariants u, v , it’s worth spelling out the explicit form of the linear functional α we consider in this work. Let us remind the reader the definition of the usual coordinates $z, \bar{z}$:

$$u = z\bar{z} \quad v = (1-z)(1-\bar{z}) \tag{5.46}$$

Then, we define the family of linear functionals α acting on an 11-dimensional vector, whose entries are functions of $z, \bar{z}$

$$\alpha[\vec{V}] = \alpha_1 V_1 \left(\frac{1}{2}, \frac{1}{2} \right) + \sum_{i=2}^6 \sum_{m=0}^{\Lambda/2} \alpha_m \partial_z^{2m} V_i(z, 1-z) \Big|_{z=\frac{1}{2}} + \sum_{i=7}^{11} \sum_{m+n < \Lambda} \alpha_{mn} \partial_z^m \partial_{\bar{z}}^n V_i(z, \bar{z}) \Big|_{z=\bar{z}=\frac{1}{2}} \quad (5.47)$$

Although we didn't write it explicitly, the linear functionals are parametrized by the integer Λ , which indicates the order of derivatives considered. Notice that the action of the functional on a vector of matrices results in a matrix, while its action on a vector of scalar functions produces a number. The existence of such a functional for a hypothetical CFT spectrum implies the inconsistency of this spectrum with crossing symmetry. In addition to any explicit assumptions placed on the allowed values of Δ , we impose that all operators must satisfy the unitarity bound

$$\Delta \geq \begin{cases} \ell + d - 2 & \ell > 0 \\ \frac{d-2}{2} & \ell = 0 \end{cases}, \quad (5.48)$$

where $d = 3$ is the spacetime dimension.

The more information about the spectrum we use in (5.45), the easier it is to find a functional $\vec{\alpha}$ that excludes the putative CFT. In this work we mainly focus on assumptions about the minimal values of operator dimensions in given sectors and the value of parameter γ defined in section 5.2.1.1.

We will review the exact SDP problem to solve case by case in the next section.

5.3 Results

In this section we present the results of our numerical investigations. In what follows $\Delta_\ell^\pm$ will denote the dimension of the first parity even/odd neutral spin ℓ operator. We will also use $(\Delta_\ell^\pm)'$ to denote the second operator in the same sector.

5.3.1 Bounds on operator dimensions

We begin our journey in the space of CFTs with global symmetries by inspecting the constraints imposed by crossing symmetry on the spectrum of scalar operators. As reviewed in Sec. 5.2.1, the OPE $J \times J$ contains both parity even and parity odd scalars. The first issue we want to address is how large can the dimensions of these operators be. To answer this question we solved the semi-definite problem (5.45) with the assumption that all scalar parity-even/odd operators have dimension larger than $\Delta_0^\pm$ correspondingly. The allowed region is shown in figure 5.4. The very first surprising result is that crossing symmetry is able to constrain the plane Δ_0^+, Δ_0^- into a closed region, meaning that all CFTs with global symmetry must have parity even and parity odd scalar operators. This is completely universal: this result is only based on unitarity and associativity of the OPE. To our knowledge this is the first completely general result for 3D

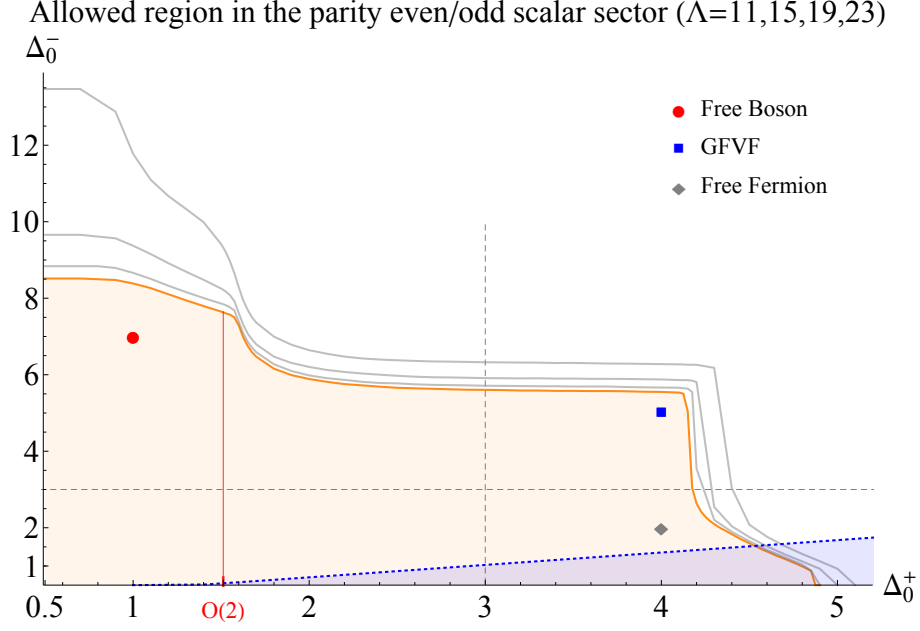


Figure 5.4: Allowed region consistent with crossing symmetry assuming that all the parity-even scalars appearing in the OPE $J \times J$ have dimension larger than Δ_0^+ and all parity-odd scalars have dimensions larger than Δ_0^- . The orange shaded region is allowed. Marks correspond to known CFTs: free complex boson $((\Delta_0^+, \Delta_0^-) = (1, 7))$, free Dirac fermion $((\Delta_0^+, \Delta_0^-) = (4, 2))$ and GFVF $((\Delta_0^+, \Delta_0^-) = (4, 5))$. The red vertical line corresponds to the approximate dimension of the lightest singlet operator in the interacting $O(2)$ model: $\Delta_0^+ = 1.5117$. The blue shading shows the region excluded by bootstrapping the four point function of identical parity odd scalars with the dimension Δ_0^- . See text for more details. The best bound has been computed at $\Lambda = 23$ while gray lines correspond to $\Lambda = 11, 15, 19$.

unitary CFT with global symmetry.¹¹

Let us now describe the shape of figure 5.4. If we regard the boundary of the allowed region as a function $(\Delta_0^-)^{\max}$ of Δ_0^+ , then it can only be a monotonic non-increasing function.¹² Hence we expect the allowed region to be shaped by existing CFTs with the largest gap in the scalar sector. There are three solvable models that we can place in the Δ_0^+, Δ_0^- plane: a free massless complex scalar field ϕ , a free massless 3d Dirac fermion ψ and a Generalized Free Vector Field (GFVF). In the free scalar field case, the $U(1)$ current OPE schematically reads:

$$J_\mu \times J_\nu \sim \underbrace{\phi^\dagger \phi}_{\text{parity-even}} + \underbrace{\epsilon^{\alpha\beta\rho} \phi^\dagger \partial_\alpha \phi \partial^\sigma \partial_\beta \phi^\dagger \partial_\sigma \partial_\rho \phi}_{\text{parity-odd}} + \dots, \quad \Delta_0^+ = 1, \Delta_0^- = 7. \quad (5.49)$$

¹¹All previous results in the bootstrap literature assumed at least the presence of a scalar or fermion operator with a given fixed dimension; theories with extended supersymmetry represent an exception: scalars are contained in certain protected super-multiplets.

¹²If we can not exclude a theory with $\Delta_0^+ = a$ and $\Delta_0^- = b$ then we cannot exclude theories with $\Delta_0^+ \leq a$ and $\Delta_0^- \leq b$.

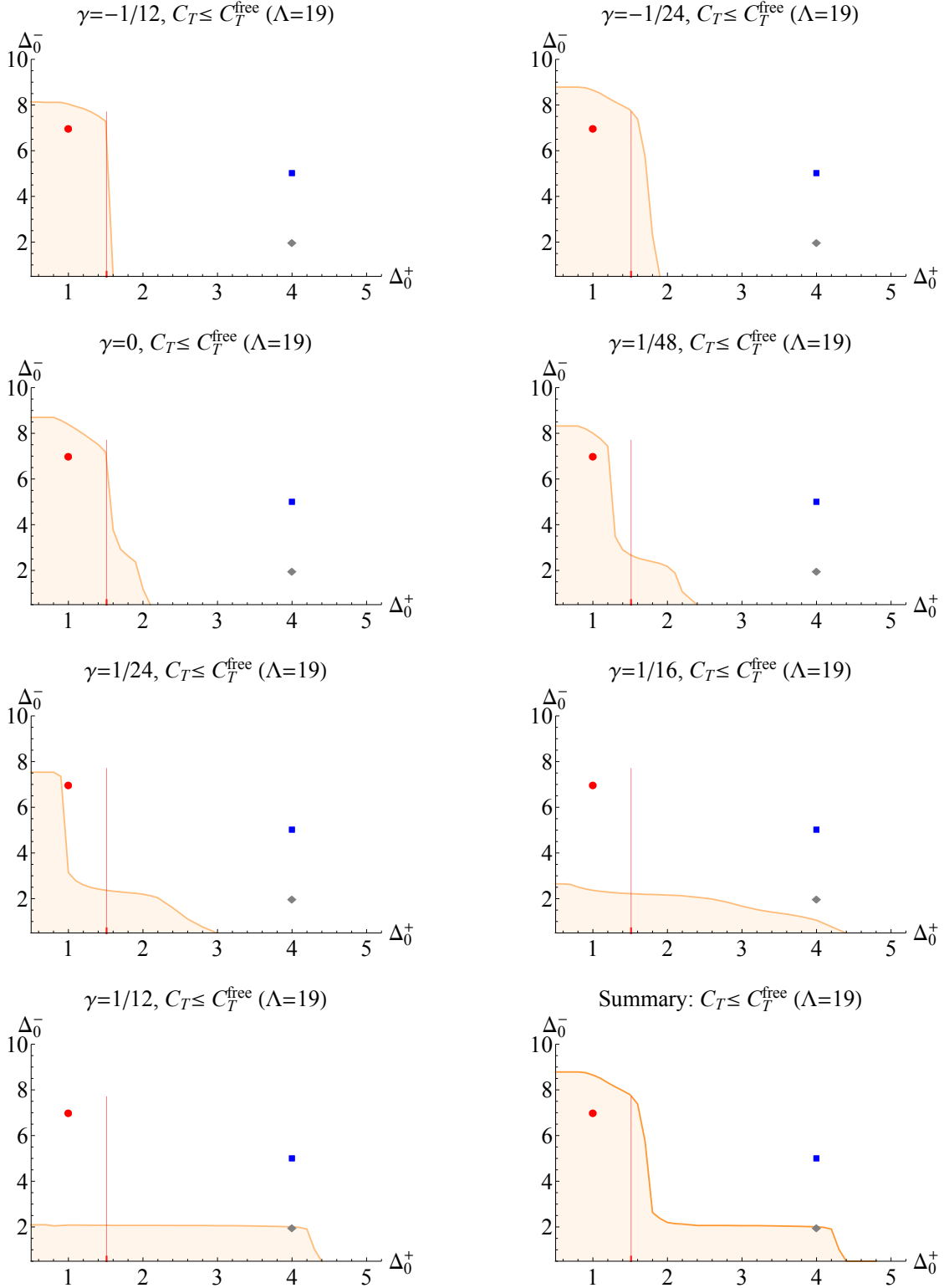


Figure 5.5: Allowed region assuming that all parity-even scalars appearing in the OPE $J \times J$ have dimension larger than Δ_0^+ and all parity-odd scalars have dimensions larger than Δ_0^- . We also impose small central charge $C_T \leq C_T^{\text{free}}$ and fix γ to specific values within the range $12|\gamma| \leq 1$. The bound has been computed at $\Lambda = 19$. See figure 5.4 for marks legend.

Allowed region in the parity even/odd scalar sector, $C_T \leq C_T^{\text{free}}$ ($\Lambda=19$)

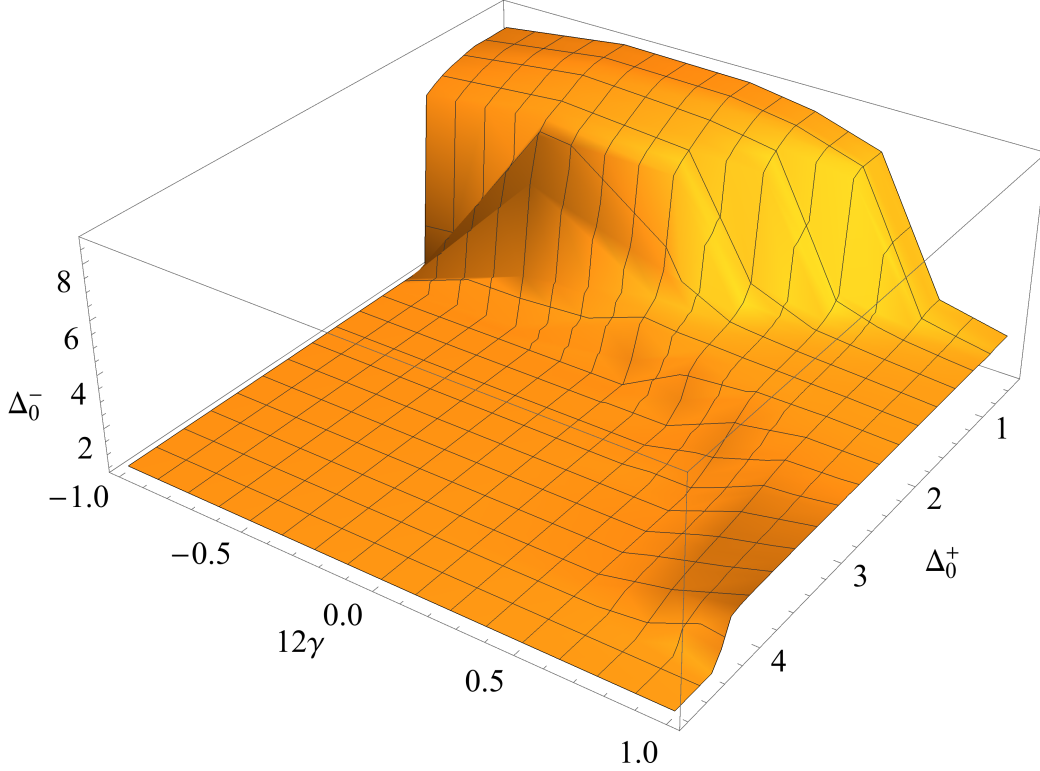


Figure 5.6: Three dimensional view of figure 5.5.

In the free fermion case, we find

$$J_\mu \times J_\nu \sim \underbrace{(\bar{\psi}\psi)^2}_{\text{parity-even}} + \underbrace{\bar{\psi}\psi}_{\text{parity-odd}} + \dots, \quad \Delta_0^+ = 4, \Delta_0^- = 2. \quad (5.50)$$

Finally, the GFVF is equivalent to a free photon in AdS_4 . From the three dimensional point of view it corresponds to a conserved current with a standard 2 point function, and all higher point correlators satisfying Wick theorem. In this case the lightest scalar operators are given by

$$J_\mu \times J_\nu \sim \underbrace{J_\mu J^\mu}_{\text{parity-even}} + \underbrace{\epsilon^{\mu\nu\rho} J_\mu \partial_\nu J_\rho}_{\text{parity-odd}} + \dots, \quad \Delta_0^+ = 4, \Delta_0^- = 5. \quad (5.51)$$

Notice that the GFVF is technically a so called *dead-end* CFT, since it doesn't contain relevant scalar operators. On the other hand it doesn't contain a local energy momentum tensor either, since it corresponds to a U(1) gauge theory on a fixed AdS background (infinite central charge C_T and no dynamical gravity).

These solvable CFTs are marked in figure 5.4 as described in the caption. While the boundary of the allowed region is close to the GFVF point, it is quite far from the point of free boson theory. Instead it starts at higher values Δ_0^- and after a small plateau it displays a kink for

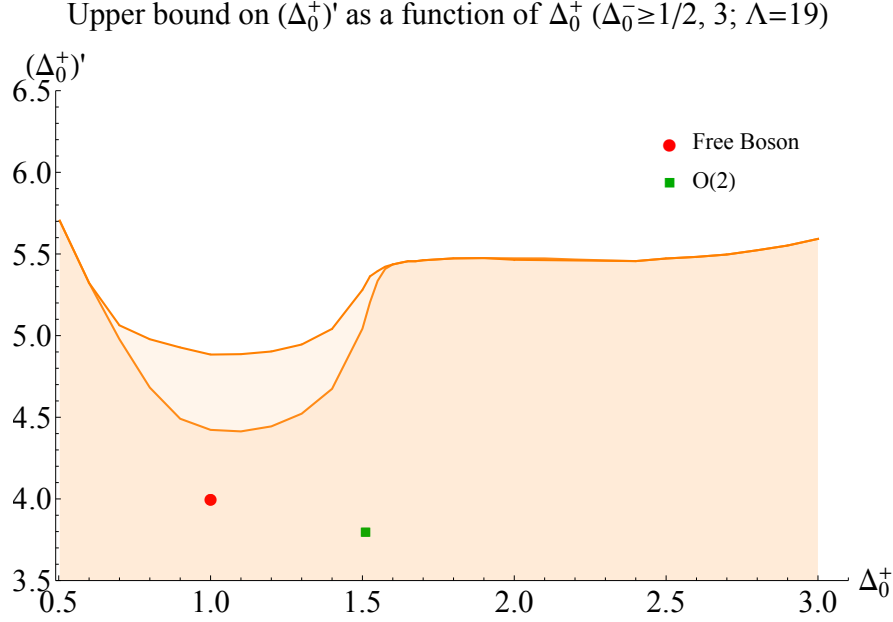


Figure 5.7: Upper bound on $(\Delta_0^+)',$ the dimension of the second lowest parity-even scalar operator appearing in the OPE $J \times J$ when we assume that the theory contains a scalar parity-even operator with dimension $\Delta_0^+.$ The red dot corresponds to the free boson $((\Delta_0^+, (\Delta_0^+)') = (1, 4)),$ and green dot corresponds to interacting $O(2)$ theory, while points for the free fermion and GFVF lie outside the range of the plot. Lower curve assumes theory has no relevant scalar parity-odd operators. The bounds have been computed with $\Lambda = 19.$

values of Δ_0^+ seemingly in correspondence to the interacting $O(2)$ model. To our knowledge the dimension of the leading parity odd scalar in this model is not known, neither in the ε -expansion nor in the $1/N$ expansion. Accordingly, we conjecture that the lightest parity odd operator in (5.49) in critical $O(2)$ theory acquires a positive anomalous dimension.

One additional feature of figure 5.4 is the region extending to values Δ_0^+ larger than 4 but requiring at the same time parity odd scalars with small dimension. Let us call ϕ^- the parity odd scalar operator with smallest dimension. The OPE of ϕ^- with itself would contain a parity even scalar operator¹³ with dimension $\Delta_0^+.$ Then, by bootstrapping the four point function $\langle \phi^- \phi^- \phi^- \phi^- \rangle$ we can obtain an independent bound of the form $\Delta_0^+ \leq f(\Delta_0^-),$ for some function $f.$ This bounds has been already obtained in past works focused on the three dimensional Ising model [61–63]. For this work we extended these results to larger values of $\Delta_0^-.$ The blue shading in figure 5.4 represents the disallowed region. We expect that the use of mixed correlators of scalars and conserved currents will shed light on the fate of this region.

The existence of a CFTs with large gaps in the scalar sector, namely the GFVF, shapes the bound shown in figure 5.4 for $1.6 \lesssim \Delta_0^+ \lesssim 4$ and could potentially hide other theories in the bulk of the allowed region. In order to better probe this region we explored the constraints on

¹³Unless there is symmetry argument preventing this from happening, this operator must coincide with the smallest dimension parity even scalar operator entering the $J \times J$ OPE.

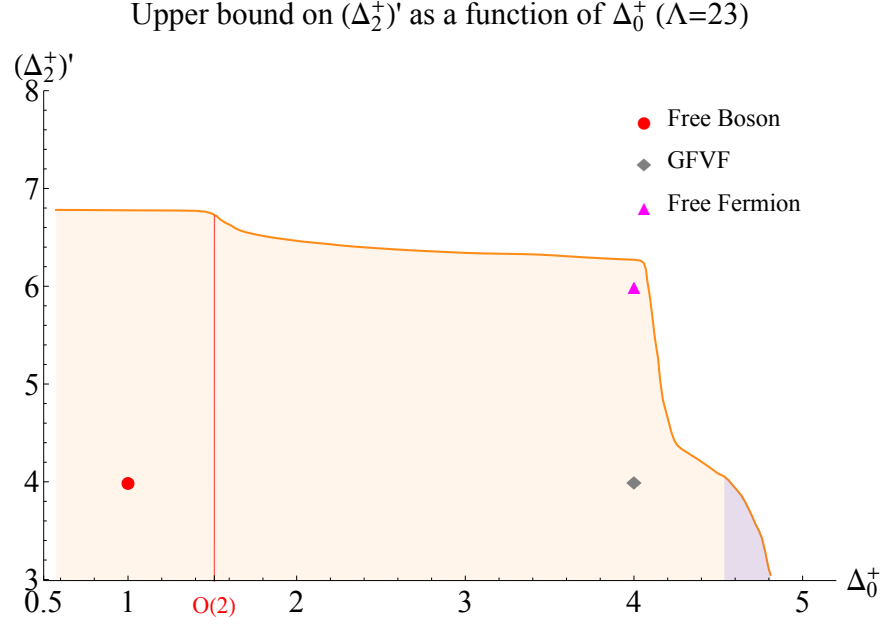


Figure 5.8: Allowed region consistent with crossing symmetry assuming that all parity-even scalars appearing in the OPE $J \times J$ have dimension larger than Δ_0^+ and all parity-even spin-2 operators (except the energy momentum tensor) have dimensions larger than Δ_2^+ . No other assumption is imposed. The shaded region is allowed. Marks correspond to known CFTs: free boson $((\Delta_0^+, (\Delta_2^+)') = (1, 4))$, free fermion $((\Delta_0^+, (\Delta_2^+)') = (4, 6))$ and GFVF $((\Delta_0^+, (\Delta_2^+)') = (4, 4))$. The vertical red line corresponds to the central value of the allowed dimensions of the smallest dimension singlet operator in the $O(2)$ model: $\Delta_0^+ = 1.5117$. The shaded blue region is excluded by bootstrapping the four point function of the first parity odd scalar operator. See figure 5.4. The bound has been computed at $\Lambda = 23$.

theories with a finite value of the central charge. To do that, we modified the conditions (5.45) and looked for a linear functional that satisfies the following requirements:

$$\begin{aligned}
 \alpha \left[\vec{V}_{0+} + b \frac{C_T^{\text{free}}}{C_T^{\text{max}}} \vec{\lambda}_{JJT}(\gamma)^T \cdot \vec{V}_{32+} \cdot \vec{\lambda}_{JJT}(\gamma) \right] &= 1, \quad (\text{normalization}) \\
 \vec{\lambda}_{JJT}(\gamma) &= \sqrt{\frac{3}{16}} \begin{pmatrix} 1 - 12\gamma \\ 5 - 12\gamma \end{pmatrix} \\
 \alpha[\vec{V}_{\Delta+}] &\geq 0, & \Delta &\geq \Delta_0^+, \ell = 0 \\
 \alpha[\vec{V}_{\Delta\ell+}] &\geq 0, & \Delta &\geq \ell + 1, \ell > 2 \text{ even} \\
 \alpha[\vec{V}_{\Delta\ell-}] &\geq 0, & \Delta &\geq \Delta_0^+, \ell = 0 \\
 \alpha[\vec{V}_{\Delta\ell-}] &\geq 0, & \Delta &\geq \ell + 1, \ell \geq 2
 \end{aligned} \tag{5.52}$$

Compared to (5.45) we have modified the normalization condition in order to input a specific value of γ and we have used (5.42). It is straightforward to show that the bound obtained with a functional satisfying (5.52) only applies to CFTs with $C_T \leq C_T^{\text{max}}$.

In figure 5.5, we again show the allowed region in the plane (Δ_0^+, Δ_0^-) but requiring small central charge $C_T \leq C_T^{\text{max}} = C_T^{\text{free}}$ and for several specific values of the parameter γ defined in

(5.3). As expected, this excludes the GFVF which effectively has infinite central charge. More interestingly, one can observe that varying the parameter γ the bounds smoothly interpolates two very different regimes. For $\gamma \simeq 1/12$, the free fermion theory drives the shape of the bound, while as we decrease γ , the allowed region is entirely concentrated at smaller Δ_0^+ but large Δ_0^- . Notice also that the maximum of Δ_0^- is not reached at the free boson theory but at slightly larger values of γ and Δ_0^+ . These results are also shown as a 3D plot in figure 5.6.

In figure 5.7 we show the upper bound on the dimension of the second lightest parity even scalar operator $(\Delta_0^+)'$ as a function of Δ_0^+ . We performed the analysis with and without forbidding relevant scalar parity odd operators. Next, in figure 5.8 we show the bound on the dimension of the first non conserved spin-2 parity even operator $(\Delta_2^+)'$. Notice that $\Delta_2^+ = 3$ because the dimension of the stress tensor is fixed.¹⁴ Interestingly, both bounds display a kink structure in the proximity of the location of the $O(2)$ model. On the other hand both the maximal allowed values of $(\Delta_0^+)'$ and $(\Delta_2^+)'$ at the kink are much larger than the ones of the free $O(2)$ model. It would be surprising if the interacting critical $O(2)$ model displayed such large anomalous dimensions. At this stage, it is unclear if the kink feature is related to the $O(2)$ model. It would be interesting to include correlations functions of charged operators in our bootstrap study to further explore this region. We leave this mixed correlator analysis for the future. Finally, notice that in figure 5.8 the region $\Delta_0^+ \gtrsim 4.52$ is excluded if we also take into account the constraints coming from the four point function of the lightest parity-odd scalar appearing in $J \times J$ (see figure 5.4).

5.3.2 Central Charge bounds

A well established feature of the conformal bootstrap is the possibility to place upper bounds on OPE coefficients, or equivalently a lower bound on C_T [64–66]. In this section we investigate the minimal value of the central charge that a CFT with a continuous local global symmetry is allowed to have, as a function of the parameter γ . To find such a bound, we search for a functional α satisfying the properties:

$$\begin{aligned}
\alpha[\vec{\lambda}_{J\bar{J}T}(\gamma)^T \cdot \vec{V}_{32+} \cdot \vec{\lambda}_{J\bar{J}T}(\gamma)] &= 1, \quad (\text{normalization}) \\
\vec{\lambda}_{J\bar{J}T}(\gamma) &= \sqrt{\frac{3}{16}} \begin{pmatrix} 1 - 12\gamma \\ 5 - 12\gamma \end{pmatrix} \\
\alpha[\vec{V}_{\Delta+}] &\geq 0, & \Delta \geq 1/2, \ell = 0 \\
\alpha[\vec{V}_{\Delta\ell+}] &\geq 0, & \Delta \geq (\Delta_2^+)', \ell = 2 \\
\alpha[\vec{V}_{\Delta\ell+}] &\geq 0, & \Delta \geq \ell + 1, \ell > 2 \text{ even} \\
\alpha[\vec{V}_{\Delta\ell-}] &\geq 0, & \Delta \geq 1/2, \ell = 0 \\
\alpha[\vec{V}_{\Delta\ell-}] &\geq 0, & \Delta \geq \ell + 1, \ell \geq 2
\end{aligned} \tag{5.53}$$

Notice that compared to (5.45) we have eliminated the assumption of the functional α being positive on the identity operator contribution. As shown later, we will instead minimize $\alpha[\vec{V}_{\Delta+}]$. Also, by fixing the normalization we input a specific value of γ . Here $\vec{\lambda}_{J\bar{J}T}(\gamma) =$

¹⁴However, we do not exclude solutions where the OPE coefficient of the stress tensor vanishes ($C_T = \infty$).

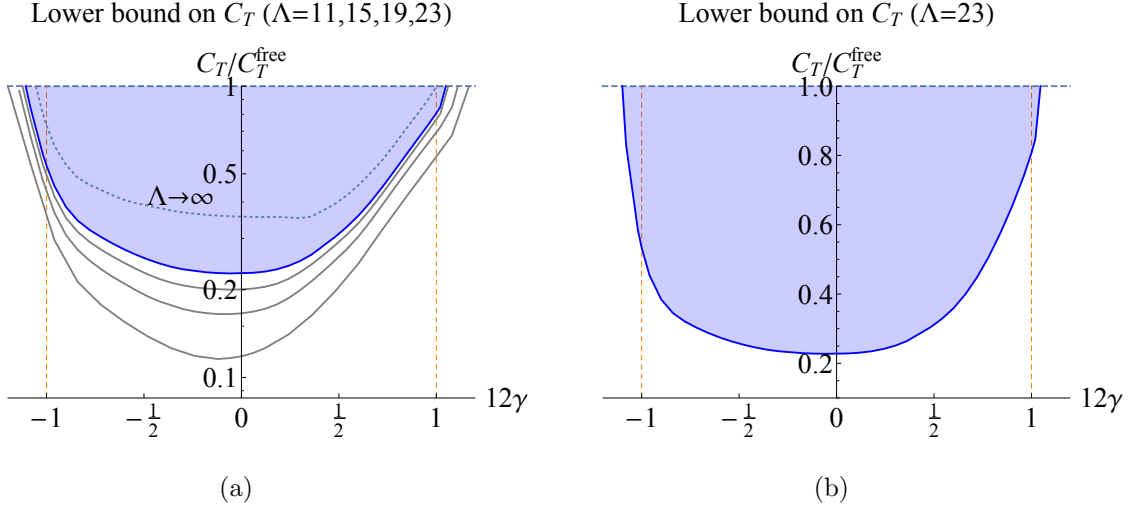


Figure 5.9: a) Zoom in in the region $|12\gamma| \leq 1$ of the lower bound on the central charge normalized to the central charge of a free complex boson as a function of the parameter γ defined in (5.3). The shaded region is allowed. Different curves corresponds to increasing the number of derivatives included in the numerical problem. The bounds have been computed at $\Lambda = 11, 15, 19, 23$. The dashed line corresponds to a linear extrapolation in Λ^{-1} . b) Best bound on central charge in linear scale.

$\sqrt{C_T/C_T^{\text{free}}} \left(\tilde{\lambda}_{JJT}^{(1)}, \tilde{\lambda}_{JJT}^{(2)} \right)$ is a two dimensional vector of OPE coefficients, with each component being a linear function of γ , and we have used (5.42). Finally, we introduced a gap in the spin 2 even sector, and assume that, besides the energy momentum tensor, whose dimension saturates the unitarity bound, all the parity-even spin-2 operators satisfy $[\mathcal{O}_{\ell=2}] \geq (\Delta_2^+)'$. We will come back to this assumption later. Applying the functional to the crossing equations (5.41) and using the results of Sec. 5.2.1.1 one obtains

$$\frac{C_T^{\text{free}}}{C_T} \leq -\alpha[\vec{V}_{0+}]. \quad (5.54)$$

Therefore, the optimal bounds on C_T will be set by the functional minimizing $\alpha[\vec{V}_{\Delta+}]$, subject to the constraints (5.53).

In figure 5.2, presented in the introduction, we show our best bound on the central charge as a function of γ and how the bound improves when increasing the numerical power Λ . As expected, inside the interval $|12\gamma| \leq 1$, the bound seems to converge to a finite value, while outside it improves by a order one factor at each step.

In figure 5.9 we display the zoomed version of the same plot. As discussed in Sec. 5.2.1.1, the two extremes of the interval $12|\gamma| \leq 1$ are saturated by the free complex boson and the free fermion theory. In [67], it was shown that when γ assumes the extremal values, the CFT must necessarily be free, i.e. all the correlators of the CFT must be equal to those of a corresponding (bosonic or fermionic) free theory. One would therefore expect the bound to approach the value of the central charge of a free complex boson or free fermion given in eq. 5.19 at the extremes

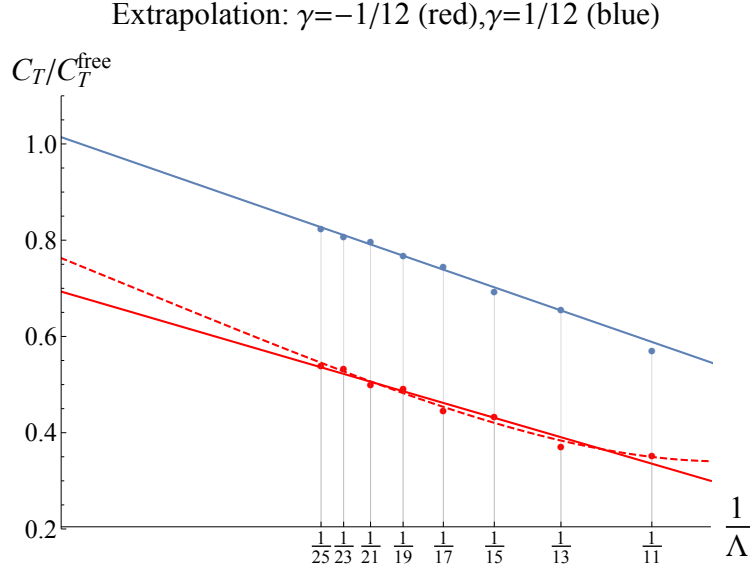


Figure 5.10: Extrapolation of the bound on central charge normalized to C_T^{free} as a function of the number of derivatives included in the semidefinite-programming Λ . The upper (blue) lines corresponds to a linear fit of the bounds at $\gamma = 1/12$. In the limit $\Lambda \rightarrow \infty$ the extrapolation approaches the value 1. The lower curves correspond to a linear (red continuous) and quadratic (red dashed) fit of the bounds at $\gamma = -1/12$. The linear fit predicts an asymptotic bound much smaller than the free theory one. Other fits may predict a value of C_T closer to C_T^{free} . This is exemplified by the quadratic fit.

of the allowed interval. This doesn't appear to be the case with the current numerical power. Nevertheless we might hope to approach the optimal bound in the limit $\Lambda \rightarrow \infty$. In figure 5.10 we show a linear extrapolation of the bounds computed at $\gamma = \pm 1/12$ for $\Lambda = 11, \dots, 25$. For $\gamma = 1/12$ a linear extrapolation (upper blue line in figure 5.10) is consistent with an asymptotic bound $C_T \geq C_T^{\text{free}}$. Extrapolating the bound for $\gamma = -1/12$ is trickier. Although we expect the bound to be $C_T \geq C_T^{\text{free}}$, the linear fit (bottom red line in figure 5.10) clearly gives an asymptotic value smaller than C_T^{free} . Most likely, the linear extrapolation in Λ simply does not capture the infinite number of derivatives limit. It is plausible that the apparent convergence of the bound to a value smaller than C_T^{free} is due to some hypothetical CFT with $C_T < C_T^{\text{free}}$ and γ close to $-1/12$. With the current numerical power we cannot make a conclusive statement confirming or ruling out such a theory.

An interesting feature of figure 5.9 is that the central charge bound is well below C_T^{free} not only near $12|\gamma| = 1$ but in the whole region $12|\gamma| \leq 1$. Based on previous works on conformal bootstrap [62, 63] we are keen to consider this as an indication that there might exist a number of CFTs whose central charge is smaller than the free theory one. A largely accepted lore suggests that the central charge measures the number of degree of freedom in the theory.¹⁵ Accordingly we expect a CFT with the central charge smaller than C_T^{free} to have minimal possible global

¹⁵This is clearly the case for free theories and CFTs that are perturbatively away from a free theory.

symmetry, i.e. only a global $U(1)$. The critical $O(2)$ -model is the only known example of such a theory with $C_T \approx 0.944$. The other possible candidate, the $N = 2$ Gross-Neveu model is in fact expected to have a central charge larger than C_T^{free} (see appendix J.1 for a review). The critical $O(2)$ -model clearly can not explain the current shape of the bound. As the numerics improves, $\Lambda \rightarrow \infty$, we expect the optimal bound to become significantly stronger and be saturated by the hypothetical new theories with $C_T \leq C_T^{\text{free}}$.

Lower bound on C_T when $(\Delta_2^+)' \geq 3, 3.5$ ($\Lambda=23$)

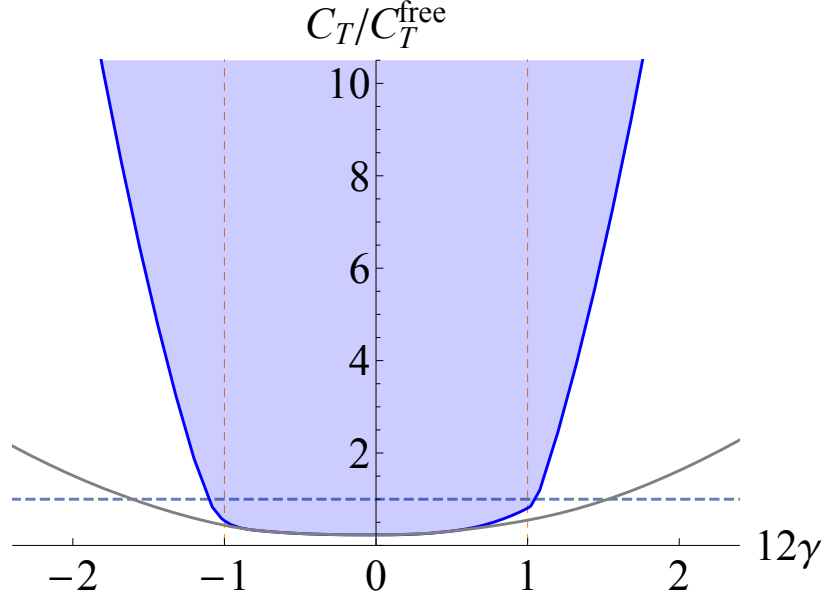


Figure 5.11: Lower bound on the central charge as a function of the parameter γ defined in (5.3). The grey line corresponds to not imposing any gap between the energy momentum tensor and the next spin-2 parity even operator. The blue line shows how a gap $(\Delta_2^+)' = 3.5$ impacts the strength of the bound. While inside the interval $|12\gamma| \leq 1$ the bound is marginally affected, the effect outside the interval is dramatic.

Let us now discuss the role of the gap in the spin-2 parity even sector. The key observation is that the proof of the conformal collider bound (5.13) elegantly obtained in [57] relies on the assumption of the existence of a *single* energy momentum tensor. If instead a CFT possesses several conserved spin-2 operators, the bound (5.13) must be replaced by a bound on a weighted sum over the corresponding γ 's:

$$-\frac{1}{12} \leq \sum_i w_i \gamma_i \leq \frac{1}{12}. \quad (5.55)$$

Unfortunately, in our bootstrap analysis with a finite truncation parameter Λ , any parity even spin-2 operator of dimension close to 3 is almost indistinguishable from another stress-tensor. This is precisely the role played by the gap $(\Delta_2^+)'$ in 5.53: imposing a single energy momentum

tensor corresponds to input a gap strictly larger than 3. In figure 5.11 we show the impact of this gap on the lower bound on the central charge of the theory. As expected, the effect is stronger in the region disallowed by the bound (5.13) because the imposed gap on the spin-2 sector implies uniqueness of the stress tensor. On the other hand, imposing a gap like $(\Delta_2^+)' = 3.5$ probably excludes most CFT's with global symmetry bigger than $U(1)$. For example, consider the OPE of two conserved currents in the $O(3)$ -model:

$$J_\mu^i \times J_\nu^k \supset \delta^{ik} \lambda_{JJT} T_{\mu\nu} + \lambda_{JJO} O_{\mu\nu}^{ik} \quad (5.56)$$

where the spin-2 operator $O_{\mu\nu}^{ik}$ transforms in the symmetric traceless representation of $O(3)$. When we restrict to a unique current, for instance to $i = k = 3$, the operator $O_{\mu\nu}^{33}$ is a singlet of the $U(1)$ generated by J_μ^3 and we expect its dimension to be perturbatively close to the unitarity bound. A similar argument holds for all $O(N > 2)$ models: generically there can be more than one spin-2 operator entering the $J \times J$ OPE, whose anomalous dimension is $1/N$ suppressed. We expect that to properly constraint these theories one has to bootstrap the four-point functions of full set of conserved currents.

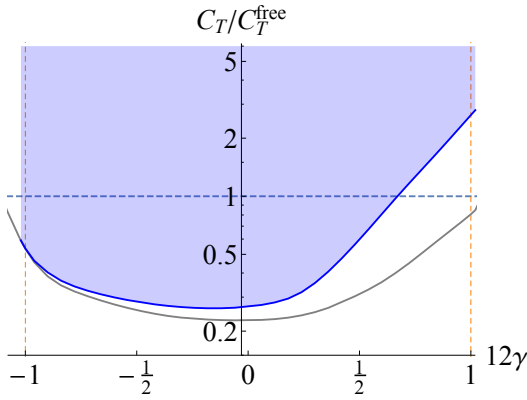
A final comment regarding the comparison between our analysis and the case of bootstrapping the stress-tensor four-point function is in order. Since the 3 point function of three stress tensors is structurally different from the one of two stress tensor and a non conserved spin-2 operator, there is no contribution in the 4 point function that could fake a second energy momentum tensor. As a consequence, the uniqueness of $T_{\mu\nu}$ is automatic and in principle there is no need to impose a gap in the spin-2 even sector.

5.3.3 Central Charge bounds with spectrum assumptions

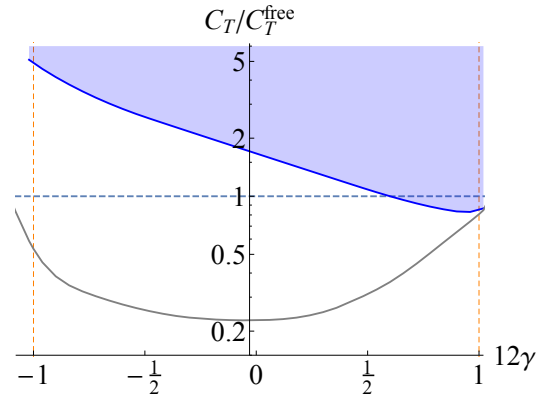
In this section we investigate how the bounds on the central charge change when we introduce additional assumptions on the spectrum of scalar operators or in the spin-4 parity-even sector. We therefore replace the conditions (5.53) with the following conditions

$$\begin{aligned} \alpha[\vec{\lambda}_{JJT}(\gamma)^T \cdot \vec{V}_{32+} \cdot \vec{\lambda}_{JJT}(\gamma)] &= 1, \quad (\text{normalization}) \\ \alpha[\vec{V}_{\Delta+}] &\geq 0, & \Delta &\geq \Delta_0^+, \ell = 0 \\ \alpha[\vec{V}_{\Delta\ell+}] &\geq 0, & \Delta &\geq (\Delta_2^+)', \ell = 2 \\ \alpha[\vec{V}_{\Delta\ell+}] &\geq 0, & \Delta &\geq \Delta_4^+, \ell = 4 \\ \alpha[\vec{V}_{\Delta\ell+}] &\geq 0, & \Delta &\geq \ell + 1, \ell > 4, \ell \text{ even} \\ \alpha[\vec{V}_{\Delta\ell-}] &\geq 0, & \Delta &\geq \Delta_0^-, \ell = 0 \\ \alpha[\vec{V}_{\Delta\ell-}] &\geq 0, & \Delta &\geq \ell + 1, \ell \geq 2 \end{aligned} \quad (5.57)$$

In figure 5.12(a) we show the impact of imposing the absence of relevant odd scalar operators in the $J \times J$ OPE. This amounts to set $\Delta_0^- = 3$ while keeping all the other gaps to their minimal value consistent with unitarity. As expected, the bound on the central charge increases for positive values of γ , excluding the free fermion theory, which is indeed ruled out by this assumption. Close to $\gamma = -1/12$ the bound is almost unaffected, consistent with the conjecture

Lower bound on C_T ($\Delta_0^+ \geq 1/2$, $\Delta_0^- \geq 3$, $\Lambda=23$)

(a)

Lower bound on C_T ($\Delta_0^+ \geq 3$, $\Delta_0^- \geq 1/2$, $\Lambda=23$)

(b)

Figure 5.12: a) Lower bound on the central charge normalized to the central charge of a free complex boson as a function of the parameter γ defined in (5.3) assuming no relevant parity-odd scalars. b) Lower bound on the central charge assuming no relevant parity-even scalars. The shaded regions are excluded. The bounds have been computed at $\Lambda = 23$. The grey lines on both plots correspond to the lower bound on C_T/C_T^{free} without any assumption.

that the left part of the plot is driven by the free boson theory and possibly by the critical $O(2)$ -model. Notice that in this analysis we haven't made any assumption about the parity-even spectrum, and in particular no assumption about the number of relevant parity-even scalars. A second investigation, shown in figure 5.12(b), solely assumes that no relevant parity-even scalar operators are present. The impact of this assumption is more dramatic: very small room is left for theories with $C_T < C_T^{\text{free}}$. Although we haven't performed a careful extrapolation we believe this window will close in the limit of infinite number of derivatives $\Lambda \rightarrow \infty$.

Finally, in figure 5.13 we combine both assumptions to study the central charge limits for the case of *dead-end* CFTs, namely those CFTs without any relevant scalar operator. As the name suggests, these CFTs would be stable under any scalar deformation and therefore would represent an attractive point for all the renormalization group flows driven by rotation-preserving deformations. While we expect such CFTs with a large central charge (from weakly coupled abelian gauge theory in AdS_4), there are no known examples with small values of C_T . Interestingly, at present, our limits do not preclude the existence of dead-end CFTs with $C_T/C_T^{\text{free}} \sim O(1)$.

We now move to exploring the dependence of the central charge bound on the gap in the spin-4 parity-even sector. This can be done by tuning the parameter Δ_4^+ in (5.58) while setting all other gaps to their minimal value consistent with unitarity. The value of the gap Δ_4^+ can be considered as a knob to interpolate from free theories to holographic CFTs. Indeed, the $J \times J$ OPE in free CFTs contains a conserved spin-4 parity even operator. When going to interacting CFTs, its dimension must be lifted [68] and the operator acquires a positive anomalous dimension. On the other hand, in holographic CFTs the lightest spin 4 operator is the “double-trace” operator $\sim J_{(\mu_1} \partial_{\mu_2} \partial_{\mu_3} J_{\mu_4)}$ of dimension 6, with corrections suppressed as $1/N$. As we increase the value of

Lower bound on C_T ($\Delta_0^+ \geq 3$, $\Delta_0^- \geq 3$, $\Lambda=23$)

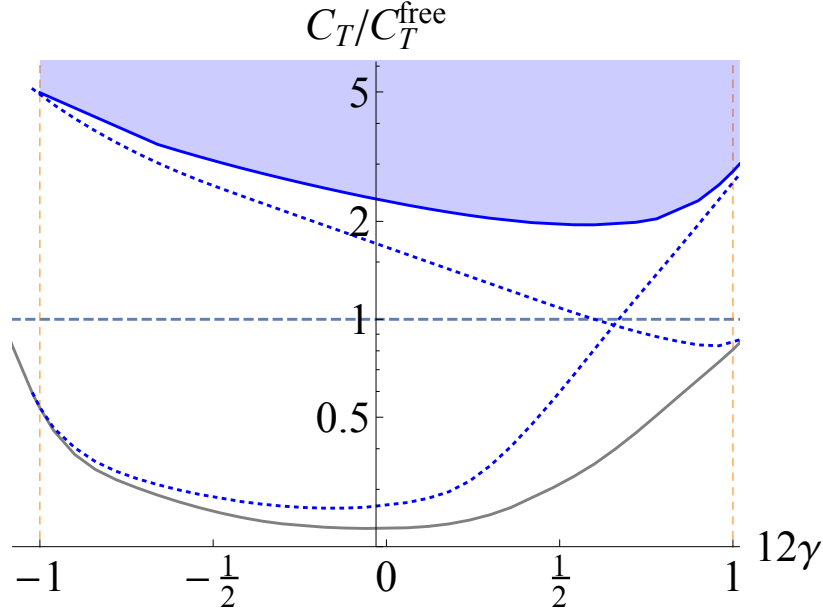


Figure 5.13: Lower bound on the central charge normalized to the central charge of a free complex boson as a function of the parameter γ assuming that both parity-even and parity-odd scalar operators are absent. The shaded region is allowed. The bounds have been computed at $\Lambda = 19$. The dashed blue line corresponds to the bounds shown in figure 5.12(a) and figure 5.12(b). The gray line corresponds to the lower bound without any assumption.

the gap, we exclude more and more theories, and it is natural to expect that the only solution still consistent with crossing symmetry are those which have a large central charge. This behavior is indeed realized in figure 5.14(a), where we show the lower bound on the central charge as a function of γ for several values of Δ_4^+ . As anticipated, the bound grows with the gap. By increasing the numerical power one can presumably make the bound much stronger. In figure 5.14(b) we performed an extrapolation in the number of derivatives of the central charge limit when $\Delta_4^+ = 6$ for the central value $\gamma = 0$. The extrapolation suggests that $\Delta_4^+ = 6$ implies $C_T = \infty$, in agreement with the holographic interpretation.¹⁶

5.3.4 Hunting the $O(2)$ -model

So far we have investigated bounds on the central charge under very general assumptions on the spectrum of CFTs. However, they do not appear to be saturated by any known CFT. The extrapolation in the number of derivatives shown in figure 5.10 suggests that in this limit we can make contact with a known result, namely the free fermion theory. On the other hand, theories such as the $O(2)$ model seem to remain in the bulk of the regions allowed by crossing symmetry.

¹⁶Recall that the anomalous dimension $\Delta_4^+ - 6 \sim 1/C_T$ must be negative due to Nachtmann's theorem [69, 70].

Lower bound on C_T with $\Delta_4^+ \geq 5.25, 5.5, 6$ ($\Lambda=19$)

Extrapolation: $\gamma=0, \Delta_4^+=6$ (Log-Log scale)

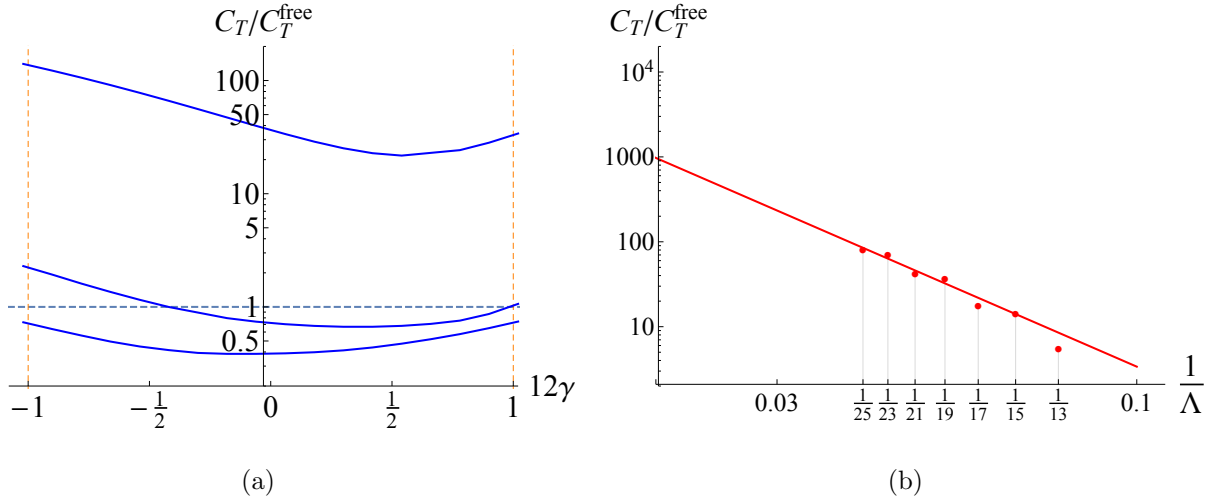


Figure 5.14: a) Central charge bound as a function of γ for several values of the gap in the spin-4 parity even sector, $\Delta_4^+ = 5.25, 5.5, 6$. As we increase the gap, solutions consistent with crossing symmetry must develop a larger central charge. b) Extrapolation of the bound on C_T for $\gamma = 0, \Delta_4^+ = 6$. Both axis are in logarithmic scale. Linear fit shown as a solid red line suggests that the bound on C_T diverges as $\Lambda \rightarrow \infty$.

In order to understand the reason for this it is useful to inspect the solution of crossing along the boundary extracted with the extremal functional¹⁷ method introduced in [71] and successfully used in [63, 72] to extract the spectrum of the three dimensional Ising model. We observe that all the extremal solutions contain odd operators with $\ell \geq 2$ and dimension saturating the unitarity bound $\Delta_\ell^- = \ell + 1$ (or very close to it). On the contrary, all known theories display a larger gap. For instance, free theories and GFVF satisfy $\Delta_\ell^- = \ell + 3$ (see appendix J.1). Basically, the extra gap comes from the need to contract ϵ -tensor indices with derivatives.

It is natural to expect that the $O(2)$ model also displays an extra gap for all parity odd operators with spin $\ell \geq 2$. Hence, in order to make contact with the $O(2)$ model, we replace the conditions (5.53) with the following requirements:

$$\begin{aligned}
 \alpha[\vec{\lambda}_{JJT}(\gamma)^T \cdot \vec{V}_{32+} \cdot \vec{\lambda}_{JJT}(\gamma)] &= 1, \quad (\text{normalization}) \\
 \alpha[\vec{V}_{\Delta+}] &\geq 0, & \Delta &\in [\Delta_\epsilon^{\min}, \Delta_\epsilon^{\max}] (\ell = 0) \\
 \alpha[\vec{V}_{\Delta+}] &\geq 0, & \Delta &\geq \Delta_0^+ (\ell = 0) \\
 \alpha[\vec{V}_{\Delta\ell+}] &\geq 0, & \Delta &\geq \Delta_2^+, \ell = 2 \\
 \alpha[\vec{V}_{\Delta\ell+}] &\geq 0, & \Delta &\geq \ell + 1, \ell > 4, \ell \text{ even} \\
 \alpha[\vec{V}_{\Delta\ell-}] &\geq 0, & \Delta &\geq \Delta_0^-, \ell = 0 \\
 \alpha[\vec{V}_{\Delta\ell-}] &\geq 0, & \Delta &\geq \ell + \tau_{\text{all}}^-, \ell \geq 2
 \end{aligned} \tag{5.58}$$

The novelty in the above conditions consists in raising the twist of all parity odd operators to

¹⁷We remind the reader that on the boundary of the allowed region the solution of the truncated crossing equation is unique and it is given by the zeros of the linear functional α .

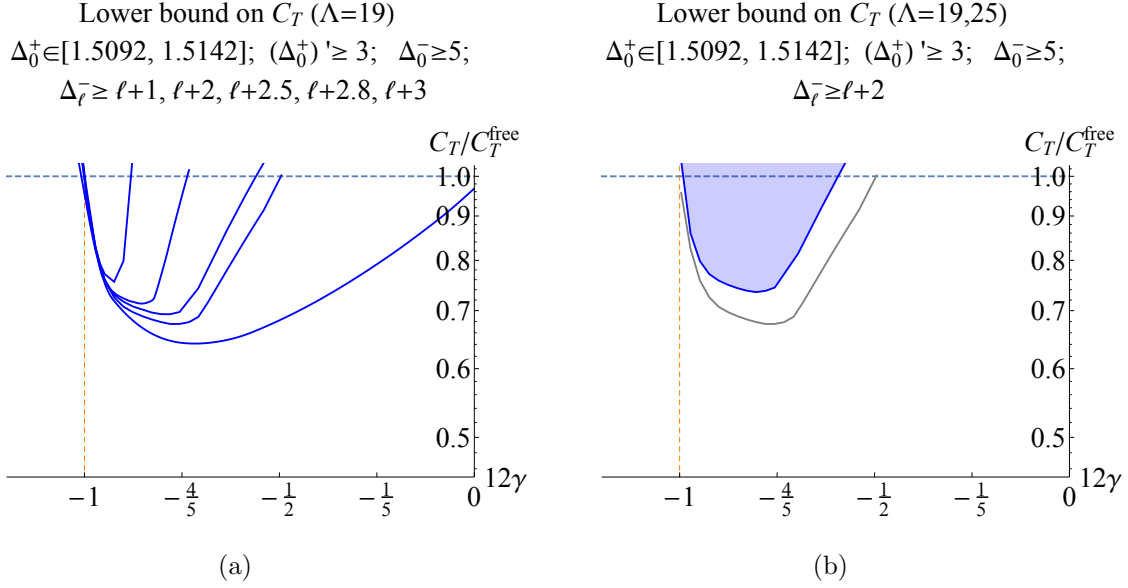


Figure 5.15: a) Lower bound on the central charge normalized to the central charge of a free complex boson as a function of the parameter γ . Different lines corresponds to increasing values of twist of the parity odd operators of spin $\ell \geq 2$: $\tau_{\text{all}}^- = 1, 2, 2.5, 2.8, 3$. b) Lower bounds on the central charge with $\tau_{\text{all}}^- = 2$ for increasing numerical power: $\Lambda = 19, 23$.

$\tau_{\text{all}}^- \geq 1$, and imposing that relevant parity even scalar must be confined in a narrow interval $\Delta \in [\Delta_\epsilon^{\min}, \Delta_\epsilon^{\max}] = [1.5092, 1.5142]$, for which we take the rigorous bound from previous bootstrap studies [73]. In figure 5.15(a) we show the impact of varying τ_{all}^- from 1 to 3. Interestingly the bounds start developing more and more pronounced minima as we increase the value of τ_{all}^- . In addition, the left part of the bound is insensitive to this parameter, while the right part heavily depends on it. Although from figure 5.15(a) it would be tempting to set $\tau_{\text{all}}^- = 3$, the large spin analysis discussed below suggests that this is not possible. Nevertheless we expect that $\tau_{\text{all}}^- = 2$ is a safe assumption. With this choice in (5.58), we can obtain a rigorous bound on the parameter γ for theories with the central charge smaller than the free theory one:

$$\gamma \in [-0.0824, -0.0494] \quad \text{rigorous (assuming 5.58, } \tau_{\text{all}}^- = 2) \quad (5.59)$$

The above interval has been computed at $\Lambda = 25$, however, as shown in figure 5.15(b), the bounds are still not converged. Using a linear extrapolation we estimate a bound

$$\gamma \in [-0.081(1), -0.060(1)] \quad \text{extrapolation (assuming 5.58, } \tau_{\text{all}}^- = 2) \quad (5.60)$$

Let us comment on the consistency of our assumption that the dimension of the leading twist parity odd operators of spin $\ell \geq 2$ in the $O(2)$ model is not too far from 3, which is the free theory value.¹⁸ The leading correction to the dimension of these operators in the large spin

¹⁸Schematically, these operators have the form $\mathcal{O}_{\mu_1 \dots \mu_\ell}^- \sim \epsilon_{\nu \rho (\mu_1} J^\nu \partial_{\mu_2} \dots \partial_{\mu_\ell}) J^\rho$.

expansion has been computed in [70] using analytic bootstrap techniques. It was found that:

$$\delta_{\mathcal{O}_\ell^-} = \Delta_{\mathcal{O}_\ell^-} - \ell - 3 = -\frac{12(1 - 144\gamma^2)}{C_T\pi^4\ell} + \dots \quad (5.61)$$

Notice that the leading correction in the above formula is negative whenever γ satisfies the conformal collider bound. Moreover, our estimate $\gamma_{O(2)} \sim -1/12$ is compatible with the assumption of small anomalous dimension $\delta_{\mathcal{O}_\ell^-}$.

Chapter 6

Conclusions

Conformal field theories are very symmetric theories which are relevant in many physical applications. Their amount of symmetry allows one to write very constraining self-consistency equations which are expected to determine the full landscape of allowed theories. A very efficient way to do this is through the requirement that the four point functions of local operators are crossing symmetric and decomposable in terms of some special functions which are called conformal blocks. These functions are completely fixed by kinematics and encode the contribution of a conformal multiplet to a four point function. The above requirements, combined with unitarity, can be analyzed numerically. This is the so called numerical bootstrap program. Recently these ideas were used in many cases in order to obtain non perturbative results on the spectrum and OPE coefficients of strongly coupled CFTs. One of the most remarkable applications is the determination of the scaling dimensions of the $3d$ Ising model at a level of accuracy which is two orders of magnitude better than the best Montecarlo results.

So far the bootstrap strategy was only implemented for four point functions of scalar operators (beside the case of external fermion in three dimension). Nonetheless it is very natural to consider four point functions of operators which have protected dimensions as conserved currents and stress tensors. This was not done yet, because of the complication of the bootstrap equations and because the conformal blocks for operators with spin were in general not known. In this thesis we obtained two systematic ways to compute conformal blocks for operators with spin in any spacetime dimension. The first one is an expansion in the radial coordinates of [38], which is straightforward to implement. The second method is a more sophisticated recurrence relation which is based on the analytic properties of the conformal blocks as functions of Δ . This is less trivial to implement, however it is extremely efficient. Using this last method we computed the conformal block to implement the numerical bootstrap for 4 equal conserved currents in $d = 3$.

In section 3 we generalized the series expansion in the radial coordinate introduced by [38] for general CBs in any spacetime dimension. To do so we defined an expansion which inherits the structure of the conformal representations exchanged in the CB. We considered the CB as a projection of a four-point function obtained by summing over a complete basis of a conformal representation. We organized the sum by grouping together all the descendant operators in the same irreducible representations $\mathbf{Y}$ of $SO(d)$. Each state gives a contribution proportional to

r^E where E is the eigenvalue of the cylinder Hamiltonian. Therefore, we naturally obtained the CB in an expansion in radial coordinates. We implemented the method in order to build CBs when the external operators are in the traceless and symmetric representation, but one could generalize it for external fermionic operators or for mixed symmetry external operators [32]. We exemplified the method in four cases, for the scalar CB, for one external vector and three scalars, two vectors and two scalars and finally for two spin two operators and two scalars.

We explain how to write closed form recurrence relations for the coefficients of the expansion of the functions $W_s(r, \eta)$ that define the CB. These can be obtained by following an algorithm, which we give. One of the nice features of these recurrence relations is that they are analytical results, therefore they can be used to obtain the conformal blocks in some limiting cases of the parameter space. For example one can consider explicitly the limits of large Δ , l or h .

The main advantage of this method is that it can be implemented for any conformal block in a very systematic way. However it may not be efficient enough to build very complicated conformal blocks (for example the ones of four conserved currents and four stress tensors). For this reason it would be very important to find a way of implementing the conservation of external operators in order to decrease the number of functions W_s to compute. If one could define a single function W_s for each conserved CB, it would be then possible to use this method in order to obtain potentially all the conformal blocks for conserved currents, stress tensors and scalars in any dimension.

In section 4 we introduced an alternative strategy to build the CBs from their radial expansion. The main idea of this method is to study the behavior of CBs in the complex Δ plane [15, 74]. From the analysis of section 4.5 we obtained that the CBs are meromorphic functions of Δ (in odd dimensions d , but it is possible to analytically continue the method to any non even d), and that the residues at all the poles can always be written as linear combination of other CBs labeled by the position of primary descendants which can appear in the conformal representation. This fact naturally leads to recursion relations. All the poles and the CBs at each residue are known in general while the specific linear combination has to be computed case by case following a systematized algorithm. The regular part of the CBs needs also to be computed case by case and it can be thought as the initial condition for the recursion relation. We showed that it is possible to implement the recurrence relations in a very efficient way. In fact we were able to obtain the conformal blocks for equal vectors in three dimensions to a very high accuracy. We also showed that conservation can be conveniently imposed in order to simplify the recurrence relations. A natural future direction is to apply this strategy to obtain the conformal blocks for conserved currents in generic spacetime dimensions. Of course a ultimate goal would be to have explicit formulae to compute all the conformal blocks for conserved tensor operator with spin one and two. Hopefully this method will be of a great help in this direction.

Finally in chapter 5 we implemented the first example of numerical bootstrap for the four point function of identical conserved currents by using the conformal blocks computed in section 4. The aim was to study the space of parity-even three dimensional conformal field theories with (at least) a global $U(1)$ symmetry. We have shown that, analogously to the case of the correlation

function of 4 scalars or 4 fermions, unitarity and OPE associativity alone let us carve out the parameter space of CFTs. Inspecting the allowed values of scalar operator dimensions we found that any CFT with a conserved spin-1 current must contain both parity even and parity odd scalars. The boundary of the allowed region displays a non trivial structure with multiple features. In particular a kink appears close to the location of the $O(2)$ model, providing an upper bound on the dimension of the first parity odd scalar $\Delta_0^- \leq 7.65(1)$. A similar kink is present in the bound on the second spin-2 parity even operator. Also, we excluded the existence of dead-end CFTs with central charge smaller than twice the central charge of a free 3d Dirac fermion. We also explored bounds on the central charge with several assumptions on the CFT spectrum. In this case we observed a slower numerical convergence. Nevertheless we found clear evidence of the conformal collider bounds for spin-1 currents [53–56].

The present work paves the way to many generalizations and extensions. Given the special role that the $O(2)$ model seems to play in our exclusion bounds, it is natural to expect that a mixed scalar-current bootstrap analysis will allow to precisely determine the spectrum of the theory [75]. Similarly, one could consider multiple correlators including external fermionic charged operators in order to narrow down the location of the $N = 2$ Gross-Neveu model.

As mentioned several times, the results of this work are very general and apply to CFTs with a continuous global symmetry that admits a local conserved current.¹ On the other hand, by studying a single current inside a larger symmetry, we loose the ability to distinguish operators that are singlets under the entire global symmetry group from those that instead are only invariant under the specific $U(1)$ considered. In this case, it will be important to bootstrap the full set of correlation functions $\langle J_{\mu_1}^{a_1} J_{\mu_2}^{a_2} J_{\mu_3}^{a_3} J_{\mu_4}^{a_4} \rangle$, with a_i spanning all the generators of the global symmetry. This set up would also allow to specify the global symmetry by inputting the group structure constants f^{abc} and to put a bound on the current central charge C_J . The analysis will require a minor modification of the present framework. All necessary conformal blocks required for this analysis have been already computed in the current work. The main difference will be represented by the higher number of crossing conditions.

Throughout this work we considered parity-invariant CFTs; as a consequence, the present analysis does not apply to many Chern-Simons-matter theories. In order to include parity breaking effects one would need to extend our analysis in several ways. Because operators would not be classified according to their parity, all three point functions will have both parity even and parity odd tensor structures. Thus, new conformal blocks should be computed. The method explained in section 4 allows to systematically perform this computation. In addition, the four point function will admit parity odd tensor structures as well. This will modify the form of conservation equations and crossing equations.

Finally, the same investigation presented in this work can be extended to higher dimensions with minor modifications. The recurrence relation of section 4 could be generalized in order to build conformal blocks in any dimension. Alternatively, the fundamental results obtained in [76–78] allows us to compute the conformal blocks in four dimensions in closed form. Moreover, the

¹A trivial example of a theory with a global symmetry but no conserved current is a free complex field in AdS_{d+1} with mass strictly larger than $-(d^2 - 4)/4$, which is the dual of a Generalized Free Field in d spacetime-dimensions with scaling dimension $\Delta > (d - 2)/2$.

analysis of the crossing equations in section 5.2.2 is valid in any spacetime dimension.

An important goal for the field of numerical bootstrap is to include all the crossing symmetry constraints given by operators with protected dimension like the stress tensor and the conserved currents associated to possible global symmetries. We believe that the addition of such constraints will be a very important ingredient in order to obtain a better classification of the landscape of CFTs. To reach this goal we think that the work summarized in this thesis will be very useful, both to compute conformal blocks and to write minimal crossing equations for conserved tensor operators. In fact this work gives in principle a general algorithm which could be extended to study any set of crossing equations. However in practice with the present technology it would be very hard both to compute some of the required conformal blocks and to obtain satisfying numerical results using the current semidefinite programs. Therefore a great effort is still needed in order to put together all these pieces. Nonetheless we believe that the payoff will be worth it.

Another interesting challenge for the numerical bootstrap is to extend the technology to cases without positivity (there are already some proposals to do so, but they are either not very systematic [14] or not extensively tested [79]). This would open a brand new world of opportunities for the application of the numerical bootstrap. In particular it would be very important for the study of non unitary theories and defect conformal field theories. For the latter ones, we found in [80] a systematic way to build all the possible conformal blocks. Remarkably, in this case the functions are much more under control than in the case without defects. We therefore believe that important achievements will be obtained in this area in the next future.

Appendix A

Differential equations in Gegenbauer basis

In the main text we explain how to trade the differential equations for the functions W_s , which come from the action of the conformal Casimir, with algebraic relations for the coefficients w_s of the expansion of the functions W_s . However, we did not stress that for one differential equation one can in principle write infinite algebraic relations to which would correspond different “dot diagrams” like the ones depicted in figures 3.4(a) and 3.4(b). This is not surprising since given a relation as for example (3.23), it is always possible to sum to it the same relation with shifted m and j . The result can still be used as a valid recurrence relation for the coefficients, but it would in general contain more terms and therefore it would be less efficient. The aim of this appendix is to explain the main manipulations which we used to simplify the algebraic relation for the w_s .

The first caveat is that in general, instead of the last relation of (3.22), it is better to use

$$\begin{aligned} \partial_\eta^n \mathfrak{f}_{m,l}(r, \eta) = \frac{1}{\eta^2 - 1} & \left[(5 - 2h - 2n) \eta \partial_\eta^{n-1} \mathfrak{f}_{m,l}(r, \eta) \right. \\ & \left. + (l - n + 2)(2h + l + n - 4) \partial_\eta^{n-2} \mathfrak{f}_{m,l}(r, \eta) \right], \quad (n > 1) \end{aligned} \quad (\text{A.1})$$

to express any derivative in η of order higher than one. Formula (A.1) comes from the usual Gegenbauer differential equation, once we derived it $n - 2$ times. This is enough to obtain formula (3.23) for the scalar conformal block.

For more general cases the procedure is more complicated. In fact, in the scalar case the Casimir equation gives just the single differential equation (3.18), while for any other case we obtain a system of coupled differential equations that we schematically write as $\text{PDE}_s = 0$ for the set of functions W_s , where the label s goes from 1 to the total number of tensor structures in the four-point function. Notice that each equation $\text{PDE}_s = 0$ arises by collecting the terms multiplying the tensor structure $Q^{(s)}$ in the Casimir equation. However, the choice of the basis $Q^{(s)}$ is not unique.

For example, in the case of one external vector the Casimir equation gives rise to a system of two coupled differential equations for the two functions W_1 and W_2 , that we schematically write

as

$$\text{PDE}_1(W_1, W_2) = 0, \quad \text{PDE}_2(W_1, W_2) = 0. \quad (\text{A.2})$$

We could write the differential equations PDE_s explicitly, but the expressions are very lengthy and they do not provide any deep insight. We want however to stress that both PDE_1 and PDE_2 can be written as a combination of the terms

$$\{W_1, \partial_\eta W_1, \partial_\eta W_2, \partial_\eta^2 W_1, \partial_\eta^2 W_2, \partial_\eta^3 W_2, \partial_r W_1, \partial_r \partial_\eta W_2, \partial_r^2 W_1, \partial_r^2 \partial_\eta W_2\}, \quad (\text{A.3})$$

with appropriate non zero coefficients depending on r and η , and on the parameters $h, \Delta, l, \Delta_{12}$ and Δ_{34} . As a first step, it is natural to try to simplify the system. To do so our criteria was to find new differential equations which involve less terms with high order derivatives. For example, if we ask for which coefficients $a(r, \eta)$ the combination $\text{PDE}_1 + a(r, \eta)\text{PDE}_2$ can be written just in terms of

$$\{W_1, \partial_\eta W_1, \partial_\eta W_2, \partial_\eta^2 W_1, \partial_r W_1, \partial_r^2 W_1\} \quad (\text{A.4})$$

we obtain that $a(r, \eta) = -\eta$. Similarly, if we want to use the set

$$\{W_1, \partial_\eta W_2, \partial_\eta^2 W_2, \partial_\eta^3 W_2, \partial_r \partial_\eta W_2, \partial_r^2 \partial_\eta W_2\} \quad (\text{A.5})$$

we find the condition $a(r, \eta) = (r^2 + 1)/(2r)$. Therefore we obtain a simpler system of equation by choosing

$$\text{PDE}_1 - \eta \text{PDE}_2 = 0, \quad 2r \text{PDE}_1 - (r^2 + 1) \text{PDE}_2 = 0. \quad (\text{A.6})$$

Applying this simplification we obtain two coupled algebraic relations AR_s for the coefficients w_s . The first of which is actually the one defined in (3.45) with $s = 1$. The second one can instead be schematically written as

$$\text{AR}_2(m, j) \equiv \sum_{s=1}^2 \sum_{(\hat{m}, \hat{j}) \in \tilde{\mathcal{S}}_s} \tilde{c}_s(\hat{m}, \hat{j}) w_s(m + \hat{m}, j + \hat{j}) = 0, \quad (\text{A.7})$$

where the coefficients $\tilde{c}_s$ are in general different from the ones of (3.45) and $\tilde{\mathcal{S}}_s$ is represented in figure A.1. Equation (A.7) can still be used to find the coefficients of the expansions of W_1 and W_2 even if there are two points with maximal $\hat{m}$. In fact, one can for example choose to express $w_s(m, j)$ with maximal m and lower j in terms of the other coefficients. But this is not enough, one also needs to assume that the expansion of W_s is of the form of figure 3.2. Therefore, at each level m one should start by expressing $w(m, j)$ starting from $j = l + m$ and decreasing j until $\max(0, l - m)$, and imposing that the coefficients with $j \notin [\max(0, l - m), l + m]$ vanish. However, one can convince oneself that the expansion of figure 3.2 implies that figure A.1 is not minimal, and that there should exist a minimal relation which has just a single dot with maximal $\hat{m}$. A relation with this feature can actually be obtained as follows. Let us first define

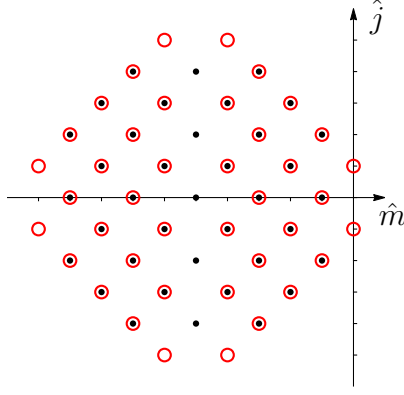


Figure A.1: The set $\tilde{\mathcal{S}}_s$ appearing in the linear combination (A.7).

a new set of algebraic relations indexed by an integer n ,

$$\mathbf{ar}^{(n)}(m, j) = \sum_{k=0}^n \mathbf{AR}_2(m, j - 1 - 2k). \quad (\text{A.8})$$

When $n = 0$, the “dot diagram” of $\mathbf{ar}^{(0)}(m, j)$, would simply be a shifted version of figure (A.1). However, increasing the value of n , we found (as a result of a computation) that in the “dot diagram” of $\mathbf{ar}^{(n)}(m, j)$ two distinct shapes of points appear and disentangle one from the other, as pictured in figure A.2. In figure A.2 we considered values of n up to $n = 6$, but increasing n further it is possible to send the lower shape of points at arbitrary large and negative values of $\hat{j}$. When we use $\mathbf{ar}^{(n)}(m, j)$ with large enough n as a recurrence relation, we can simply forget about the lower shape of points since they are just combinations of coefficients $w_s(m, j)$ with $j < 0$, which are zero. Since the coefficients of the higher shape do not change by increasing n , we can just consider the relation $\mathbf{ar}^{(n)}(m, j)$ for any n with $n \geq 4$ (when the two shapes are really disentangled) and set the lower shape to zero. In this way we obtain the algebraic relation (3.45), which only has one dot with maximal $\hat{m}$.

When we consider two external vectors more complicated structures can arise but it is enough to use the ideas of this section to simplify the recurrence relations.

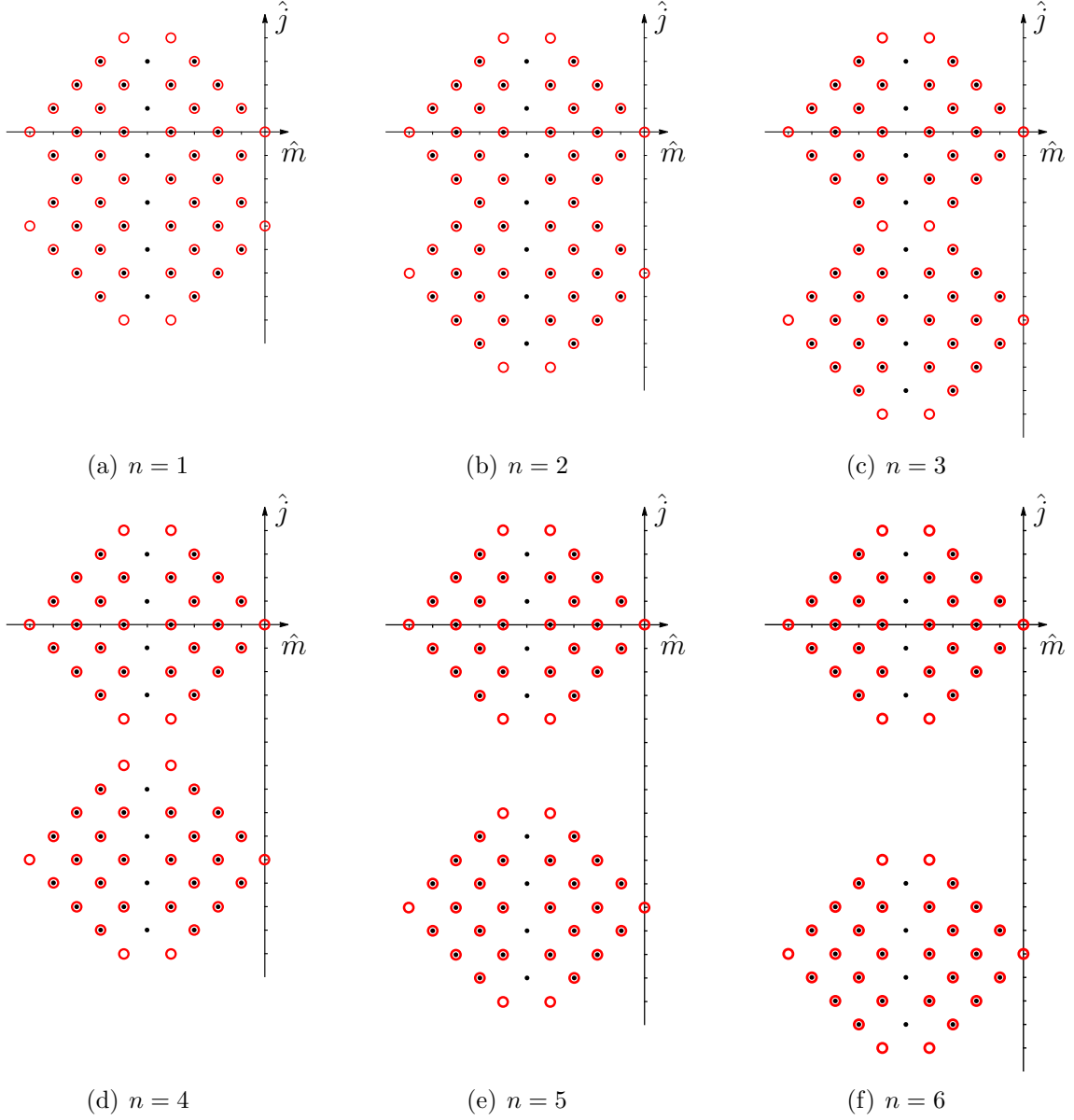


Figure A.2: The algebraic relation $\text{ar}^{(n)}(m, j)$. Increasing the value of n it is possible to disentangle two relations.

Appendix B

Primary Descendant States

In this appendix we aim to demonstrate that a descendant $\mathcal{O}_A$ of a primary $\mathcal{O}$ becomes a primary when $\Delta = \Delta_A^*$. Moreover we provide more details on the computation of the norm of $|\mathcal{O}_A\rangle$.

B.1 Type I

We first want to demonstrate that a descendant of type I becomes a primary when $\Delta = \Delta_{\text{I},n}^*$, namely

$$(K \cdot z')(P \cdot z)^n |\Delta, l; z\rangle = 0, \quad \text{if } \Delta = \Delta_{\text{I},n}^* \equiv 1 - l - n. \quad (\text{B.1})$$

Commuting $(K \cdot z')$ through all the $(P \cdot z)$ we obtain

$$\begin{aligned} [(K \cdot z'), (P \cdot z)^n] &= \sum_{j=1}^n (P \cdot z)^{n-j} [(K \cdot z'), (P \cdot z)] (P \cdot z)^{j-1} \\ &= 2i \sum_{j=1}^n (P \cdot z)^{n-j} [(z \cdot z')D - (z' \cdot J \cdot z)] (P \cdot z)^{j-1} \\ &= 2i \sum_{j=1}^n (P \cdot z)^{n-1} [(z \cdot z')(D + 2i(j-1)) - (z' \cdot J \cdot z)] \\ &= 2i n (P \cdot z)^{n-1} [(z \cdot z')(D + i(n-1)) - (z' \cdot J \cdot z)]. \end{aligned} \quad (\text{B.2})$$

Acting with (B.2) on a primary state we get

$$(K \cdot z')(P \cdot z)^n |\Delta, l; z\rangle = -2n(\Delta + n + l - 1)(z \cdot z')(P \cdot z)^{n-1} |\Delta, l; z\rangle, \quad (\text{B.3})$$

which proves (B.1).

We can then compute the norm $N_{\text{I},n}$ defined by

$$N_{\text{I},n} \equiv \frac{\langle \Delta + n, l - n; z' | \Delta + n, l - n; z \rangle}{(z \cdot z')^{n+l}} = \frac{\langle \Delta, l; z' | (K \cdot z')^n (P \cdot z)^n | \Delta, l; z \rangle}{(z \cdot z')^{n+l}}. \quad (\text{B.4})$$

Using (B.3) it is straightforward to obtain the recurrence relation

$$N_{I,n} = -2n(\Delta + n + l - 1)N_{I,n-1} . \quad (\text{B.5})$$

Iterating equation (B.5) up to $N_{I,0} = 1$, we obtain

$$N_{I,n} = (-2)^n n! (\Delta + l)_n . \quad (\text{B.6})$$

B.2 Type II

We now wish to demonstrate that a descendant of type II becomes a primary when $\Delta = \Delta_{II,n}^*$,

$$(K \cdot z')(D_z \cdot P)^n |\Delta, l; z\rangle = 0 , \quad \text{if } \Delta = \Delta_{II,n}^* \equiv l + d - 1 - n . \quad (\text{B.7})$$

Using (B.2) and replacing z with D_z , we obtain the following commutator,

$$[(K \cdot z'), (P \cdot D_z)^n] = 2i n (P \cdot D_z)^{n-1} [(z' \cdot D_z)(D + i(n-1)) - (z' \cdot J \cdot D_z)] . \quad (\text{B.8})$$

The eigenvalue of $(z' \cdot J \cdot D_z)$ on a primary state is

$$(z' \cdot J \cdot D_z) |\Delta, l; z\rangle = i(l - 2 + d)(z' \cdot D_z) |\Delta, l; z\rangle . \quad (\text{B.9})$$

Therefore when we act with (B.8) on $|\Delta, l; z\rangle$ we find

$$(K \cdot z')(D_z \cdot P)^n |\Delta, l; z\rangle = -2n(\Delta - l - d + 1 + n)(z' \cdot D_z)(P \cdot D_z)^{n-1} |\Delta, l; z\rangle , \quad (\text{B.10})$$

which proves (B.7).

The norm $N_{II,n}$ can be defined as

$$N_{II,n} \equiv \frac{\langle \Delta + n, l + n; z' | \Delta + n, l + n; z \rangle}{(z \cdot z')^{l-n}} = \frac{\langle \Delta, l; z' | \frac{(K \cdot \overleftarrow{D}_z)^n}{(2-h-l)_n (-l)_n} \frac{(P \cdot \vec{D}_z)^n}{(2-h-l)_n (-l)_n} | \Delta, l; z \rangle}{(z \cdot z')^{l-n}} . \quad (\text{B.11})$$

where $\overleftarrow{D}_z$ and $\vec{D}_z$ act respectively on the left and on the right. Using the following identity

$$\frac{(D_z \cdot D_{z'})^n}{(2-h-l)_n^2 (-l)_n^2} (z \cdot z')^l = \frac{(3-d-l)_n}{(-l)_n} \frac{(d+2l-2)}{(d+2l-2n-2)} (z \cdot z')^{l-n} , \quad (\text{B.12})$$

and applying recursively (B.10) (with $z' \rightarrow \overleftarrow{D}_{z'}$) as we did for the norm of type I, we obtain

$$N_{II,n} = (-2)^n n! (\Delta - d - l + 2)_n \frac{(3-d-l)_n}{(-l)_n} \frac{(d+2l-2)}{(d+2l-2n-2)} . \quad (\text{B.13})$$

B.3 Type III

The descendants of type III are more complicated than the ones of the previous two cases. In fact at level $2n$, there are $1 + \min(l, n)$ multiplets of spin l , which can be written as

$$(P^2)^{n-j} (P \cdot z)^j (P \cdot D_z)^j |\Delta, l; z\rangle, \quad (\text{B.14})$$

where $j = 0, 1, \dots, \min(l, n)$. To find which is the correct linear combination of the states (B.14), we impose

$$(K \cdot z') |\Delta + 2n, l; z\rangle = 0, \quad \text{if } \Delta = \Delta_{\text{III}, n}^* \equiv h - n. \quad (\text{B.15})$$

From numerical experiments, we conjecture the following form

$$|\Delta + 2n, l; z\rangle = \mathcal{D}_{\text{III}, n} |\Delta, l; z\rangle \equiv \sum_{j=0}^{\min(l, n)} a(j) (P^2)^{n-j} (P \cdot z)^j (P \cdot D_z)^j |\Delta, l; z\rangle \quad (\text{B.16})$$

with coefficients

$$a(j) \equiv \frac{l!n!}{j!(l-j)!(n-j)!} \frac{(-2)^j}{(h+l+n-j-1)_j} \frac{1}{(2-h-l)_j(-l)_j}. \quad (\text{B.17})$$

Formula (B.16) can be also written as (4.18), which is given in the main text.

We would like to compute the norm $N_{\text{III}, n}$ defined by

$$\langle \Delta + 2n, l; z' | \Delta + 2n, l; z \rangle = N_{\text{III}, n} (z \cdot z')^l. \quad (\text{B.18})$$

We did not find a closed formula. However when $l = 0$, the only multiplet is $(P^2)^n |\Delta, 0\rangle$, therefore we can compute the norm

$$N_{\text{III}, n} \big|_{l=0} = 16^n n! (h)_n (\Delta)_n (\Delta - h + 1)_n. \quad (\text{B.19})$$

Moreover by inspection we found, for example,

n	$N_{\text{III},n}$
1	$\frac{16(h-1)(\Delta-h+1)(h+l)}{(h+l-2)(h+l-1)^2} [(\Delta-h+1)((h+l-1)^2-(h-1)) + (h-2)(h+l-1)^2] ,$
2	$\frac{512(h-1)h(\Delta-h+2)(\Delta-h+1)(h+l+1)}{(h+l-3)(h+l-2)(h+l-1)^2(h+l)} [(\Delta-1)\Delta(h+l-2)(h+l-1)^2(h+l)$ $+2h(\Delta-h+2)((h-1)(\Delta-h+1)-(\Delta-1)(h+l-1)^2)] ,$
3	$-\frac{24576(h-1)h(h+1)(-\Delta+h-3)(-\Delta+h-2)(-\Delta+h-1)(h+l+2)}{(h+l-4)(h+l-3)(h+l-2)(h+l-1)^2(h+l)(h+l+1)} [\Delta(\Delta^2-1)l^6 + 2(\Delta-1)\times$ $\times(h-1)l(-6\Delta(\Delta+2)+3(\Delta+1)(\Delta+2)h^4-6(\Delta+2)(3\Delta+2)h^3+(5\Delta(4\Delta+7)+6)h^2+(\Delta(23\Delta+68)+36)h)$ $+4(\Delta-1)\Delta(h-1)l^3(h(-13\Delta+(5\Delta+8)h-16)-3(\Delta+3))+\Delta(\Delta+1)(\Delta+2)(h-4)(h-3)(h-2)(h-1)h(h+1)$ $+(\Delta-1)l^2(\Delta^2(3h(h(5h-26)+28)+9)-26)+\Delta(3h((h-1)h(11h-37)+36)-56)+6(h-3)(h-2)h(h+1))$ $+6\Delta(\Delta^2-1)(h-1)l^5+(\Delta-1)\Delta l^4(7\Delta+3h(-11\Delta+(5\Delta+6)h-12)+1)] ,$

Motivated by these results, we came to the conjectured result

$$\frac{1}{N_{\text{III},n}} \approx -\frac{n}{16^n(n!)^2(h)_n(1-h)_n} \frac{(h+l-n-1)(h+n-1)}{(h+l+n-1)(h-n-1)} \frac{1}{\Delta-h+n} . \quad (\text{B.20})$$

It would be nice to prove this result analytically.

Appendix C

Computation of M_A

In this section we explain one possible way to compute the coefficients M_A which are needed to determine $R_A = M_A^{(L)} Q_A M_A^{(R)}$. In particular we will obtain M_A defined in (4.26) for the scalar-scalar OPE, and in (4.53) for the vector-scalar OPE.

C.1 Scalar-scalar OPE

We first compute M_A defined in (4.26) by

$$\mathcal{D}_A \frac{(-x \cdot z)^l}{(x^2)^\alpha} = M_A \frac{(-x \cdot z)^{l_A}}{(x^2)^{\alpha_A}}, \quad (\text{C.1})$$

where $\alpha = \frac{\Delta + \Delta_{12} + l}{2}$ and $\alpha_A = \frac{\Delta + n_A + \Delta_{12} + l_A}{2}$. Throughout this appendix we will adopt the notation

$$A = T, n, \quad \begin{cases} T &= \text{I, II, III} \\ n &= 1, 2, \dots \end{cases}. \quad (\text{C.2})$$

The method we use to compute M_A relies on the observation that the differential operators $\mathcal{D}_{T,n}$ can be expressed as $(\mathcal{D}_{T,1})^n$ (this statement is exact only for $T = \text{I, II}$, while for $T = \text{III}$ one has to slightly correct it). Hence roughly speaking, it is sufficient to act with the differential operator $\mathcal{D}_{T,1}$ on $\frac{(-x \cdot z)^l}{(x^2)^\alpha}$ only once, and then “multiply” n times the resulting coefficient.

The exact procedure is as follows. We define a coefficient $m_T(j)$ for the types $T = \text{I, II, III}$ by

$$\begin{aligned} \mathcal{D}_{\text{I},1} \frac{(-x \cdot z)^{l+j}}{(x^2)^{\alpha+j}} &= m_{\text{I}}(j) \frac{(-x \cdot z)^{l+j+1}}{(x^2)^{\alpha+j+1}}, \\ \mathcal{D}_{\text{II},1} \frac{(-x \cdot z)^{l-j}}{(x^2)^\alpha} &= m_{\text{II}}(j) \frac{(-x \cdot z)^{l-j-1}}{(x^2)^\alpha}, \\ \mathcal{V}_j \frac{(-x \cdot z)^l}{(x^2)^{\alpha+j}} &= m_{\text{III}}(j) \frac{(-x \cdot z)^l}{(x^2)^{\alpha+j+1}}, \end{aligned} \quad (\text{C.3})$$

where we opportunely shifted α and l in such a way to obtain M_A simply as the product

$$M_{T,n} = \prod_{j=0}^{n-1} m_T(j) . \quad (\text{C.4})$$

We find the coefficients

$$\begin{aligned} m_{\text{I}}(j) &= 2i(\alpha + j) , \\ m_{\text{II}}(j) &= -\frac{i(2h - j + l - 3)(-\alpha + h - j + l - 1)}{h - j + l - 2} , \\ m_{\text{III}}(j) &= \frac{4(h - j - 2)(h + j - 1)(\alpha + j)(-\alpha + h - j + l - 1)}{(h - j + l - 2)(h + j + l - 1)} , \end{aligned} \quad (\text{C.5})$$

which, using (C.4), give (4.27).

C.2 Scalar-vector OPE

We now generalize the previous procedure to compute the matrix M_A defined in (4.53). We first compute the 2×2 matrix $m_T(j)$ for the types $T = \text{I, II, III}$

$$\mathcal{D}_{\text{I},1} \frac{t_{l+j}^{(q)}(x, z, z_1)}{(x^2)^{\alpha+j}} = \sum_{q'=1}^2 (m_{\text{I}}(j))_{qq'} \frac{t_{l+j+1}^{(q')}(x, z, z_1)}{(x^2)^{\alpha+j+1}} , \quad (\text{C.6})$$

$$\mathcal{D}_{\text{II},1} \frac{t_{l-j}^{(q)}(x, z, z_1)}{(x^2)^{\alpha}} = \sum_{q'=1}^2 (m_{\text{II}}(j))_{qq'} \frac{t_{l-j-1}^{(q')}(x, z, z_1)}{(x^2)^{\alpha}} , \quad (\text{C.7})$$

$$\mathcal{V}_j \frac{t_l^{(q)}(x, z, z_1)}{(x^2)^{\alpha+j}} = \sum_{q'=1}^2 (m_{\text{III}}(j))_{qq'} \frac{t_l^{(q')}(x, z, z_1)}{(x^2)^{\alpha+j+1}} . \quad (\text{C.8})$$

The resulting matrices $m_T(j)$ are

$$\begin{aligned} m_{\text{I}}(j) &= -2i \begin{pmatrix} j + \alpha - 1 & -1/2 \\ 0 & j + \alpha - 2 \end{pmatrix} , \\ m_{\text{II}}(j) &= \frac{i}{h-j+l-1} \begin{pmatrix} -h + \alpha + p_j & \frac{j-l}{2} \\ \frac{2(h-1)(\alpha-1)}{j-l-1} & \frac{(j-l)p_j}{j-l-1} \end{pmatrix} , \\ m_{\text{III}}(j) &= \frac{4(\alpha+j-1)(\alpha+j)}{(h-j+l-2)(h+j+l-1)} \begin{pmatrix} \frac{q_j}{j+\alpha-1} & \frac{l(h^2+(-2j+l-2\alpha-1)h+j(j+3)-l+2\alpha)}{2(j+\alpha-1)(j+\alpha)} \\ 2(h-1) & \frac{h^2-(2j+2\alpha+1)h+j(j+3)+2\alpha+q_j}{j+\alpha} \end{pmatrix} , \end{aligned} \quad (\text{C.9})$$

where

$$\begin{aligned} p_j &\equiv \alpha + 2h^2 + h(-2\alpha - 3j + 3l) + j^2 + j(\alpha - 2l + 1) + l^2 - \alpha l - l - 1 , \\ q_j &\equiv -2\alpha + h^3 - h^2(\alpha + j - l + 3) + h(3\alpha - (j - 2)j - 2l + 2) - 2j + l \\ &\quad + j(j + 1)(\alpha + j - l) . \end{aligned} \quad (\text{C.10})$$

The matrix $M_{T,n}$ can be obtained by multiplying n matrices $m_T(j)$ as follows

$$M_{T,n} = m_T(0)m_T(1) \cdots m_T(n-1) . \quad (\text{C.11})$$

For the type I, one can diagonalize $m_I(j)$ using a matrix that does not depend on j , therefore it is trivial to find a closed form for $M_{I,n}$. In the other cases this procedure does not work. However it is not hard to guess a closed form for M_A and then use formula (C.11) to prove the guess by induction. The final result is

$$\begin{aligned} M_{I,n} &= (-2i)^n (\alpha)_{n-1} \begin{pmatrix} n + \alpha - 1 & -\frac{n}{2} \\ 0 & \alpha - 1 \end{pmatrix} , \\ M_{II,n} &= \frac{(-i)^n (-2h-l+3)_{n-1} (-h-l+\alpha+1)_{n-1}}{(-h-l+2)_n} \begin{pmatrix} P_n & \frac{n}{2}(n-l) \\ \frac{2n(1-\alpha)(h-1)}{l} & \frac{(l-n)(P_n-n\alpha+nh)}{l} \end{pmatrix} , \\ M_{III,n} &= \frac{(h-n)_{2n-1} (\alpha)_{n-1} (h+l-n-\alpha+1)_{n-1}}{2^{-2n} (h+l-n-1)_{2n}} \begin{pmatrix} (\alpha+n-1)Q_n & l/2 (R_n + nl) \\ 2n(\alpha-1)(\alpha+n-1) & (\alpha-1)(Q_n + R_n) \end{pmatrix} , \end{aligned} \quad (\text{C.12})$$

where

$$\begin{aligned} P_n &\equiv 2h^2 + l^2 + n + (2+n)\alpha - l(2+n+\alpha) + h(3l - 2(1+n+\alpha)) , \\ Q_n &\equiv \alpha + h(h+l-n-1) + \alpha(n-h) - l , \\ R_n &\equiv n(1+h-n-2\alpha) . \end{aligned} \quad (\text{C.13})$$

C.3 Efficient strategy to compute the matrices M_A

While it was easy to compute M_A in the scalar case, it was much harder to find in a closed form M_A with a matrix structure (for one external vector) by means of the strategy proposed in the previous section. This is due to the fact that M_A are defined by the product (C.11) of n matrices, which is not easy to compute in a closed form. In fact this problem would become too hard for cases in which the number of OPE tensor structures is higher. However there exists a simple way to compute the matrices $(M_A)_{pp'}$ in a closed form, which we are going to explain in this section. For this purpose we define the conformal blocks $\hat{G}_{\mathcal{O}}^{(p,q)}$ in the basis of the differential operators (see appendix F)

$$\hat{G}_{\mathcal{O}}^{(p,q)} \equiv \mathfrak{D}_{\text{left}}^{(p)} \mathfrak{D}_{\text{right}}^{(q)} G_{\mathcal{O}} \quad (\text{C.14})$$

where $G_{\mathcal{O}}$ is the scalar block. The blocks used for the recurrence relation are related to $\hat{G}_{\mathcal{O}}^{(p,q)}$ by the a change of basis of the form

$$\hat{G}_{\mathcal{O}}^{(p,q)} = \sum_{p'q'} (\mathfrak{n}_{\mathcal{O}})_{pp'} (\mathfrak{n}_{\mathcal{O}})_{qq'} G_{\mathcal{O}}^{(p',q')} , \quad (\text{C.15})$$

where $\mathbf{n}_{\mathcal{O}}$ are square matrices of coefficients independent of the cross ratios. In the differential basis the residues at the pole are diagonal in the labels p and q , in fact

$$\begin{aligned}
\hat{G}_{\mathcal{O}}^{(p,q)} &= \mathfrak{D}_{\text{left}}^{(p)} \mathfrak{D}_{\text{right}}^{(q)} \frac{M_A^{(L)} Q_A M_A^{(R)}}{\Delta - \Delta_A^*} G_{\mathcal{O}_A} + O((\Delta - \Delta_A^*)^0) \\
&= \frac{\mathfrak{M}_A^{(L,p)} Q_A \mathfrak{M}_A^{(R,q)}}{\Delta - \Delta_A^*} \mathfrak{D}_{\text{left}}^{(p)} \mathfrak{D}_{\text{right}}^{(q)} G_{\mathcal{O}_A} + O((\Delta - \Delta_A^*)^0) \\
&= \frac{\mathfrak{R}_A^{(p,q)}}{\Delta - \Delta_A^*} \hat{G}_{\mathcal{O}_A}^{(p,q)} + O((\Delta - \Delta_A^*)^0) ,
\end{aligned} \tag{C.16}$$

where $M_A^{(L)} Q_A M_A^{(R)} \equiv R_A$ is the residue of the scalar block and $\mathfrak{R}_A^{(p,q)} \equiv \mathfrak{M}_A^{(L,p)} Q_A \mathfrak{M}_A^{(R,q)}$. The coefficient $\mathfrak{M}_A^{(p)}$ is morally the same as M_A since the differential part of the operator $\mathfrak{D}^{(p)}$ clearly does not act on R_A . However the operators $\mathfrak{D}^{(p)}$ also act by shifting Δ_{12} (or Δ_{34}) by some units. Therefore the coefficient $\mathfrak{M}_A^{(p)}$ is trivially obtained by implementing such shifts on M_A . Knowing $\mathfrak{M}_A^{(p)}$, it is trivial to obtain $(M_A)_{pp'}$ in a closed form, just by performing a change of basis from the differential basis to the one that we need. For example we can reproduce the result of C.2 by

$$(M_A)_{pq} = \sum_{p'=1}^2 \left(\mathbf{n}_{\Delta_A^*, l}^{-1} \right)_{pp'} \mathfrak{M}_A^{(p')} (\mathbf{n}_{\Delta_A, l_A})_{p'q} , \quad A = \begin{cases} \text{I}, n \\ \text{II}, n \\ \text{III}, n \end{cases} \tag{C.17}$$

where

$$\mathfrak{M}_A^{(1)} = M_A \Big|_{\Delta_{12} \rightarrow \Delta_{12}+1} , \quad \mathfrak{M}_A^{(2)} = M_A \Big|_{\Delta_{12} \rightarrow \Delta_{12}-1} , \tag{C.18}$$

where M_A are the scalar coefficients defined in formula (4.27). The 2×2 matrix $\mathbf{n}_{\Delta, l}$ implements the change of basis from the differential operator basis to the basis used in appendix C.2. It takes the form

$$\mathbf{n}_{\Delta, l} \equiv \frac{i(-1)^{-l}}{\sqrt{2}} \begin{pmatrix} 1 - \Delta + \Delta_{12} + l & -l \\ 1 - \Delta - \Delta_{12} - l & l \end{pmatrix} . \tag{C.19}$$

Appendix D

One External Conserved Current

In this appendix we write down the formulas for $\tilde{h}_{\Delta,l,1}^\infty(r, \eta)$ and $\tilde{h}_{\Delta,l,2}^\infty(r, \eta)$ obtained solving the Casimir equation at subleading in large Δ ,

$$\begin{aligned} \tilde{h}_{\Delta,l,1}^\infty(r, \eta) = & \frac{2^{-l} l!}{l (r^2 - 2\eta r + 1) (r^2 + 2\eta r + 1) (h - 1)_l} \times \left\{ 2(h - 1) C_{l-1}^{(h)}(\eta) \times \right. \\ & \left[\eta (r^4 + (2 - 4\eta^2) r^2 + 1) (\Delta(1 - r^2) + h((h - 1)r^2 - 2) + r^2 + 1) \right. \\ & + \Delta_{12}^2 \eta r^2 (r^2 - 1) (-2\eta^2 + r^2 + 1) \\ & + \Delta_{34} r \left(\Delta_{34} \eta r (r^2 - 1) (-2\eta^2 + r^2 + 1) + 2(\eta^2 - 1) (r^2 + 1)^2 \right) \\ & + \Delta_{12} \eta \left(-2\Delta_{34} \eta (r^2 - 1)^2 r + r^2 (- (8\eta^2 + r^4 - 3r^2 - 5)) + 1 \right) \Big] \\ & \left. - l (r^2 + 1) (2h + l - 2) (r^2 + 2\eta r + 1) (r^2 - 2\eta r + 1) C_l^{(h-1)}(\eta) \right\} , \end{aligned} \quad (D.1)$$

$$\begin{aligned} \tilde{h}_{\Delta,l,2}^\infty(r, \eta) = & \frac{2^{-l} (r^2 + 1) l!}{l (r^2 - 2\eta r + 1) (r^2 + 2\eta r + 1) (h - 1)_l} \times \left\{ - \frac{(h - 1)}{r} C_{l-1}^{(h)}(\eta) \times \right. \\ & \left[(r^4 + (2 - 4\eta^2) r^2 + 1) (\Delta(1 - r^2) + h((h - 1)r^2 - 2) + r^2 + 1) \right. \\ & + \Delta_{34}^2 r^2 (r^2 - 1) (-2\eta^2 + r^2 + 1) \\ & + \Delta_{12}^2 (r^2 - 1) r^2 (-2\eta^2 + r^2 + 1) \\ & + \Delta_{12} (r^2 (- (r^4 - 4\eta^2 (r^2 - 1) + r^2 - 8\eta^3 r + 8\eta r - 1)) + 1) \\ & \left. - 2\Delta_{34} r \left(2(\eta^2 - 1) (r^3 + r) + \Delta_{12} \eta (r^2 - 1)^2 \right) \right] \\ & \left. - l(2h + l - 2) (r^2 - 2\eta r + 1) (r^2 + 2\eta r + 1) C_l^{(h-1)}(\eta) \right\} . \end{aligned} \quad (D.2)$$

Moreover the condition for the conservation of the blocks can be written as

$$\mathfrak{D}_\Delta [h_{\Delta,l,s}(r, \eta)] = 0 ,$$

where

$$\begin{aligned}
\mathfrak{D}_\Delta [h_{\Delta l, s}(r, \eta)] \equiv & \tag{D.3} \\
& + (r - r^3) \partial_r \tilde{h}_{\Delta l, 1}(r, \eta) + 2 (\eta^2 - 1) r \partial_\eta \tilde{h}_{\Delta l, 1}(r, \eta) \\
& + \eta r (r^2 - 1) \partial_r \tilde{h}_{\Delta l, 2}(r, \eta) + (\eta^2 - 1) (r^2 + 1) \partial_\eta \tilde{h}_{\Delta l, 2}(r, \eta) \\
& + \left[\Delta(1 - r^2) + \frac{4r (\Delta_{43} + 2h - 1) (\eta + r(\eta r - 2)) - \Delta_2 (r^4 + (4\eta^2 - 6) r^2 + 1)}{r^2 - 2\eta r + 1} \right] \tilde{h}_{\Delta l, 1}(r, \eta) \\
& + ((2h - 1)(\eta + r(\eta r - 2)) + \Delta_2(\eta + r(\eta r + 2)) + \Delta \eta (r^2 - 1) + 4r \Delta_{34} r) \tilde{h}_{\Delta l, 2}(r, \eta) .
\end{aligned}$$

Appendix E

Scalar-vector recurrence relation

This appendix is devoted to the computation of all the elements which enter formulae (4.81) and (4.84). In the first section we compute the coefficients R_A and in the second section we find the large Δ behavior of the blocks. Finally, in the last section we perform the leading OPE computation for the blocks in the natural normalization provided by the method of [81].

E.1 The residues R_A

In this appendix we want to give more details on the computation of all the coefficients R_A in formulas (4.81) and (4.84). We start by a complete derivation of $(R_{I_2,1})_{pq}$ and $(R_{II_2,1})_{pq}$. Then we explain how to obtain $(R_A)_{pp'qq'}$ of (4.81) from the results of [81]. Finally we write the coefficients R_A of (4.84), which we obtained by matching the expansion in r .

In the following we obtain from firsts principles the coefficients $(R_{I_2,1})_{pq}$ and $(R_{II_2,1})_{pq}$ of (4.82) and (4.85). Notice that these coefficients arise from new poles which were predicted in [81], but still not observed. These arise from (null) primary descendant states which are in a different representation with respect to the primaries. In particular, the primary descendant of type $I_2, 1$ is in the representation $(\Delta + 1, l, 1)$, and it is a descendant at the level one of a primary state in the representation (Δ, l) . Conversely, the descendant of type $II_2, 1$ lives in $(\Delta + 1, l)$, and is a descendant at the level one of a primary in $(\Delta, l, 1)$. Schematically

$$\begin{aligned} (\Delta, l) &\xrightarrow{I_2,1} (\Delta + 1, l, 1), \\ (\Delta, l, 1) &\xrightarrow{II_2,1} (\Delta + 1, l). \end{aligned} \tag{E.1}$$

Using the language of [81], we generate the descendant states acting with an operator $\mathcal{D}_A$ on primary states. These descendant states become primary (and therefore null) when the conformal dimension Δ of the primaries takes specific values,

$$\begin{aligned} |I_2, 1\rangle &\equiv \mathcal{D}_{I_2,1} |(\Delta, l); z\rangle, & \text{becomes primary when } \Delta = \Delta_{I_2,1}^* \equiv 1, \\ |II_2, 1\rangle &\equiv \mathcal{D}_{II_2,1} |(\Delta, l, 1); z, w\rangle, & \text{becomes primary when } \Delta = \Delta_{II_2,1}^* \equiv 2h - 2. \end{aligned} \tag{E.2}$$

The primary states $|(\Delta, l); z\rangle$ and $|(\Delta, l, 1); z, w\rangle$ are contracted with polarization vectors as

follows

$$\begin{aligned} |(\Delta, l); z\rangle &\equiv |(\Delta, l); \overline{\mu_1 \mu_2 \dots \mu_l}\rangle z^{\mu_1} z^{\mu_2} \dots z^{\mu_l} \\ |(\Delta, l, 1); z, w\rangle &\equiv \left| (\Delta, l, 1); \overline{\mu_1 \dots \mu_l} \right\rangle z^{\mu_1} \dots z^{\mu_l} w^\nu \end{aligned} \quad (\text{E.3})$$

and the new operators $\mathcal{D}_A$ are

$$\begin{aligned} \mathcal{D}_{\text{I}_{2,1}} &= \frac{-2i}{(h+l-2)(2h+l-4)} (z_{[\nu} w_{\mu]}) (P^{[\nu} D_{z,w}^{\mu]}), \\ \mathcal{D}_{\text{II}_{2,1}} &= \frac{-i}{(h-2)(h-1)} (P \cdot D_{w,z}), \end{aligned} \quad (\text{E.4})$$

where the overall normalizations are chosen for convenience, and $D_{z,w}$ and $D_{w,z}$ are the differential operators which generate projectors to traceless mixed-symmetry tensors [33]. They are defined as

$$D_{z,w}^\mu \equiv d_{00} \partial_z^\mu + d_{-11} \partial_w^\mu + z^\mu d_{-20} + w^\mu d_{-1-1}, \quad (\text{E.5})$$

where d_{mn} are differential operators with weight m in the variable z and n in the variable w ,

$$\begin{aligned} d_{00} &\equiv (1-h) [(2h-3) + 3(z \cdot \partial_z) + (w \cdot \partial_w)] - (z \cdot \partial_z)(w \cdot \partial_w) - z^\alpha (z \cdot \partial_z) \partial_{z\alpha}, \\ d_{-20} &\equiv \frac{1}{2} [2(h-1) + (w \cdot \partial_w) + (z \cdot \partial_z)] (\partial_z \cdot \partial_z), \\ d_{-11} &\equiv -2(h-1)(w \cdot \partial_z) - (w \cdot \partial_z)(w \cdot \partial_w) - (z \cdot \partial_z)(w \cdot \partial_z), \\ d_{-1-1} &\equiv [(h-1) + (z \cdot \partial_z)] (\partial_z \cdot \partial_w) + \frac{1}{2} [(w \cdot \partial_z)(\partial_w \cdot \partial_w) - (z \cdot \partial_w)(\partial_z \cdot \partial_z)], \end{aligned} \quad (\text{E.6})$$

and similarly for $D_{w,z}^\mu$ with $z \leftrightarrow w$.

Using the operators $\mathcal{D}_A$ in (E.4) we can proceed to the computation of the coefficients R_A as explained in [81]. The following computations are done in a new normalization, with respect to the main text, which is conveniently chosen to apply the techniques introduced in [81]. In the end of the appendix we explain how to return to the conventions used in the main text.

First we compute Q_A , the residue at the pole Δ_A^* of the inverse of norms $N_{\text{I}_{2,1}} \equiv \langle \text{I}_2, 1 | \text{I}_2, 1 \rangle$ and $N_{\text{II}_{2,1}} \equiv \langle \text{II}_2, 1 | \text{II}_2, 1 \rangle$,

$$\frac{1}{N_A} \sim \frac{Q_A}{\Delta - \Delta_A^*}. \quad (\text{E.7})$$

This follows from the commutation relations of the conformal algebra, and the result is

$$Q_{\text{I}_{2,1}} = \frac{1}{l(l+1)}, \quad Q_{\text{II}_{2,1}} = \frac{h-1}{2(2h+l-3)(2h+l-2)}. \quad (\text{E.8})$$

Next we need to obtain the relation between the OPE coefficients of the primary descendant and the OPE coefficients of the original primary. It is convenient to introduce the leading OPE of the three point function of a vector operator $\mathcal{O}_1$, a scalar operator $\mathcal{O}_2$ and an operator $\mathcal{O}$ in

the representations (Δ, l) and $(\Delta, l, 1)$,

$$\mathcal{O}(x, z)\mathcal{O}_1(0, z_1) = \frac{\mathcal{O}_2(0)}{(x^2)^{\alpha_{\Delta_{12}}}} \times \begin{cases} c_{12\mathcal{O}}^{(0)} t_l^{(0)}(x, z, w, z_1) & \text{if } \mathcal{O} \in (\Delta, l, 1) \\ c_{12\mathcal{O}}^{(1)} t_l^{(1)}(x, z, z_1) + c_{12\mathcal{O}}^{(2)} t_l^{(2)}(x, z, z_1) & \text{if } \mathcal{O} \in (\Delta, l) \end{cases} \quad (\text{E.9})$$

with $\alpha_x \equiv \frac{\Delta+x+l+1}{2}$ and

$$\begin{aligned} t_l^{(0)}(x, z, w, z_1) &= \frac{1}{2}|x|(x \cdot z)^{l-1}[(w \cdot z_1)(z \cdot x) - (z \cdot z_1)(w \cdot x)] , \\ t_l^{(1)}(x, z, z_1) &= (x \cdot z)^l(x \cdot z_1) , \\ t_l^{(2)}(x, z, z_1) &= (x \cdot z)^{l-1}x^2(z \cdot z_1) . \end{aligned} \quad (\text{E.10})$$

Therefore we can define the vectors $M_{\text{I}_{2,1}}$ and $M_{\text{II}_{2,1}}$ from the following computation [81]

$$\begin{aligned} \mathcal{D}_{\text{I}_{2,1}} \frac{t_l^{(p)}(x, z, z_1)}{(x^2)^{\alpha_\delta}} &= \left(M_{\text{I}_{2,1}}(\delta)\right)_p \frac{t_l^{(0)}(x, z, w, z_1)}{(x^2)^{\alpha_\delta + \frac{1}{2}}} , \quad (p = 1, 2) \\ \mathcal{D}_{\text{II}_{2,1}} \frac{t_l^{(0)}(x, z, w, z_1)}{(x^2)^{\alpha_\delta}} &= \sum_{p=1}^2 \left(M_{\text{II}_{2,1}}(\delta)\right)_p \frac{t_l^{(p)}(x, z, z_1)}{(x^2)^{\alpha_\delta + \frac{1}{2}}} . \end{aligned} \quad (\text{E.11})$$

The result is

$$M_{\text{I}_{2,1}}(\delta) = \left(l, 2\alpha_\delta - 2\right) , \quad M_{\text{II}_{2,1}}(\delta) = \left(2\alpha_\delta - 1, 2h - 2\alpha_\delta + l - 1\right) . \quad (\text{E.12})$$

The residues are then given by

$$\begin{aligned} (\hat{R}_{\text{I}_{2,1}})_{pq} &= Q_{\text{I}_{2,1}} \left(M_{\text{I}_{2,1}}(\Delta_{12})\right)_p \left(M_{\text{I}_{2,1}}(\Delta_{34})\right)_q \Big|_{\Delta=1} , \\ (\hat{R}_{\text{II}_{2,1}})_{pq} &= Q_{\text{II}_{2,1}} \left(M_{\text{II}_{2,1}}(\Delta_{12})\right)_p \left(M_{\text{II}_{2,1}}(\Delta_{34})\right)_q \Big|_{\Delta=2h-2} . \end{aligned} \quad (\text{E.13})$$

However, to obtain (4.82) and (4.85) we need to return to the normalization of the conformal blocks used in the main text.

To relate the normalization of the conformal blocks g_λ of the main text to the normalization of the conformal blocks $\hat{g}_\lambda$ in this appendix, it is convenient to compare their leading OPE behavior. To find the leading OPE for $\hat{g}_\lambda$, we use the techniques proposed in [81] which are extended in appendix H.1.1. Meanwhile, the leading OPE of g_λ can be trivially found taking the $O(r^0)$ coefficient of the r expansion according to (3.52) and (3.53). When $\lambda = (\Delta, l)$ we obtain

$$\hat{g}_{(\Delta, l), s}^{(p, q)}(r, \eta) = \mathfrak{a}_l \sum_{p', q'=1}^2 (\mathfrak{m}_l^{(L)})_{p, p'} (\mathfrak{m}_l^{(R)})_{q, q'} g_{(\Delta, l), s}^{(p', q')}(r, \eta) . \quad (\text{E.14})$$

where

$$\mathfrak{a}_l \equiv \frac{(2h-1)_{l-1}}{(-2)^{l-1}(h)_{l-1}} , \quad \mathfrak{m}_l^{(L)} \equiv \begin{pmatrix} -1 & 0 \\ -1 & \frac{1}{l} \end{pmatrix} , \quad \mathfrak{m}_l^{(R)} \equiv \begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{l} \end{pmatrix} . \quad (\text{E.15})$$

Similarly, when $\lambda = (\Delta, l, 1)$, we have

$$\hat{g}_{(\Delta, l, 1), s}(r, \eta) = \mathfrak{b}_l g_{(\Delta, l, 1), s}(r, \eta) , \quad \mathfrak{b}_l \equiv \frac{\mathfrak{a}_l}{-2l(2h+l-3)} . \quad (\text{E.16})$$

Equations (E.14) and (E.16) give rise to the following definition of the coefficients $(R_A)_{pq}$,

$$\begin{aligned} (R_{\text{I2},1})_{pq} &= \mathfrak{a}_l^{-1} Q_{\text{I2},1} \mathfrak{b}_l \left((\mathfrak{m}_l^{(L)})^{-1} M_{\text{I2},1}(\Delta_{12}) \right)_p \left((\mathfrak{m}_l^{(R)})^{-1} M_{\text{I2},1}(\Delta_{34}) \right)_q \Big|_{\Delta=1} , \\ (R_{\text{II2},1})_{pq} &= \mathfrak{b}_l^{-1} Q_{\text{II2},1} \mathfrak{a}_l \left(M_{\text{II2},1}(\Delta_{12}) \mathfrak{m}_l^{(L)} \right)_p \left(M_{\text{II2},1}(\Delta_{34}) \mathfrak{m}_l^{(R)} \right)_q \Big|_{\Delta=2h-2} , \end{aligned} \quad (\text{E.17})$$

which finally gives (4.82) and (4.85), once we replace the values of Q_A and M_A computed in (E.8) and (E.12).

Similarly, we obtain the coefficients $(R_A)_{pp'qq'}$ of (4.81),

$$(R_A)_{pp'qq'} = \mathfrak{a}_l^{-1} Q_A \mathfrak{a}_{l_A} \left((\mathfrak{m}_l^{(L)})^{-1} M_A^{(L)} \mathfrak{m}_{l_A}^{(L)} \right)_{pp'} \left((\mathfrak{m}_l^{(R)})^{-1} M_A^{(R)} \mathfrak{m}_{l_A}^{(R)} \right)_{qq'} \Big|_{\Delta=\Delta_A^*} , \quad (\text{E.18})$$

where Q_A and M_A are presented in formulae (34-36) and (189) of [81].

The coefficients R_A of (4.84) could in principle be computed using the same technology that we used in this appendix, but we decided to find them matching terms in the expansion in r of the conformal blocks. This can be easily done since these coefficients do not present any matrix structure. Moreover, they are not conceptually new since they arise from null states created by the operators $\mathcal{D}_{\text{I},n}$, $\mathcal{D}_{\text{II},n}$ and $\mathcal{D}_{\text{III},n}$ already defined in [81]. The result is

$$\begin{aligned} R_{\text{I1},n} &= -\frac{l \left(\frac{-n+\Delta_{12}+1}{2} \right)_n \left(\frac{-n+\Delta_{34}+1}{2} \right)_n (2h+l-3)_n}{(n-1)!n!(l+n)(h+l-1)_n} , \\ R_{\text{II1},n} &= -\frac{(-1)^n(-l-1)_n \left(\frac{-n+\Delta_{12}+1}{2} \right)_n \left(\frac{-n+\Delta_{34}+1}{2} \right)_n (2h+l-n+1-2)_n}{n!(n-1)!(h+l-n+1-1)_n(2h+l-n+1-3)_n} , \\ R_{\text{III},n} &= \frac{\left(\frac{h-n+1}{2} \right)_n \left(\frac{h-n-2}{2} \right)_n \left(\frac{\Delta_{12}+h+l-n}{2} \right)_n \left(\frac{\Delta_{12}-h-l-n+2}{2} \right)_n \left(\frac{\Delta_{34}+h+l-n}{2} \right)_n \left(\frac{\Delta_{34}-h-l-n+2}{2} \right)_n}{(-1)^{n+1}4^{-n}n!(n-1)!(h+l-n)_{2n}(h+l-n-1)_{2n}} . \end{aligned} \quad (\text{E.19})$$

E.2 Conformal Block at Large Δ

The goal of this section is to find $h_{(\infty, l), s}^{(p, q)}(r, \eta)$ of (4.81) and $h_{(\infty, l, 1), s}(r, \eta)$ of (4.84). They can be obtained solving the leading term in Δ of the Casimir equation (2.80).

First we replace $g_{\lambda, s}(r, \eta) \equiv (4r)^\Delta h_{\lambda, s}(r, \eta)$ in (2.80), then we keep the term linear in Δ . We obtain five coupled first order differential equations in the variable r for the functions $h_{\infty, \lambda, s}(r, \eta) \equiv \lim_{\Delta \rightarrow \infty} h_{\lambda, s}(r, \eta)$. We then make the ansatz for the function $h_{\infty, \lambda, s}$,

$$h_{\infty, \lambda, s}(r, \eta) = \mathcal{A}^{\Delta_{12}, \Delta_{34}}(r, \eta) \sum_{t=1}^5 F_s^t(r, \eta) h_{\lambda, t}(0, \eta) , \quad (\text{E.20})$$

where

$$\mathcal{A}^{\Delta_{12}, \Delta_{34}}(r, \eta) = \frac{(1 - r^2)^{-h-1}}{(1 + r^2 - 2r\eta)^{\frac{-\Delta_{12} + \Delta_{34} + 1}{2}} (1 + r^2 + 2r\eta)^{\frac{\Delta_{12} - \Delta_{34} + 3}{2}}} , \quad (\text{E.21})$$

and $F_s^t(0, \eta) = \delta_{s,t}$ so that $h_{\infty, \lambda, s}(0, \eta) = h_{\lambda, s}(0, \eta)$. Using the ansatz (G.20), the five coupled differential equations can be schematically written as

$$\partial_r F_s^t(r, \eta) = \sum_{s'=1}^5 \mathcal{M}_{ss'}(r, \eta) F_{s'}^t(r, \eta) , \quad (\text{E.22})$$

where $\mathcal{M}_{ss'}$ is a known 5×5 matrix with entries dependent on r and η . The label $t = 1, \dots, 5$ of F_s^t parametrize the five independent solutions of (E.22). Even if solving a generic system of five coupled first order differential equations is usually very difficult, in this case we can easily obtain the full solution. This is possible since the choice of the ansatz (E.21) turns the solution F_s^t into polynomials of r and η that can therefore be obtained by expanding (E.22) in series of r , and then solving it term by term. The choice of (E.21), which makes this possible, is an educated guess inspired by the large Δ behavior of the scalar conformal block. The result is

$$F(r, \eta) = \begin{pmatrix} (r^2 - 1)^2 (2r\eta + A_3) & 0 & 0 & 0 & 0 \\ -2r^2 A_1^2 & -A_1 A_2 A_3 & -2r A_1 A_3 & 2r^2 \eta A_1 A_2 & 4r^3 \eta A_1 \\ -2r (2\eta r^3 - r^2 - 1) A_1 & -2r^2 \eta A_1 A_3 & A_1 A_3^2 & 4r^4 \eta^2 A_1 & -2r^2 \eta A_1 A_3 \\ 2r (3r^2 - 2\eta r - 1) A_1 & 2r A_1 A_2 & 4r^2 A_1 & A_1 A_2^2 & 2r A_1 A_2 \\ -2r^2 A_1^2 & 4r^3 \eta A_1 & -2r A_1 A_3 & 2r^2 \eta A_1 A_2 & -A_1 A_2 A_3 \end{pmatrix} , \quad (\text{E.23})$$

where

$$A_1 \equiv r^2 - 2\eta r + 1 , \quad A_2 \equiv r^2 - 2\eta r - 1 , \quad A_3 \equiv r^2 + 1 . \quad (\text{E.24})$$

Notice that the matrix F_s^t is independent of the $SO(d)$ representation of the exchanged operator. This is not surprising since this information only appears in the eigenvalue c_λ in terms of order $O(\Delta^0)$, which do not contribute at leading order at large Δ .

We can finally obtain $h_{(\infty, l), s}^{(p, q)}(r, \eta)$ of (4.81), and $h_{(\infty, l, 1), s}(r, \eta)$ of (4.84), simply by taking the setting the appropriate initial conditions for the functions $G_{\Delta, l}^{(p, q)}$ and $G_{\Delta, l, 1}$. In particular this can be obtained from the leading OPE limit (H.14) of the partial waves. To match the notation of section 4.4.3 we also need to implement the change of normalization of the three point functions defined in equations (E.14) and (E.16).

Appendix F

Differential Operators Method

In this appendix we implement the method of [41] to match some of the results obtained in the main text. The main idea of [41] is that we can obtain conformal blocks for external operators with spin, by acting with differential operators on the scalar conformal block. Schematically one can write

$$G_{\Delta,l}^{(p,q)}(P_i, Z_i) = \mathfrak{D}_{\text{Left}}^{(p)} \mathfrak{D}_{\text{Right}}^{(q)} G_{\Delta,l}(P_i), \quad (\text{F.1})$$

where the operators $\mathfrak{D}_{\text{Left}}^{(p)}$ and $\mathfrak{D}_{\text{Right}}^{(q)}$ are explicitly defined in [41] in terms of multiplication and derivatives using the embedding vectors P_i and Z_i , and by some shifts of the external dimensions. Using formula (F.1) one can obtain all the blocks for the exchange of a symmetric and traceless representation. Unfortunately, with this method it is not possible to obtain the blocks for more complicated exchanges. However, one can pursue the philosophy of acting with some differential operators on simple *seed* blocks, which exchange a representation λ in order to obtain the most generic blocks labeled by the exchange of the same λ . This idea was fruitfully followed in four dimensions in [82].

It is instructive to write explicitly how the differential operators look in radial coordinates. To do so we first express the scalar block using equation (3.16) and then collect the coefficients multiplying each tensor structure in the four point function according to (2.76). In the case of the conformal block decomposition (3.26) of the four point function with one external vector, we find

$$h_{(\Delta,l),s}^{(p)}(r, \eta) = \sum_{q=1}^2 (\mathbf{m})_{pq} D_s^{(q)} h_{(\Delta,l)}(r, \eta), \quad (\text{F.2})$$

where

$$\mathbf{m} = \frac{-1}{2(\Delta-1)} \begin{pmatrix} 1 & 1 \\ 1-\Delta-\Delta_{12} & \Delta-1-\Delta_{12} \end{pmatrix} \quad (\text{F.3})$$

and the functions $h_{(\Delta,l)}$ are related to the usual conformal blocks as follows $g_{(\Delta,l)} = (4r)^\Delta h_{(\Delta,l)}$ (and similarly for $h_{(\Delta,l),s}^{(p)}$). The matrix $\mathbf{m}$ is introduced in order to match the normalization of

the conformal blocks that we used in section 3.2.1. The operators $D_s^{(p)}$ are defined as follows

$$\begin{aligned}
D_1^{(1)} h &= \left(\frac{2(1+r^2)(1+\Delta_{12})}{1+r^2+2r\eta} + \frac{1+\Delta+r^2(-1+\Delta-\Delta_{12})+\Delta_{12}}{-1+r^2} \right) h^{[1]} \\
&\quad - \eta \partial_\eta h^{[1]} + \frac{(r+r^3) \partial_r h^{[1]}}{-1+r^2} , \\
D_2^{(1)} h &= \left(-\frac{2r\Delta}{-1+r^2} - \frac{4r(1+\Delta_{12})}{1+r^2+2r\eta} \right) h^{[1]} - \partial_\eta h^{[1]} - \frac{2r^2 \partial_r h^{[1]}}{-1+r^2} , \\
D_1^{(2)} h &= \frac{(1+r^2-2r\eta)(-1+\Delta+r^2(1+\Delta-\Delta_{12})+\Delta_{12})h^{[-1]}}{(-1+r^2)(1+r^2+2r\eta)} \\
&\quad + \frac{\eta(1+r^2-2r\eta)\partial_\eta h^{[-1]}}{1+r^2+2r\eta} + \frac{(r+r^3)(1+r^2-2r\eta)\partial_r h^{[-1]}}{(-1+r^2)(1+r^2+2r\eta)} , \\
D_2^{(2)} h &= \frac{2r(2(1+\Delta_{34}-\Delta_{12})(1-r^2)-\Delta(r^2+1-2r\eta))h^{[-1]}}{(-1+r^2)(1+r^2+2r\eta)} \\
&\quad + \left(-1 + \frac{2(1+r^2)}{1+r^2+2r\eta} \right) \partial_\eta h^{[-1]} - \frac{2r^2(1+r^2-2r\eta)\partial_r h^{[-1]}}{(-1+r^2)(1+r^2+2r\eta)} ,
\end{aligned} \tag{F.4}$$

where $h^{[k]}$ means that the variable Δ_{12} in the function h is shifted by k , according to $\Delta_{12} \rightarrow \Delta_{12} + k$. Formula (F.2) is in agreement with the radial expansion of section 3.2.1 and with the results of [81].

Similarly, for the case of two external vectors we can write $h_{(\Delta,l),s}^{(p,q)}(r,\eta) = D_s^{(p,q)} h_{(\Delta,l)}(r,\eta)$. In this case we do not present the explicit action of the operators since it would be too long. However, we define them in a `Mathematica` file included with the submission. It is worth mentioning that the result matches the computation of the symmetric and traceless blocks $g_{(\Delta,l),s}^{(p,q)}(r,\eta)$ obtained both from the r expansion and from the analytic structure in Δ .

Appendix G

Four vectors in three dimensions

G.1 Definitions

As a first step we explain our convention for the labels p, q of the OPE. We define the leading OPE in terms of a linear combination of tensor structures

$$\mathcal{O}^\pm(x, z) J_1(0, z_1) \sim \frac{J_2(0, \partial_{z_2})}{(x^2)^{\alpha_\pm}} \sum_q c_{12\mathcal{O}^\pm}^{(q)} t_{l\pm}^{(q)}(x, z, z_1, z_2) . \quad (\text{G.1})$$

We also define $\alpha_+ \equiv \frac{\Delta + \Delta_1 - \Delta_2 + l + 2}{2}$ and $\alpha_- \equiv \frac{\Delta + \Delta_1 - \Delta_2 + l + 1}{2}$. The various OPE coefficients $(c_{12\mathcal{O}^\pm})_q$ respectively multiply the tensor structures $t_{l\pm}^{(q)}(x, z, z_2, z_3)$, which are defined to be the Lorentz invariant which satisfy

$$t_{l+}^{(q)}(\mu x, \lambda z, \lambda_1 z_1, \lambda_2 z_2) = \mu^{l+2} \lambda^l \lambda_1 \lambda_2 t_{l+}^{(q)}(x, z, z_1, z_2) , \quad (\text{G.2})$$

$$t_{l-}^{(q)}(\mu x, \lambda z, \lambda_1 z_1, \lambda_2 z_2) = \mu^{l+1} \lambda^l \lambda_1 \lambda_2 t_{l-}^{(q)}(x, z, z_1, z_2) . \quad (\text{G.3})$$

The sum over the q in (G.1) runs from one to five for parity even operators, since we can build the following five structures

$$\begin{aligned} t_{l+}^{(1)}(x, z, z_1, z_2) &\equiv (x \cdot z)^l (z_1 \cdot z_2) x^2 , \\ t_{l+}^{(2)}(x, z, z_1, z_2) &\equiv (x \cdot z)^l (x \cdot z_1)(x \cdot z_2) , \\ t_{l+}^{(3)}(x, z, z_1, z_2) &\equiv (x \cdot z)^{l-1} (z \cdot z_1)(x \cdot z_2) x^2 , \\ t_{l+}^{(4)}(x, z, z_1, z_2) &\equiv (x \cdot z)^{l-1} (z \cdot z_2)(x \cdot z_1) x^2 , \\ t_{l+}^{(5)}(x, z, z_1, z_2) &\equiv (x \cdot z)^{l-2} (z \cdot z_1)(z \cdot z_2) x^4 . \end{aligned} \quad (\text{G.4})$$

Notice that for $l = 0$ only $t^{(1)}$ and $t^{(2)}$ exist and for $l = 1$ they are all allowed except $t^{(5)}$. Similarly for parity odd operators there are five allowed tensor structures which can be build by

using the epsilon tensor (see appendix G.5.2)

$$\begin{aligned}
t_{l-}^{(1)} &= \epsilon(x, z_1, z_2)(x \cdot z)^l \\
t_{l-}^{(2)} &= \epsilon(x, z, z_1)(x \cdot z_2)(x \cdot z)^{l-1} \\
t_{l-}^{(3)} &= \epsilon(x, z, z_2)(x \cdot z_1)(x \cdot z)^{l-1} \\
t_{l-}^{(4)} &= [\epsilon(x, z, z_1)(z \cdot z_2) + \epsilon(x, z, z_2)(z \cdot z_1)](x \cdot z)^{l-2} x^2 .
\end{aligned} \tag{G.5}$$

Again it is clear that for $l=0$ there exists just $t^{(1)}$ and for $l = 1$ only $t^{(1)}, t^{(2)}, t^{(3)}$.

G.2 Null States

In this section we write all the possible primary descendant states that can be exchanged when the external operators are all vectors. In $d = 3$ the rotation group is $SO(3)$, so only the symmetric and traceless representation is allowed.

The primary descendant states are of four possible types which are labeled also by an integer n (n runs over all positive integers for type I and type II, and over a finite set for type III and IV). We define each descendant state $|\mathcal{O}_A; z\rangle$ by the action of an operator $\mathcal{D}_A$ (built as a linear combination of many P^μ , the generators of the translations) on a primary state $|\mathcal{O}; z\rangle$

$$|\mathcal{O}_A; z\rangle = \mathcal{D}_A |\mathcal{O}; z\rangle . \tag{G.6}$$

The operators $\mathcal{D}_A$ can be fixed by asking that $K_\mu |\mathcal{O}_A; z\rangle = 0$ when $\Delta = \Delta_A^*$. This in fact ensures that when $\Delta = \Delta_A^*$, the descendants $|\mathcal{O}_A; z\rangle$ become primaries. The operators $\mathcal{D}_A$ can be written in the following compact form

$$\mathcal{D}_{\text{I},n} \equiv c_{\text{I},n} (z \cdot P)^n , \tag{G.7}$$

$$\mathcal{D}_{\text{II},n} \equiv c_{\text{II},n} (D_z \cdot P)^n , \tag{G.8}$$

$$\mathcal{D}_{\text{III},n} \equiv c_{\text{III},n} \mathcal{V}_{\frac{1}{2}} \mathcal{V}_{\frac{3}{2}} \cdots \mathcal{V}_{n-\frac{1}{2}} , \tag{G.9}$$

$$\mathcal{D}_{\text{IV},n} \equiv c_{\text{IV},n} \mathcal{E} \mathcal{V}_1 \mathcal{V}_2 \cdots \mathcal{V}_{n-1} , \tag{G.10}$$

where $\mathcal{V}_j$ and $\mathcal{E}$ are defined by

$$\mathcal{V}_j = (j+l)(l-j)P^2 - 2(P \cdot z)(P \cdot D_z) , \tag{G.11}$$

$$\mathcal{E} = \epsilon_{\mu\nu\sigma} P^\mu z^\nu D_z^\sigma . \tag{G.12}$$

where for convenience we fix the normalization coefficients c as follows

$$c_{\text{I},n} = 1 = c_{\text{IV},n} , \quad c_{\text{II},n} = \frac{1}{\left(\frac{1}{2} - l\right)_n (-l)_n} , \quad c_{\text{III},n} = \frac{1}{\left(l - n + \frac{1}{2}\right)_{2n}} . \tag{G.13}$$

Actually the first three types are the same as the ones discussed in section 4.2. However we presented them again for clarity.

We want to stress that the operators of the type I, II and III, do not change the parity of the

state on which they are applied, while the one of type IV does, namely

$$\mathcal{D}_A|\mathcal{O}^\pm\rangle = \begin{cases} |\mathcal{O}_A^\pm\rangle & A = (\text{I}, n), (\text{II}, n), (\text{III}, n) \\ |\mathcal{O}_A^\mp\rangle & A = (\text{IV}, n) \end{cases} . \quad (\text{G.14})$$

From the definitions (G.7-G.10) we can compute the residue Q_A at the pole Δ_A^* of the inverse of the norm of the primary descendant states

$$\langle \mathcal{O}_A; z | \mathcal{O}_A; z \rangle^{-1} = \frac{Q_A}{\Delta - \Delta_A^*} \langle \mathcal{O}; z | \mathcal{O}; z \rangle^{-1} + O((\Delta - \Delta_A^*)^0) . \quad (\text{G.15})$$

The result for the four types is

$$\begin{aligned} Q_{\text{I},n} &= -\frac{2^{-n}}{(n-1)!n!} , \\ Q_{\text{II},n} &= \frac{(-2)^{-n}}{(n-1)!n!} \frac{(2n-2l-1)}{(2l+1)} \frac{(-l)_n}{(l-n+1)_n} , \\ Q_{\text{III},n} &= -\frac{(-4)^{-2n}}{(n-1)!n!} \frac{(l-n+\frac{1}{2})}{(l+n+\frac{1}{2})} \frac{1}{(\frac{1}{2}-n)_{2n}} , \\ Q_{\text{IV},n} &= \frac{2}{(2n-1)!(2n-2)!} \frac{1}{(1-2l)^2} \frac{1}{(-l-n+1)_{2n-2}(l-n+1)_{2n}} . \end{aligned} \quad (\text{G.16})$$

The norm of the first three types was already computed in section 4.2. The norm of type IV can be also computed in a similar way.

G.3 The Residue R_A

The residue R_A is obtained using formula $(R_A)_{pp'qq'} \equiv (M_A^{(L)})_{pp'} Q_A (M_A^{(R)})_{qq'}$ as explained in section 4.3. The coefficients Q_A are defined in (G.16). The matrices M_A can be defined by the action of differential operator $\mathcal{D}_A$ on the tensor structures appearing in the leading OPE. For the first three types we have

$$\mathcal{D}_A \frac{t_{l\pm}^{(q)}(x, z, z_1, z_2)}{(x^2)^{\alpha_\pm}} = \sum_{q'} \left(M_{\pm A}^{(L)} \right)_{qq'} \frac{t_{l_A \pm}^{(q')}(x, z, z_1, z_2)}{(x^2)^{\alpha_{\pm A}}} \quad A = \begin{cases} \text{I}, n \\ \text{II}, n \\ \text{III}, n \end{cases} , \quad (\text{G.17})$$

where the matrices M_{+A} are 5×5 while the M_{-A} are 4×4 . The exponent $\alpha_{\pm A}$ is equal to $\alpha_\pm$ where we replace $l \rightarrow l_A$ and $\Delta \rightarrow \Delta + n_A$.

The type IV is slightly different since it changes the parity of the primary state, therefore

$$\mathcal{D}_A \frac{t_{\pm l}^{(q)}(x, z, z_1, z_2)}{(x^2)^{\alpha_\pm}} = \sum_{q'} \left(M_{\pm A}^{(L)} \right)_{qq'} \frac{t_{\mp A}^{(q')}(x, z, z_1, z_2)}{(x^2)^{\alpha_{\mp A}}} \quad A = \text{IV}, n . \quad (\text{G.18})$$

In this case M_{+IV} is a rectangular matrix 4×5 while M_{-IV} is 5×4 .

Actually M_A are the obtained as the residue at the pole Δ_A^* . Therefore, to be precise, the M_A defined above are correct only after we replace $\Delta \rightarrow \Delta_A^*$.

The definitions given here can be directly used to compute M . However in appendix C we present some very efficient strategies to implement this computation (and we exemplify them for the case of one external vector). It is worth commenting that by direct computation one can check that $M_{IV,n}$ vanishes for $n > 2$ which means that only two poles of type (IV, n) contribute, namely $\Delta = 0, 1$ as mentioned in appendix G.6.3.

G.4 Conformal Block at Large Δ

In this section we explain how to compute h_∞ , the regular part in Δ of the conformal blocks. To do so, we are going to solve the Casimir differential equation at the leading term for large Δ with the appropriate initial condition for $G_{\Delta l \pm}^{(p,q)}$ when $x_{12}, x_{34} \rightarrow 0$.

We consider the leading order in Δ of the Casimir equation (2.80), where we substitute the definition $g_{\Delta l \pm, s}^{(p,q)}(r, \eta) \equiv (4r)^\Delta h_{\Delta l \pm, s}^{(p,q)}(r, \eta)$. The result is a set of 43 coupled first order differential equations for the functions h_s

$$\partial_r h_s = \sum_{t=1}^{43} \mathcal{M}_{st}(r, \eta) h_t, \quad (\text{G.19})$$

where $\mathcal{M}$ is a 43×43 matrix of explicitly known functions of r and η and where we dropped all the labels of h_s (which will be reintroduced when we will fix the initial condition of the Casimir equation). Since the 43 equations (G.19) are of the first order, we have 43 independent solutions $h_s^{(s')}$, labeled by $s' = 1 \dots 43$. We then use the ansatz

$$h_s^{(s')}(r, \eta) \equiv \mathcal{A}(r, \eta) P_s^{(s')}(r, \eta), \quad (\text{G.20})$$

$$\mathcal{A}(r, \eta) \equiv \frac{(1 - r^2)^{-3 - \frac{d}{2}}}{(1 + r^2 - 2r\eta)^{\frac{1}{2} - \Delta_{12} + \Delta_{34}} (1 + r^2 + 2r\eta)^{\frac{5}{2} + \Delta_{12} - \Delta_{34}}}, \quad (\text{G.21})$$

to obtain a set of differential equations for the 43 functions $P_s^{(s')}$. This ansatz, inspired by the solution of the scalar Casimir at large Δ , has the property of eliminating completely the dependence on Δ_{12} and Δ_{34} from the differential equation. Moreover it turns out that we can easily fix all the functions $P_s^{(s')}(r, \eta)$ since they are simple polynomials in r (of maximal degree 12). Notice also that we are leaving d unfixed: in fact the solution that we find works in any dimension. We can further choose a basis such that

$$P_s^{(s')}(0, \eta) = \delta_s^{s'}, \quad s, s' = 1, \dots, 43. \quad (\text{G.22})$$

The functions h_∞ can then be written as a linear combination of the 43 functions $h_s^{(s')}$ as follows

$$h_{\infty l \pm, s}^{(p,q)}(r, \eta) = \sum_{s'=1}^{43} h_s^{(s')}(r, \eta) f_{l \pm, s'}^{(p,q)}(\eta), \quad (\text{G.23})$$

where the functions f are constants of integrations that can be fixed by imposing the correct initial condition for the differential equation. In particular with our conventions $f_{l\pm,s}^{(p,q)}(\eta) \equiv h_{\infty l\pm,s}^{(p,q)}(0, \eta)$. Therefore we can fix them by studying the OPE limit (namely $r \rightarrow 0$) of $G_{\Delta l\pm}^{(p,q)}$. As explained in appendix H, by studying the OPE limit of $G_{\Delta l\pm}^{(p,q)}$ we obtain the following equation that can be used to define the functions f ,

$$\frac{t_{l\pm}^{(p)}(\hat{x}_{12}, I(x_{24}) \cdot D_z, I(x_{12}) \cdot z_1, z_2) t_{l\pm}^{(q)}(\hat{x}_{34}, z, I(x_{34}) \cdot z_3, z_4)}{l!(h-1)_l} \approx \sum_s f_{l\pm,s}^{(p,q)}(\eta) Q^{(s)}. \quad (\text{G.24})$$

As a last remark we want to stress that from this computations, all the functions $h_{\infty l+,s}^{(p,q)}$ were found in generic dimensions. However the leading term of the blocks $h_{\infty l-,s}^{(p,q)}$ is by construction related to the three dimensional case. Nevertheless to generalize it to any dimension it is trivial. It is sufficient to replace $f_{l-,s}^{(p,q)}$ with the OPE limit of the higher dimensional conformal blocks for the exchange of operators in the $\text{SO}(d)$ representations $(l, 1)$, $(l, 2)$ and $(l, 1, 1)$.

G.5 Tensor structures for equal and conserved currents

In this section we obtain the matrices $m_{\pm}$ which are needed in order to obtain the blocks of conserved equal currents, according to (5.39).

G.5.1 Parity even

When we consider the case $\mathcal{O}_1 = \mathcal{O}_2$, the OPE structures have to satisfy extra conditions. The invariance under the exchange of $\mathcal{O}_1$ and $\mathcal{O}_2$ can be easily formulated in terms of the OPE by asking that the a linear combination $\sum_{q=1}^5 a_q t_{l+}^{(q)}$ of the OPE structures is invariant under the map

$$\begin{aligned} x &\rightarrow -x \\ z_1 &\rightarrow I(x) \cdot z_2 \\ z_2 &\rightarrow I(x) \cdot z_1 \end{aligned} \quad (\text{G.25})$$

This automatically gives the following set of constraints on the set of constants

$$\begin{cases} a_1 ((-1)^l - 1) = 0 \\ a_2 ((-1)^l - 1) + 2(a_3 + a_4 + 2a_5)(-1)^l = 0 \\ (a_4 + 2a_5)(-1)^l + a_3 = 0 \\ (a_3 + 2a_5)(-1)^l + a_4 = 0 \\ a_5 ((-1)^l - 1) = 0 \end{cases} \quad (\text{G.26})$$

Solving this set of constraints one can define a new set of allowed tensor structures. From (G.26) it is clear that for $l = 0$ we have just two possible structures, for $l > 0$ and even we have four, while for l odd we always have one.

When $\mathcal{O}_1 = J_1$ is a conserved current, we have

$$\partial_{z_1} \cdot \partial_{x_1} J_1(x_1, z_1) = 0 \quad \implies \quad \partial_{z_1} \cdot \partial_{x_1} \frac{\sum_{q=1}^5 a_q t_{l+}^{(q)}(-x_1, z, z_1, z_2)}{(x_1^2)^{\alpha_+}} = 0 . \quad (\text{G.27})$$

The conservation condition applied to the OPE gives constraint on the allowed combinations of OPE coefficients

$$\begin{cases} -2(\alpha_+ - 1)a_1 - 2(\alpha_+ - 1)a_3 + a_2(-2\alpha_+ + d + l + 1) = 0 \\ -2(\alpha_+ - 2)a_5 + a_4(-2\alpha_+ + 2h + l + 1) + a_1l + a_3 = 0 \end{cases} \quad (\text{G.28})$$

This implies that for $l = 0$ there exists only one allowed structure, for $l = 1$ there are two, and for $l > 1$ there are always three.

To find all the allowed structures for conserved equal currents we can solve simultaneously the systems (G.26) and (G.28). We decide to define the basis τ of the tensor structures for conserved and equal currents as the following linear combination of the basis of generic external vectors

$$\tau_{\Delta l+}^{(i)}(x, z, z_1, z_2) = \sum_{p=1}^5 (m_+)_{ip} t_{+l}^{(p)}(x, z, z_1, z_2) , \quad (i = 1, 2) , \quad (\text{G.29})$$

where

$$m_+ = \begin{pmatrix} (2 - \Delta)(l + \Delta) & (\Delta - l)(l + \Delta) & 2l(\Delta - 2) & 0 & -l(\Delta - 2) \\ l - \Delta + 2 & 0 & -l + \Delta - 2 & \Delta - l & l - \Delta + 1 \end{pmatrix} . \quad (\text{G.30})$$

In particular for odd values of l there are no allowed structures while in the case $l = 0$ only $\tau^{(1)}$ is allowed. Instead for all $l > 0$ even both the structures are allowed.

G.5.2 Parity Odd Structures in three dimensions

In equation (G.5) we defined the leading OPE of two spin one operators J_1 and J_2 with a pseudo-tensor $\mathcal{O}^-$ of spin l and dimension Δ . Notice that we did not include the structures

$$\epsilon(z, z_1, z_2)x^2(x \cdot z)^{l-1} , \quad \epsilon(x, z, z_1)(z \cdot z_2) , \quad (\text{G.31})$$

since they can be written as linear combinations of the previous structures t_{l-} . In fact we have the following two constraints

$$0 = \det \begin{pmatrix} x^2 & x \cdot z & x \cdot z_1 & x \cdot z_2 \\ \left(x\right)_\mu & \left(z\right)_\mu & \left(z_1\right)_\mu & \left(z_2\right)_\mu \end{pmatrix} = \begin{aligned} & x^2 \epsilon(z, z_1, z_2) - (x \cdot z) \epsilon(x, z_1, z_2) \\ & + (x \cdot z_1) \epsilon(x, z, z_2) - (x \cdot z_2) \epsilon(x, z, z_1) , \end{aligned} \quad (\text{G.32})$$

$$0 = \det \begin{pmatrix} z \cdot x & 0 & z \cdot z_1 & z \cdot z_2 \\ \left(x\right)_\mu & \left(z\right)_\mu & \left(z_1\right)_\mu & \left(z_2\right)_\mu \end{pmatrix} = \begin{aligned} & (z \cdot x) \epsilon(z, z_1, z_2) + (z \cdot z_1) \epsilon(x, z, z_2) \\ & - (z \cdot z_2) \epsilon(x, z, z_1) , \end{aligned} \quad (\text{G.33})$$

which reduce the space of six possible tensor structures to just four independent ones. We also remind that in the case of $l = 0$ it exists only the structure $t_{l-}^{(1)}$, while for $l = 1$ we have three possible structures $t_{l-}^{(1)}, t_{l-}^{(2)}, t_{l-}^{(3)}$.

In the case of equal operators we need to find the linear combinations of (G.5) that are invariant under the map (G.25). We obtain that for $l = 0, 1$ we can only have a single structure, while for $l > 1$ there are two. For two different conserved currents one would have one single structure for $l = 0$, two for $l = 1$ and three for $l > 1$. For conserved equal currents in three dimensions we obtain just one structure $\tau_{\Delta l-}$ that takes two different forms for l even and l odd,

$$\tau_{\Delta l-}(x, z, z_1, z_2) = \sum_p (m_{l-})_p t_{l-}^{(p)}(x, z, z_1, z_2) \quad (\text{G.34})$$

with

$$m_{l-} = \begin{cases} (\Delta - 3, l, l, 0) & l \text{ even,} \\ (0, \Delta - l - 3, \Delta + l + 1, 1 - \Delta) & l > 1, \text{ odd.} \end{cases} \quad (\text{G.35})$$

We remark that for the special case $l = 1$ there are no allowed tensor structures. For $l = 0$ instead $\tau_{\Delta l-}$ is still allowed.

G.6 Comments on the recurrence relation

In this appendix we explain some technical developments on the recurrence relations which were understood in the present work. The comments in these sections may be important in order to fulfill the future goal of computing the full set of conformal blocks needed to implement the numerical bootstrap in any spacetime dimension and for any “interesting” external operator.

G.6.1 Simplifications of the recurrence relations

In this appendix we mention some interesting ways to simplify the recurrence relations which were not adopted in this work, but could be very important for future computations.

The first remarkable simplification of the recurrence relation comes from equation (C.16). Using the basis of the differential operators it is possible to define a fully diagonal recurrence relation of the form

$$\mathfrak{h}_{\Delta \ell, s}^{(p, q)}(r, \eta) = \mathfrak{h}_{\infty \Delta \ell, s}^{(p, q)}(r, \eta) + \sum_A (4r)^{n_A} \frac{\mathfrak{R}_{\ell_A}^{(p, q)}}{\Delta - \Delta_A^*} \mathfrak{h}_{\Delta_A \ell_A, s}^{(p, q)}(r, \eta). \quad (\text{G.36})$$

where $\mathfrak{R}_A^{(p, q)}$ is equal to the constant R_A written in (4.36) after shifting Δ_{12} and Δ_{34} by some units (fixed in terms of p and q). We use the notation $\mathfrak{h}$ instead of h to make clear that we are using a different basis for the three point functions. From (C.16) it is also clear that the labels A, ℓ_A, n_A and Δ_A appearing in this recurrence relation would be the same of the scalar relation (4.35). The form (G.36) should be much more convenient to compute complicated conformal blocks (for example the four external stress tensors ones) since it decouples all the recurrence relations. However, in order to obtain such a beautiful result, one has to pay a price (which should be still worth to pay in most cases). In fact in this basis the function $\mathfrak{h}_\infty$ is a polynomial

in Δ . The order of such polynomial depends on the choice of the external operators and it is related to the matrix that changes basis from the usual Δ -independent three point functions basis (for example the leading OPE basis (G.4) used in this paper) to the differential one (which is Δ -dependent). The fact that $\mathfrak{h}_\infty$ is a polynomial in turns implies that to obtain $\mathfrak{h}_\infty$ we need to solve the Casimir equation at sub-leading orders in large Δ .

Similarly one could implement the recurrence relation for the conformal blocks directly in the basis of conserved equal currents. This was already successfully tried for one external conserved current in generic dimension (4.4.2). This would also give rise to important simplifications since we would just have 5 coupled recurrence relations instead of 41 (an even more drastic simplification would appear in the case of four external stress tensors). However, also in this case, the behaviour at large Δ of the blocks would change. In particular since the coefficients of (G.30) and (G.35) are polynomial of order two in Δ (and we have to contract twice these matrices with the conformal blocks), we would need to solve the Casimir equation at four subleading orders in large Δ .

It is also interesting to notice that one could implement the recurrence relation for the conformal blocks in the basis of equal vectors obtained by solving for example (G.26). This basis is very convenient since it does not depend on Δ and therefore it does not modify the behaviour at large Δ of the conformal blocks. It is therefore very simple to implement such a change and it would introduce a significant simplification.

In this work we decided to stick the most conservative recurrence relation valid for any external vector even if we had all the ingredients necessary to apply any of the previous simplifications (in fact we managed to solve the Casimir equation to many subleading orders in large Δ). We did not further analyzed the simplified recurrence relations since the actual algorithm was already efficient enough for the computations done in this work. However it would be very interesting to obtain very efficient formulas for this and for more complicated conformal blocks valid in any dimensions. To do so it would be worth to implement the ideas mentioned in this appendix.

G.6.2 A new implementation of the recurrence relation

In this section we want to comment on a new way to implement the recurrence relations, which was adopted in this work. We will exemplify the method for the case of four external vectors in three dimensions, but one can implement it also in any other case. It is convenient to introduce the notation

$$h_{\Delta\ell\pm,s}^{(p,q)}(r,\eta) \equiv \sum_{m=0}^{\infty} r^m h_{\Delta\ell\pm,s}^{(p,q)}[m](\eta) \quad (\text{G.37})$$

where $h_{\Delta\ell\pm,s}^{(p,q)}[m](\eta)$ are the coefficient of the expansion in r of the conformal blocks, which are just functions of η . We shall drop the dependence on η in the following formulae for convenience in the notation. The recurrence relation can be easily casted in the following r -independent

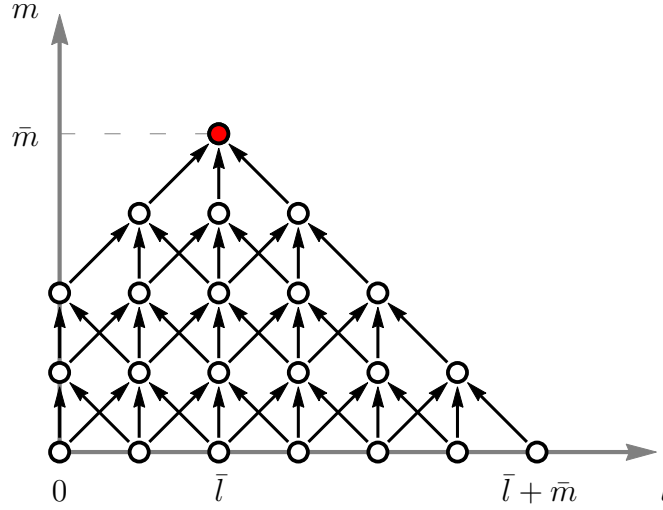


Figure G.1: In this plot it is shown which information it is used to compute a block $h_{\Delta\bar{l},s}^{(p,q)}[\bar{m}]$, represented as the red dot. In order to build this block one needs to know all the $h_{\Delta l,s}^{(p',q')}[m]$ for $l = \max[0, \bar{l} - \bar{m}], \dots, \bar{l} + \bar{m}$ and $m = \bar{m} - |l - \bar{l}|$.

form¹

$$h_{\Delta l_{\pm},s}^{(p,q)}[m] = h_{\infty l_{\pm},s}^{(p,q)}[m] + \sum_A \sum_{p',q'} (4)^{n_A} \frac{(R_{\pm A})_{pp'qq'}}{\Delta - \Delta_A^*} h_{\Delta_A l_{A\pm A},s}^{(p',q')}[m - n_A]. \quad (\text{G.38})$$

Since $n_A > 1$ it is clear how to use the formula to build $h[\bar{m}]$ from the knowledge of $h[m]$ with $m < \bar{m}$. However it is interesting to notice that the blocks $h_{\Delta l_{\pm},s}^{(p,q)}[m]$ for $m < \bar{m}$ are called in the recurrence relation with different spins l_A . In particular in order to know $h_{\Delta\bar{l}+,s}^{(p,q)}[\bar{m}]$ (or $h_{\Delta\bar{l}-,s}^{(p,q)}[\bar{m}]$) one needs to know² $h_{\Delta l_{\pm},s}^{(p',q')}[\bar{m} - 1]$ with $l = \bar{l} - 1, \bar{l}, \bar{l} + 1$, which can be computed knowing $h_{\Delta l_{\pm},s}^{(p'',q'')}[\bar{m} - 2]$ with $l = \bar{l} - 2, \bar{l} - 1, \bar{l}, \bar{l} + 1, \bar{l} + 2$ and so on, until $h_{\Delta l_{\pm},s}^{(p''',q''')}[0]$ with $l = \max[0, \bar{l} - \bar{m}], \dots, \bar{l} + \bar{m}$. Therefore formula (G.38) can be conveniently implemented as a recurrence relation in spin. In figure G.1 it is shown how to build $h_{\Delta\bar{l}+,s}^{(p,q)}[\bar{m}]$ using the information of the conformal blocks at different spins and lower value of m .

The code that we implemented uses this strategy to compute all the blocks with various spins at once. Given the inputs of $\bar{m}$ and $\bar{l}$, it starts by computing all the blocks for $m = 0$ and $l = 0, \dots, \bar{l} + \bar{m}$, then it computes all the blocks with $m = 1$ and $l = 0, \dots, \bar{l} + \bar{m} - 1$, and so on, until one obtain the blocks with $m = \bar{m}$ and $l = 0, \dots, \bar{l}$. The output of the algorithm is therefore a list of all the blocks with $m \leq \bar{m}$ and $l \leq \bar{l}$, but it also generates blocks with higher spins and lower m . With this algorithm we were able to obtain the blocks with $\bar{m} = 50$ and $\bar{l} = 40$, as showed in figure G.2. Each dot in the picture corresponds to 41 exact functions of Δ and η : $h_{\Delta l_{\pm},s}^{(p,q)}[m]$ for $p, q = 1, \dots, 5$ and $h_{\Delta l_{\pm},s}^{(p,q)}[m]$ for $p, q = 1, \dots, 4$ for a given structure s . This plot should be repeated 19 times, which correspond to the number of structures s that we considered.

¹Formula (G.38) is a more compact way to write (4.91-4.92). We hope not to confuse the reader with this change of notation. The symbol $\pm_A = \pm$ for the first three types, while $\pm_{IV,n} = \mp$.

²We put primes in the label p and q to stress that we need all the blocks for any p and q , since (G.38) couples those labels. On the other hand s is diagonal so we can study one vale of s at the time.



Figure G.2: Schematic representation of the computation of the blocks performed for four equal external vectors in three dimensions.

G.6.3 General V.S. integer dimensions

In this section comment on some subtleties involving the dependence on d of the recurrence relations. In particular we are interested on the difference features of the recurrence relations if we fix the spacetime dimension to an odd integer or if we consider generic d (the even dimensional case is obtained as a limit of the generic d case).

First we want to comment on the types I_k, II_k . We want to show that there may exist more poles associated to these types when we do not fix the spacetime dimension. In fact the label k is bounded to be in the set $1, \dots, [\frac{d}{2}]$. While for integer d this range is finite, when d is not fixed this range is virtually infinite. When d is not fixed the range only truncates because of the choice of the external operators. In fact, at a given pole of type A , the residue is a conformal block $h_{\mathcal{O}_A}$. However $\mathcal{O}_A$ may be labeled by an $SO(d)$ representation which cannot possibly couple to the external operators, namely $\mathcal{O}_A \notin (\mathcal{O}_1 \times \mathcal{O}_2) \cap (\mathcal{O}_3 \times \mathcal{O}_4)$. This phenomenon is particularly relevant for the types (I_k, n) (or (II_k, n)) when $k > 1$. In fact in these cases the $SO(d)$ representation of $\mathcal{O}_A$ is different from the one of $\mathcal{O}$, since the Young tableau associated to $\mathcal{O}_A$ has n more boxes (or n less boxes) in its k -th row. As an example, if we consider any choice of traceless and symmetric external operators in any dimension, the label k is bounded to take the values $k = 1, 2, 3$. When d is fixed to an odd integer instead we have a finite number of k independently of the choice of the external operators. For example in three dimensions the types I_k and II_k exist just for $k = 1$ and similarly for $d = 5$ we only have $k = 1, 2$.

The reverse phenomenon happens for the type IV. In generic dimensions the type (IV, n) cannot appear since n runs in the set $[1, l_{[\frac{d}{2}]}]$. In fact, since d is not specified, the label $l_{[\frac{d}{2}]}$ of the exchanged operator has to vanish for any choice of the external operators. Instead, for d

integer, $l_{[\frac{d}{2}]}$ is well defined and the poles of type IV may appear depending on the choice of the external operators.

The fact that some poles exist only in integer dimensions and that others exist only when d is not integer may seem paradoxical, since we claim that the recurrence relations are analytic functions of d . The resolution of the paradox comes from the fact that the poles of type I_k, II_k which disappear at integer values of d , are exactly replaced by the poles of type IV (which exist only at d integer).

For example in the case of 4 external vectors in generic d , beside the poles of the type I_1, II_1 and III (which are present also in the scalar case), there exist also new poles coming from the type $I_2, n = 1, 2, II_2, n = 1, 2, I_3, n = 1, II_3, n = 1$. We can read their position from the table (4.94). Once we set the value of $d = 3$, we obtain three new poles at positions 0, 1, 2. On the other hand when we consider the table (4.94) directly in three dimensions, we predict l new poles of type IV at positions $1, 0, -1, \dots, -l + 2$. One can check that, rather magically, the residues R correspondent to the type IV vanish for $n > 2$ giving allowing poles only at 1, 0. In a similar way one can prove that the pole at position 2 predicted in general dimension has a vanishing residue when $d = 3$.

Appendix H

CBs in the leading OPE limit

While CBs are usually complicated functions, we can easily obtain them in one limit. In fact the hard part of the CBs comes from the fact that they resum the infinite series of the OPE which encode the contribution of all the descendant states. However we can consider their limit when just the primary is exchanged, this is called the leading OPE limit of the CBs.

H.1 Generic CB for external operators with spin

This limit is obtained by considering the OPE of the two pairs of operators $\mathcal{O}_1, \mathcal{O}_2$ and $\mathcal{O}_3, \mathcal{O}_4$, which therefore implies the following behavior of the cross ratios

$$x_{12}, x_{34} \rightarrow 0 \implies \begin{cases} r \rightarrow 0 \\ \eta \approx -\hat{x}_{12} \cdot I(x_{24}) \cdot \hat{x}_{34} \end{cases} . \quad (\text{H.1})$$

In fact we have $x_{13} \approx x_{23} \approx x_{24} \approx x_{14}$ and $(4r)^2 \approx x_{12}^2 x_{34}^2 / (x_{24}^2)^2$.

We consider the four point function (2.66) of traceless and symmetric operators $\mathcal{O}_i$ with dimensions Δ_i and spin l_i . We want to study the leading OPE of the conformal block $G_{\mathcal{O}}^{(p,q)}$ associated to the exchange of an operator $\mathcal{O}$ labeled by Δ and the $SO(d)$ representation $\mathbf{Y}$, where the labels p, q are defined by (2.64) and the two-point function are normalized as (2.48). In this limit we schematically get $\mathcal{O}_1 \times \mathcal{O}_2 \sim t^{(p)} \mathcal{O}$ and $\mathcal{O}_3 \times \mathcal{O}_4 \sim t^{(q)} \mathcal{O}$, which give the following result

$$G_{\mathcal{O}}^{(p,q)}(\{P_i, Z_i\}) \approx (4r)^\Delta \pi(I(x_{24}) \cdot \mathbf{Y}', \mathbf{Y}) \frac{t^{(p)}(\hat{x}_{12}, z_1, z_2, \mathbf{Y}') t^{(q)}(\hat{x}_{34}, z_3, z_4, \mathbf{Y})}{(x_{12}^2)^{\frac{\Delta_1 + \Delta_2}{2}} (x_{34}^2)^{\frac{\Delta_3 + \Delta_4}{2}}} . \quad (\text{H.2})$$

The first simple remark is that the leading term of the CB scale like r^Δ . We can therefore define a new function $h_{\mathcal{O},s}^{(p,q)}(r, \eta)$ which is finite for $r \rightarrow 0$,

$$g_{\mathcal{O},s}^{(p,q)}(r, \eta) = (4r)^\Delta h_{\mathcal{O},s}^{(p,q)}(r, \eta) \quad (\text{H.3})$$

On the other hand we can apply the OPE to the conformal block decomposition (2.74). To do

so, we write the leading OPE limit of the structures $V_{i,jk}$, H_{ij} ,

$$\begin{aligned} V_{a,bi} &\approx z_a \cdot \hat{x}_{ab} , & V_{i,ja} &\approx z_i \cdot \hat{x}_{ij} , \\ V_{i,12} &\approx -z_i \cdot I(x_{24}) \cdot \hat{x}_{12} , & V_{a,34} &\approx -z_a \cdot I(x_{24}) \cdot \hat{x}_{34} , \\ H_{12} &\approx z_1 \cdot I(x_{12}) \cdot z_2 , & H_{34} &\approx z_3 \cdot I(x_{34}) \cdot z_4 , \\ H_{ai} &\approx z_a \cdot I(x_{24}) \cdot z_i , \end{aligned} \quad (\text{H.4})$$

where $a, b \in \{1, 2\}$ and $i, j \in \{3, 4\}$. Using (H.4) one can determine $h_{\mathcal{O},s}^{(p,q)}(0, \eta)$ by matching the two sides of the equation

$$\pi(I(x_{24}) \cdot \mathbf{Y}', \mathbf{Y}) t^{(p)}(\hat{x}_{12}, z_1, z_2, \mathbf{Y}') t^{(q)}(\hat{x}_{34}, z_3, z_4, \mathbf{Y}) \approx \sum_s h_{\mathcal{O},s}^{(p,q)}(0, \eta) Q_s(\{P_i, Z_i\}) . \quad (\text{H.5})$$

Notice that the left hand side of (H.5) can be trivially written in a closed form if we know how to write the projector into $\mathbf{Y}$. Therefore we are able to find the leading OPE of all the conformal blocks which exchange operators in the representations enumerated in section 2.1.3.2.

H.1.1 Traceless and symmetric exchange

Let us consider the case of a conformal block $G_{\mathcal{O}}^{(p,q)}(\{P_i, Z_i\})$, where $\mathcal{O}$ has dimension Δ and spin l . In order to agree with the chapter (4) we define here the leading OPE as follows,

$$\mathcal{O}_1(x_1, z_1) \mathcal{O}_2(x_2, z_2) \approx \frac{1}{l!(h-1)_l} \frac{\mathcal{O}(x_2, D_z)}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2-\Delta}{2}}} \sum_q c_{12\mathcal{O}}^{(q)} t_l^{(q)}(-\hat{x}_{12}, z, I(x_{12}) \cdot z_1, z_2) . \quad (\text{H.6})$$

where $\mathcal{O}$ has spin l and conformal dimension Δ . With this convention (H.5) reduces to

$$\frac{t_l^{(p)}(\hat{x}_{12}, I(x_{24}) \cdot D_z, I(x_{12}) \cdot z_1, z_2) t_l^{(q)}(\hat{x}_{34}, z, I(x_{34}) \cdot z_3, z_4)}{l!(h-1)_l} \approx \sum_s f_{l,s}^{(p,q)}(\eta) Q_s , \quad (\text{H.7})$$

where $f_{l,s}^{(p,q)}(\eta) \equiv h_{\Delta l, s}^{(p,q)}(0, \eta)$.

We now want to understand how it is possible to find the structures listed above on the left hand side of (H.7). The simplest building blocks that appear in the left hand side of (H.7) are all the scalar combination of left variables $\hat{x}_{12}, I(x_{12}) \cdot z_1, z_2$ only, and that of the the right variables $\hat{x}_{34}, I(x_{34}) \cdot z_3, z_4$ only. These combinations generates $\mathfrak{v}_{a,bi}, \mathfrak{v}_{i,ja}, \mathfrak{h}_{12}, \mathfrak{h}_{34}$. We can also build more complicated structures that mix the left variables with the right ones. In fact when a combination $\hat{x}_{12} \cdot I(x_{24}) \cdot D_z, z_1 \cdot I(x_{12}) \cdot I(x_{24}) \cdot D_z, z_2 \cdot I(x_{24}) \cdot D_z$ appears on the left structure, it then acts on one of the three possible combinations on the right $\hat{x}_{34} \cdot z, z_3 \cdot I(x_{34}) \cdot z, z_4 \cdot z$. This action creates structures that can be written as linear combinations of (H.4), with the single exception of $\hat{x}_{12} \cdot I(x_{24}) \cdot \hat{x}_{34}$. In fact,

$$\eta \approx -\hat{x}_{12} \cdot I(x_{24}) \cdot \hat{x}_{34} , \quad (\text{H.8})$$

thus this combination actually gives the dependence on η , which is needed for the identifications

of the functions $f_{l,s}^{(p,q)}(\eta)$.

In order to find the closed form expression of $f_{l,s}^{(p,q)}(\eta)$ for any l it is important to use a closed form formula for the projector onto symmetric and traceless representation (2.16) as we did for the scalar case in section (4.2.3). Actually for generic structures $t^{(p)}$ and $t^{(q)}$ we need to compute expression of the form

$$\mathcal{F}(\{u_i, v_i\}, x, y) = \frac{(u_1 \cdot D_z) \cdots (u_j \cdot D_z)(x \cdot D_z)^{l-j}}{l!(h-1)_l} (y \cdot z)^{l-k} (v_1 \cdot z) \cdots (v_k \cdot z) . \quad (\text{H.9})$$

However we can compute (H.9) after rewriting it as a set of derivative acting on $\frac{(x \cdot D_z)^l}{l!(h-1)_l} (y \cdot z)^l$, which is known in a closed form from equation (2.16),

$$\begin{aligned} \mathcal{F}(\{u_i, v_i\}, x, y) &= \left(\prod_{i=1}^j \frac{(u_i \cdot \partial_x)}{l-i+1} \right) \left(\prod_{i=1}^k \frac{(v_i \cdot \partial_y)}{l-i+1} \right) \frac{(x \cdot D_z)^l (y \cdot z)^l}{l!(h-1)_l} \\ &= \left(\prod_{i=1}^j \frac{(u_i \cdot \partial_x)}{l-i+1} \right) \left(\prod_{i=1}^k \frac{(v_i \cdot \partial_y)}{l-i+1} \right) \frac{l!(x^2 y^2)^{l/2} C_l^{h-1}(\hat{x} \cdot \hat{y})}{2^l (h-1)_l} . \end{aligned} \quad (\text{H.10})$$

Moreover one can use the identities for derivatives of Gegenbauer polynomial such as

$$\partial_x C_l^h(x) = 2h C_{l-1}^{h+1}(x) . \quad (\text{H.11})$$

Putting together all these ingredients one can find the functions $f_{l,s}^{(p,q)}(\eta)$ that define the OPE limit of the conformal blocks for any traceless and symmetric operator exchange. We can generalize this idea to other exchanges by using the opportune projector.

H.1.2 Exchange of $(l, 1)$

In this section we consider the example of the four point function with two vector operators at the points x_1 and x_3 and two scalars. In particular we will refer to the notation of appendix E. From the previous section we can obtain the blocks $\hat{h}_{(\Delta,l),s}^{(p,q)}(0, \eta)$ for symmetric and traceless exchange using the definition of the leading OPE structures $t_l^{(1)}$ and $t_l^{(2)}$ given in (E.10). Instead, the case of the exchange of $(\Delta, l, 1)$ is new. We can implement the contraction of the indices of (H.5) in two ways, either using the operators $D_{z,w}$, $D_{w,z}$ of (E.5), or using the closed form for the projector proposed in (3.56). Following the first way we obtain

$$\begin{aligned} \mathcal{N}_{l,1} t_l^{(0)}(\hat{x}_{12}, I(x_{24}) \cdot D_{z,w}, I(x_{24}) \cdot D_{w,z}, I(x_{12}) \cdot z_1) t_l^{(0)}(\hat{x}_{34}, z, \omega, I(x_{34}) \cdot z_3) \\ \approx \sum_s \hat{h}_{(\Delta,l,1),s}(0, \eta) Q_s , \end{aligned} \quad (\text{H.12})$$

where $t_l^{(0)}$ is defined in (E.10). Here we introduced the factor $\mathcal{N}_{l,1} \equiv \frac{4l(-1)^{-l-1}}{(l+1)!(h-1)_l(2h-4)_{l+2}}$ in order to properly normalize the projector into the $(l, 1)$ irrep, as shown in [33]. Equivalently, we can

use the closed form representation (3.56) of the double vector harmonic

$$(I_\mu^\beta(x_{24})I_{\beta\alpha}(\hat{x}_{12})z_1^\alpha)(I_{\nu\delta}(\hat{x}_{34})z_3^\delta)\mathcal{V}^{\mu\nu}(I(x_{24})\cdot\hat{x}_{12},\hat{x}_{34})\approx\sum_s\hat{h}_{(\Delta,l,1),s}(0,\eta)Q_s. \quad (\text{H.13})$$

From these formulas one can completely fix $\hat{h}_{(\Delta,l,1),s}(0,\eta)$ as

$$\begin{aligned} \hat{h}_{(\Delta,l,1),2}(0,\eta) &= \hat{h}_{(\Delta,l,1),5}(0,\eta) = c_l C_{l-2}^{(h+1)}(\eta), \\ \hat{h}_{(\Delta,l,1),4}(0,\eta) &= \frac{2(1-h)}{l-1} c_l C_{l-3}^{(h+1)}(\eta) - \frac{(-2h-l+3)}{l-1} \hat{h}_{(\Delta,l,1),3}(0,\eta), \\ \hat{h}_{(\Delta,l,1),3}(0,\eta) &= \eta \hat{h}_{(\Delta,l,1),2}(0,\eta), \\ \hat{h}_{(\Delta,l,1),1}(0,\eta) &= \eta \hat{h}_{(\Delta,l,1),4}(0,\eta) - \hat{h}_{(\Delta,l,1),2}(0,\eta), \end{aligned} \quad (\text{H.14})$$

where $c_l = -\frac{(-2)^{1-l}(l-1)!}{(2h+l-3)(h+1)_{l-2}}$.

Appendix I

Further Discussions on Jantzen's Criterion

In this appendix we discuss the Jantzen's criterion in more detail (see section 4.5.1).

First notice that for any weight λ that belongs to a wall Ω_γ , its Weyl reflection $s_\beta \cdot \lambda$ is contained in the Weyl reflected wall $\Omega_{s_\beta(\gamma)}$, where $s_\beta(\gamma)$ is the undotted Weyl action

$$s_\beta(\gamma) \equiv \gamma - 2 \frac{\langle \gamma, \beta \rangle}{\langle \beta, \beta \rangle} \beta . \quad (\text{I.1})$$

This simple observation is useful to obtain a very efficient way to see when (4.102) holds. First we check if λ belongs to some wall Ω_γ for some root γ . As we already explained in section 4.5.1, if such a wall does not exist either Ψ_λ^+ is empty and $M_{\mathfrak{p}}(\lambda)$ is trivially simple, or Ψ_λ^+ is not empty and $M_{\mathfrak{p}}(\lambda)$ is trivially not simple. Then we apply Weyl transformations s_β for $\beta \in \Psi_\lambda^+$ to the wall Ω_γ . This generates a new set of walls Ω_{γ_β} with

$$\Omega_{\gamma_\beta} \equiv \Omega_{s_\beta(\gamma)} , \quad \beta \in \Psi_\lambda^+ . \quad (\text{I.2})$$

The last step is to find for each $\beta \in \Psi_\lambda^+$ a Weyl transformation $\omega_\beta \in W_{\mathfrak{p}}$ with odd length that acts on Ω_{γ_β} , in such a way the set of walls maps to itself

$$\{\Omega_{\omega_\beta(\gamma_\beta)}\}_{\beta \in \Psi_\lambda^+} = \{\Omega_{\gamma_\beta}\}_{\beta \in \Psi_\lambda^+} . \quad (\text{I.3})$$

If this set of ω_β exists, then $M_{\mathfrak{p}}(\lambda)$ is simple. As we will see there is one more simplification: anytime that $\gamma_\beta \in \Phi_{\mathfrak{p}}$ then it is trivial to find a ω_β such that γ_β is invariant under ω_β . Thus one needs to find a transformation ω_β to fulfill (I.3) only for the roots $\gamma_\beta \notin \Phi_{\mathfrak{p}}$.

The full criterion can be written as follows. When $\lambda \in \Omega_\gamma$, the parabolic Verma module $M_{\mathfrak{p}}(\lambda)$ is simple if for any $\beta \in \Psi_\lambda^+$, it exists $\omega_\beta \in W_{\mathfrak{p}}$ with odd length such that

$$\{\omega_\beta(s_\beta(\gamma))\}_{\beta \in \Psi_\lambda^+} = \{\pm s_\beta(\gamma)\}_{\beta \in \Psi_\lambda^+} . \quad (\text{I.4})$$

Where $\pm$ means that the two sets of roots have to be equal up to signs since $\Omega_\gamma = \Omega_{-\gamma}$.

I.0.3 Example: $\mathfrak{so}(5)$

Following the algorithm (I.4) we first find that $\Psi_\lambda^+ = \{(1, 1), (1, 0)\}$. Then we obtain the roots that parametrize the walls reflected by the two Weyl reflections s_β with $\beta \in \Psi_\lambda^+$

$$\begin{aligned} s_{(1,1)}((1, -1)) &= (1, -1) , \\ s_{(1,0)}((1, -1)) &= (-1, -1) . \end{aligned} \tag{I.5}$$

As a last step we check that the walls $\Omega_{(1,-1)}$ and $\Omega_{(-1,-1)}$ can be reflected into each other by a Weyl transformation of $W_{\mathfrak{p}}$ with odd length. In fact $s_{(0,1)}((1, -1)) = -(-1, -1)$. Again the minus sign is not important since $\Omega_\gamma = \Omega_{-\gamma}$.

I.0.4 Odd Spacetime Dimension

We first consider the case in which λ belongs to the wall of the Weyl chamber $\Omega_{\alpha_{0k}^{++}}$, given by

$$\Delta = 2N + 1 + l_k - k \quad (k = 1, \dots, N) . \tag{I.6}$$

We obtain $\Psi_\lambda^+ = \{\alpha_{01}^{++}, \alpha_{02}^{++}, \dots, \alpha_{0,k-1}^{++}\}$. Notice that $s_{\alpha_{0j}^{++}}(\alpha_{0k}^{++}) = \alpha_{jk}^{-+}$ for any $1 \leq j < k \leq N$. Therefore each reflected wall is parametrized by a root α_{jk}^{-+} , which can be Weyl reflected to itself using a Weyl symmetry $s_{\alpha_{jk}^{++}} \in W_{\mathfrak{p}}$ with odd length. Therefore $M_{\mathfrak{p}}(\lambda)$ is simple. The other special case is when λ belongs to the wall $\Omega_{\alpha_{0k}^{-+}}$,

$$\Delta = k - l_k \quad (k = 1, 2, \dots, N) . \tag{I.7}$$

In this case $\Psi_\lambda^+ = \{\alpha_0^+, \alpha_{01}^{++}, \alpha_{02}^{++}, \dots, \alpha_{0N}^{++}, \alpha_{0,k+1}^{+-}, \dots, \alpha_{0N}^{+-}\}$. When we Weyl-reflect the wall $\Omega_{\alpha_{0k}^{-+}}$ with s_β for all the roots $\beta \in \Psi_\lambda^+$, we obtain walls parametrized by the following roots:

$$s_{\alpha_{0j}^{+-}}(\alpha_{0k}^{-+}) = \alpha_{jk}^{-+} \quad (j \neq k) , \tag{I.8}$$

$$s_{\alpha_{0j}^{++}}(\alpha_{0k}^{-+}) = \alpha_{jk}^{++} \quad (j \neq k) , \tag{I.9}$$

$$s_{\alpha_{0k}^{++}}(\alpha_{0k}^{-+}) = \alpha_{0k}^{-+} , \tag{I.10}$$

$$s_{\alpha_0^+}(\alpha_{0k}^{-+}) = \alpha_{0k}^{++} . \tag{I.11}$$

Notice that the cases (I.8) and (I.9) are trivial since the transformed roots live in $\Phi_{\mathfrak{p}}$, so we can map each one of them to itself using a transformation in $W_{\mathfrak{p}}$ with odd length, namely $s_{\alpha_{jk}^{++}}(\alpha_{jk}^{+-}) = \alpha_{jk}^{+-}$ and $s_{\alpha_{jk}^{+-}}(\alpha_{jk}^{++}) = \alpha_{jk}^{++}$. The two remaining cases (I.10) and (I.11) are less trivial because the roots α_{0k}^{-+} and α_{0k}^{--} do not live in $\Phi_{\mathfrak{p}}$. This means that we cannot map each one to itself, but we can still map one into the other $s_{\alpha_k^+}(\alpha_{0k}^{-+}) = -\alpha_{0k}^{++}$ (up to a sign, which is irrelevant since $\Omega_\alpha = \Omega_{-\alpha}$). Therefore $M_{\mathfrak{p}}(\lambda)$ is simple.

I.0.5 Even Spacetime Dimension

The even dimensional case looks much less straightforward then the odd dimensional one. This is both because the roots α_k^+ are absent and because l_N can now be also negative. The result can be sketched as follows

$$\lambda \in \Omega_{\alpha_{0k}^{++}} = \{\lambda : \Delta = 2N + l_k - k\} \implies \begin{array}{|c|cc|} \hline & l_N \geq 0 & l_N < 0 \\ \hline k = 1, \dots, N-1 & \text{simple} & \text{simple} \\ k = N & \text{simple} & \text{non simple} \\ \hline \end{array} \quad (\text{I.12})$$

$$\lambda \in \Omega_{\alpha_{0k}^{+-}} = \{\lambda : \Delta = k - l_k\} \implies \begin{array}{|c|cc|} \hline & l_N > 0 & l_N \leq 0 \\ \hline k = 1, \dots, N-1 & \text{non simple} & \text{non simple} \\ k = N & \text{non simple} & \text{simple} \\ \hline \end{array} \quad (\text{I.13})$$

Let us first consider the case in which λ belongs to the wall $\Omega_{\alpha_{0k}^{++}}$ with $k = 1, \dots, N$. We find that

$$\Psi_\lambda^+ = \begin{cases} \{\alpha_{01}^{++}, \alpha_{02}^{++}, \dots, \alpha_{0,N-1}^{++}, \alpha_{0,N}^{+-}\} & \text{if } l_N < 0 \text{ and } k = N, \\ \{\alpha_{01}^{++}, \alpha_{02}^{++}, \dots, \alpha_{0,k-1}^{++}\} & \text{otherwise.} \end{cases} \quad (\text{I.14})$$

As we already discussed in the odd dimensional case, all the roots α_{0j}^{++} for $j = 1, \dots, k$ trivially satisfy Jantzen's criterion since $s_{\alpha_{0j}^{++}}(\alpha_{0k}^{++}) \in \Phi_{\mathfrak{p}}$. Thus when we only have roots α_{0j}^{++} with for $j = 1, \dots, k$, the module $M_{\mathfrak{p}}(\lambda)$ is simple. When $l_N < 0$ and $k = N$, we also have the root α_{0N}^{+-} which reflects the wall as $s_{\alpha_{0N}^{+-}}(\alpha_{0N}^{++}) = \alpha_{0N}^{++}$. Clearly there is no odd transformation in $W_{\mathfrak{p}}$ which maps this wall to itself. We can then conclude that the module is not simple. The other special case is when λ belongs to the wall $\Omega_{\alpha_{0k}^{+-}}$. We obtain

$$\Psi_\lambda^+ = \begin{cases} \{\alpha_{01}^{++}, \alpha_{02}^{++}, \dots, \alpha_{0,N-1}^{++}\} & \text{if } l_N \leq 0 \text{ and } k = N, \\ \{\alpha_{01}^{++}, \alpha_{02}^{++}, \dots, \alpha_{0N}^{++}, \alpha_{0,k+1}^{+-}, \dots, \alpha_{0N}^{+-}\} & \text{otherwise.} \end{cases} \quad (\text{I.15})$$

When we Weyl-reflect the wall $\Omega_{\alpha_{0k}^{+-}}$ with s_β for all the roots $\beta \in \Psi_\lambda^+$, we obtain walls parametrized by the following roots

$$s_{\alpha_{0j}^{+-}}(\alpha_{0k}^{+-}) = \alpha_{jk}^{-+} \quad (j = 1, \dots, N, j \neq k), \quad (\text{I.16})$$

$$s_{\alpha_{0j}^{++}}(\alpha_{0k}^{+-}) = \alpha_{jk}^{++} \quad (j = 1, \dots, N, j \neq k), \quad (\text{I.17})$$

$$s_{\alpha_{0k}^{++}}(\alpha_{0k}^{+-}) = \alpha_{0k}^{-+}. \quad (\text{I.18})$$

Again (I.16) and (I.17) are trivial since they live in $\Phi_{\mathfrak{p}}$. Only the single root α_{0k}^{-+} does not belong to $\Phi_{\mathfrak{p}}$. Since there are no more candidate roots in Ψ_λ^+ which we can map into α_{0k}^{-+} under a weyl reflection in $W_{\mathfrak{p}}$, we conclude that $M_{\mathfrak{p}}(\lambda)$ is not simple. On the other hand when $l_N \leq 0$ and $\lambda \in \Omega_{\alpha_{0N}^{+-}}$, it is crucial that the root α_{0N}^{++} is not in Ψ_λ^+ . Therefore in this case $M_{\mathfrak{p}}(\lambda)$ is simple.

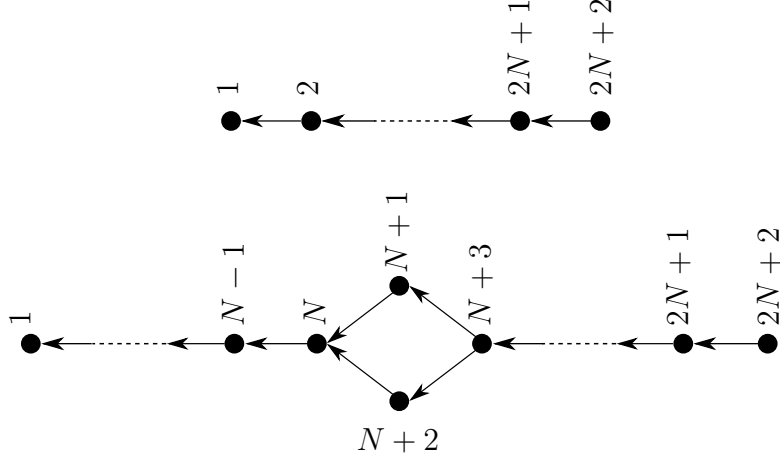


Figure I.1: Bruhat ordering for elements of $W_{\mathfrak{p}} \setminus W$, for $B_{N+1} = \mathfrak{so}(2N+3)$ (above) and $D_{N+1} = \mathfrak{so}(2N+2)$ (below).

I.1 The Kazhdan-Lusztig Conjecture

In this appendix let us be more explicit about the KL polynomial $P_{w,x}(q)$ in (4.107).

I.1.1 Odd Spacetime Dimension

Let us first consider the odd-dimensional case: $\mathfrak{so}(2N+3)$. Then $W_{\mathfrak{p}} \setminus W$ contains $2N+2$ elements $w_1, \dots, w_{2N+2}$, and the Bruhat ordering among them is shown in figure I.1, namely $w_i \leq w_j$ when $i \leq j$.

In this case, the value of the KL polynomial at $q = 1$ is known to be [83, section 5]

$$P_{w_i, w_j}(q = 1) = \begin{cases} 1 & (w_i \geq w_j) , \\ 0 & (\text{otherwise}) . \end{cases} \quad (\text{I.19})$$

This means

$$\text{ch } L(w_i \cdot \lambda) = \sum_{j \leq i} (-1)^{j-i} \text{ch } M_{\mathfrak{p}}(w_j \cdot \lambda) , \quad (\text{I.20})$$

and hence by inverting the relations we obtain

$$\text{ch } M_{\mathfrak{p}}(w_1 \cdot \lambda) = \text{ch } L(w_1 \cdot \lambda) , \quad (\text{I.21})$$

$$\text{ch } M_{\mathfrak{p}}(w_i \cdot \lambda) = \text{ch } L(w_i \cdot \lambda) + \text{ch } L(w_{i-1} \cdot \lambda) \quad (i \geq 2) . \quad (\text{I.22})$$

Namely each generalized Verma module contains either one or two irreducible components, and we do not need to worry about the null descendants within null descendants. This explains the absence of higher order poles in the corresponding conformal blocks.

Let us take the example of $\mathfrak{so}(5)$. The Bruhat ordering is shown in picture I.2. Notice that the chambers below the origin do not have any label because there is no element in $W_{\mathfrak{p}} \setminus W$

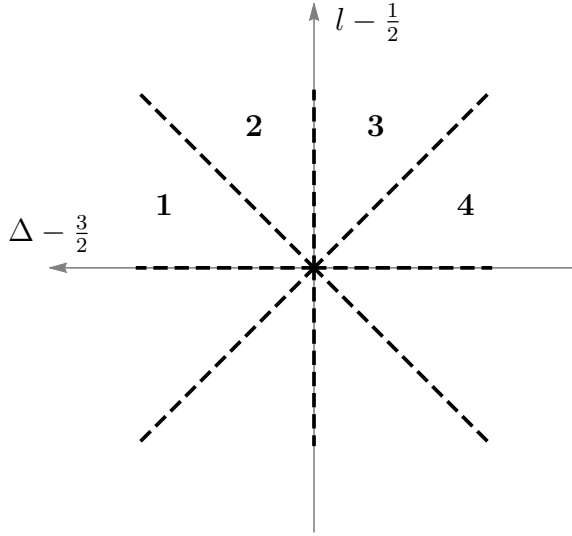


Figure I.2: Bruhat ordering for $\mathfrak{so}(5)$.

which can map the upper half plane to the lower one (since the root $(0, 1)$ is in $\Phi_{\mathfrak{p}}$). Defining $M_{\mathfrak{p}n} \equiv M_{\mathfrak{p}}(w_n \cdot \lambda)$ and $L_n \equiv L(w_n \cdot \lambda)$ and applying formula (I.20) we obtain

$$\text{ch } M_{\mathfrak{p}1} = \text{ch } L_1 , \quad (\text{I.23})$$

$$\text{ch } M_{\mathfrak{p}2} = \text{ch } L_2 + \text{ch } L_1 , \quad (\text{I.24})$$

$$\text{ch } M_{\mathfrak{p}3} = \text{ch } L_3 + \text{ch } L_2 , \quad (\text{I.25})$$

$$\text{ch } M_{\mathfrak{p}4} = \text{ch } L_4 + \text{ch } L_3 , \quad (\text{I.26})$$

which matches formulas (4.112-4.114) in the main text.

I.1.2 Even Spacetime Dimension

The situation is different for even dimensions: $\mathfrak{so}(2N + 2)$. In this case, $W_{\mathfrak{p}} \setminus W$ contains $2N + 2$ elements, and the Bruhat ordering among them is a bit more involved, as shown in figure I.1: $w_i < w_j$ for $i < j$, except that w_{N+1} and w_{N+2} are uncomparable. The $q = 1$ value of the KL polynomial is known to be

$$P_{w_i, w_j}(q = 1) = \begin{cases} 1 & (w_i \leq w_j) , \\ 2 & (n + 2 \leq j \leq 2n - 1, 1 \leq i \leq 2n - j) , \\ 0 & (\text{otherwise}) . \end{cases} \quad (\text{I.27})$$

For example, for $\mathfrak{so}(6)$ the Bruhat ordering of the Weyl chambers is shown in figure I.3. Again we only labeled the Weyl chambers of the parabolic module. We also show in red the shape of

a Weyl chamber, which now are three dimensional objects. Applying formula I.27 we get

$$\begin{aligned}
\text{ch } L_1 &= \text{ch } M_{\mathfrak{p}_1} , \\
\text{ch } L_2 &= \text{ch } M_{\mathfrak{p}_2} - \text{ch } M_{\mathfrak{p}_1} , \\
\text{ch } L_3 &= \text{ch } M_{\mathfrak{p}_3} - \text{ch } M_{\mathfrak{p}_2} + \text{ch } M_{\mathfrak{p}_1} , \\
\text{ch } L_4 &= \text{ch } M_{\mathfrak{p}_4} - \text{ch } M_{\mathfrak{p}_2} + \text{ch } M_{\mathfrak{p}_1} , \\
\text{ch } L_5 &= \text{ch } M_{\mathfrak{p}_5} - \text{ch } M_{\mathfrak{p}_4} - \text{ch } M_{\mathfrak{p}_3} + \text{ch } M_{\mathfrak{p}_2} - 2 \text{ch } M_{\mathfrak{p}_1} , \\
\text{ch } L_6 &= \text{ch } M_{\mathfrak{p}_6} - \text{ch } M_{\mathfrak{p}_5} + \text{ch } M_{\mathfrak{p}_4} + \text{ch } M_{\mathfrak{p}_3} - \text{ch } M_{\mathfrak{p}_2} + \text{ch } M_{\mathfrak{p}_1} .
\end{aligned} \tag{I.28}$$

and hence

$$\begin{aligned}
\text{ch } M_{\mathfrak{p}_1} &= \text{ch } L_1 , \\
\text{ch } M_{\mathfrak{p}_2} &= \text{ch } L_2 + \text{ch } L_1 , \\
\text{ch } M_{\mathfrak{p}_3} &= \text{ch } L_3 + \text{ch } L_2 , \\
\text{ch } M_{\mathfrak{p}_4} &= \text{ch } L_4 + \text{ch } L_2 , \\
\text{ch } M_{\mathfrak{p}_5} &= \text{ch } L_5 + \text{ch } L_4 + \text{ch } L_3 + \text{ch } L_2 + \text{ch } L_1 , \\
\text{ch } M_{\mathfrak{p}_6} &= \text{ch } L_6 + \text{ch } L_5 + \text{ch } L_1 .
\end{aligned} \tag{I.29}$$

In particular, the parabolic Verma modules contain in general more than two irreducible components.

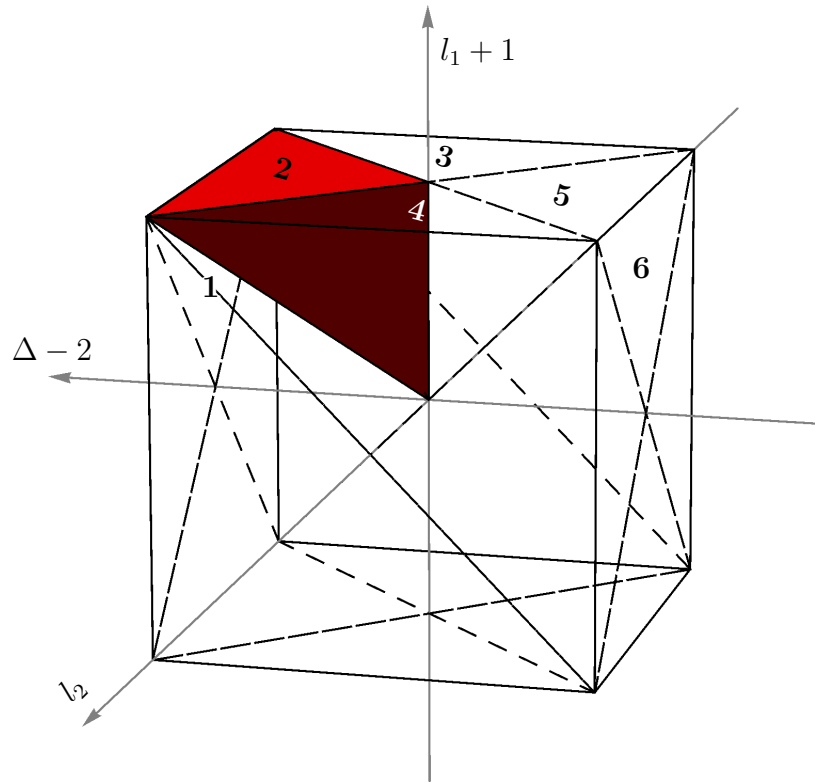


Figure I.3: Bruhat ordering of $\mathfrak{so}(6)$. The Weyl chambers are the cones spanned by the lines originating from the center of the cube and passing through the three points of the triangles in the faces. We labeled them with numbers in the faces (for example we colored in red the Weyl chamber number 2).

Appendix J

Charting the space of 3D CFTs with a continuous global symmetry

J.1 Spectrum of Simple Theories

J.1.1 Free Scalar Theories

The simplest example of CFT in 3 dimensions with $U(1)$ global symmetry is the theory of free complex scalar field φ . The central charge of this theory $C_T = C_T^{\text{free}}$ was given in (5.19). The global $U(1)$ current is given by the conventional expression $J_\mu = i\bar{\varphi}\partial_\mu\varphi - i\varphi\partial_\mu\bar{\varphi}$. The lightest parity even neutral scalar $\bar{\varphi}\varphi$ has dimension $\Delta_0^+ = 1$. The lightest parity-odd scalar is more complicated. Normally, one can build a parity-odd scalar out of two vectors

$$\epsilon^{\mu\nu\lambda} J_\mu \partial_\nu J_\lambda , \quad (\text{J.1})$$

but in case of one complex field this combination vanishes. Hence the lightest parity-odd scalar has more derivatives

$$i\epsilon_{\mu\nu\lambda} J_\mu (\partial_\rho \partial_\nu \varphi) (\partial_\rho \partial_\lambda \bar{\varphi}) \quad (\text{J.2})$$

and is of dimension $\Delta_0^- = 7$.

A complex field φ can be decomposed into two real fields. One can consider a more general case of N free real fields φ^i . This theory has $O(N)$ global symmetry, $N(N-1)/2$ currents $J_\mu^{[ij]} = \varphi^i \partial_\mu \varphi^j$ and

$$C_T = \frac{N}{2} C_T^{\text{free}} . \quad (\text{J.3})$$

The lightest parity-even scalar $\varphi_i \varphi^i$ still has the same dimension $\Delta_0^+ = 1$. When $N \geq 4$ one can combine two mutually commuting currents $J_\mu^{[ij]}$ and $J_\nu^{[kl]}$ with four distinct i, j, k, l into a

dimension $\Delta_0^- = 5$ parity-odd scalar

$$O_-^{[ijkl]} = 4\epsilon^{\mu\nu\rho}\phi^{[i}\partial_\mu\phi^j\partial_\nu\phi^k\partial_\rho\phi^{l]} = -\epsilon^{\mu\nu\rho}J_\mu^{[ij]}\partial_\nu J_\rho^{[kl]} . \quad (\text{J.4})$$

This operator is charged under full $O(N)$ but is neutral under some generators, including $J_\mu^{[ij]}$ and $J_\mu^{[kl]}$. Depending on the choice of the generator $J_\mu = \omega_{ij}J_\mu^{ij}$ the OPE two identical J_μ will or will not include (J.4). For example the OPE of two $J_\mu = J_\mu^{[12]}$ will remain the same as in the theory of one complex boson, with $\Delta_0^- = 7$, while the OPE of two $J_\mu = (J_\mu^{[12]} + J_\mu^{[34]})/2^{1/2}$ will include (J.4), leading to $\Delta_0^- = 5$.

In the theory of a free complex boson, the four-point function of currents can be easily calculated explicitly. Using the symmetry properties of Table J.1 and the crossing symmetry conditions (5.24) together with the definitions in (J.38), the vector of 43 structures in d -dimensions can be compactly written as:

$$f_s = \left\{ f_1, f_2, \hat{f}_1, f_4, f_5, f_5, f_7, f_8, f_9, \hat{f}_9, f_8, \hat{f}_5, \hat{f}_4, \hat{f}_7, \hat{f}_5, \hat{f}_5, \hat{f}_4, \hat{f}_7, \hat{f}_5, f_8, f_9, \hat{f}_9, f_8, f_7, f_5, f_5, \right. \\ \left. f_4, f_{28}, f_{29}, f_{29}, f_{31}, \hat{f}_{29}, \hat{f}_{28}, f_{34}, f_{29}, \hat{f}_{29}, f_{34}, \hat{f}_{28}, f_{29}, \hat{f}_{31}, \hat{f}_{29}, \hat{f}_{29}, f_{28} \right\} \quad (\text{J.5})$$

where $\hat{f}_i \equiv u^{-1+d}v^{1-d}f_i(v, u)$. Finally, the 11 independent functions appearing in the above equation are:

$$\begin{aligned} f_1 &= ((-2+d)^2v^{-d/2}(2uv^{d/2} + u^{d/2}(v + v^{d/2}))) / (2u), & f_9 &= \frac{1}{2}(-2+d)^3u^{\frac{1}{2}(-1+d)}\sqrt{v} \\ f_2 &= \frac{1}{2}(-2+d)^2u^{-1+\frac{d}{2}}v^{-d/2}(uv^{d/2} + u^{d/2}(v + 2v^{d/2})), & f_{28} &= \frac{1}{2}(-2+d)^4u^{d/2}v^{1-\frac{d}{2}}, \\ f_4 &= \frac{1}{2}(-2+d)^3u^{d/2}v^{\frac{1}{2}-\frac{d}{2}}, & f_{29} &= -\frac{1}{2}(-2+d)^4u^{\frac{1}{2}(-1+d)}v^{\frac{3}{2}-\frac{d}{2}}, \\ f_5 &= -\frac{1}{2}(-2+d)^3u^{\frac{1}{2}(-1+d)}v^{1-\frac{d}{2}}, & f_{31} &= \frac{1}{2}(-2+d)^4u^{-1+\frac{d}{2}}v^{1-\frac{d}{2}}(v + v^{d/2}), \\ f_7 &= \frac{1}{2}(-2+d)^3u^{-1+\frac{d}{2}}v^{\frac{1}{2}-\frac{d}{2}}(v + v^{d/2}), & f_{34} &= \frac{1}{2}(-2+d)^4u^{d/2}v^{1-\frac{d}{2}}, \\ f_8 &= 0. \end{aligned} \quad (\text{J.6})$$

First few terms in the conformal block decomposition of (J.6) are summarized in the table below. In particular it shows that the second lightest parity-odd scalar

$$\bar{\varphi}\varphi\partial_\mu\bar{\varphi}\partial_\mu\varphi \quad (\text{J.7})$$

appearing in the OPE of two currents has dimension $(\Delta_0^+)' = 4$. This is because dimension 3 operator $\partial_\mu\bar{\varphi}\partial_\mu\varphi$ is not a primary. Similarly, second lightest spin-2 operator $\bar{\varphi}\varphi T_{\mu\nu}$ also has

dimension $(\Delta_2^+)' = 4$.

Δ	ℓ	$(\tilde{\lambda}_{JJ\mathcal{O}^+}^{(1)})^2$	$\tilde{\lambda}_{JJ\mathcal{O}^+}^{(1)}\tilde{\lambda}_{JJ\mathcal{O}^+}^{(2)}$	$(\tilde{\lambda}_{JJ\mathcal{O}^+}^{(2)})^2$
1	0	4	0	0
4	0	4	0	0
6	0	$\frac{256}{105}$	0	0
8	0	$\frac{48}{77}$	0	0
10	0	$\frac{131072}{75075}$	0	0
3	2	12	72	432
4	2	0	0	768
6	2	$\frac{1112}{245}$	$-\frac{1984}{245}$	$\frac{117248}{245}$
8	2	$\frac{963584}{72765}$	$-\frac{2023424}{24255}$	$\frac{6815744}{8085}$
10	2	$\frac{1196480}{231231}$	$-\frac{3650560}{231231}$	$\frac{197918720}{231231}$
5	4	35	980	27440
6	4	0	0	$\frac{442368}{7}$
8	4	$\frac{171392}{7623}$	$-\frac{3384320}{7623}$	$\frac{144711680}{7623}$
10	4	$\frac{132841472}{2147145}$	$-\frac{20922368}{17745}$	$\frac{71189430272}{2147145}$
7	6	$\frac{231}{2}$	7623	503118
8	6	0	0	$\frac{14376960}{11}$
10	6	$\frac{6208}{55}$	$-\frac{3861504}{715}$	$\frac{263061504}{715}$
9	8	$\frac{6435}{16}$	$\frac{96525}{2}$	5791500
10	8	0	0	$\frac{2424963072}{143}$

Δ	ℓ	$(\tilde{\lambda}_{JJ\mathcal{O}^-})^2$
7	0	$\frac{768}{5}$
9	0	$\frac{8192}{91}$
5	2	768
7	2	$\frac{18432}{35}$
9	2	$\frac{7577600}{7007}$
6	3	$\frac{3072}{35}$
8	3	$\frac{4096}{11}$
10	3	$\frac{507904}{1573}$
7	4	$\frac{9728}{3}$
9	4	$\frac{41656320}{11011}$
8	5	$\frac{737280}{539}$
10	5	$\frac{6356992}{1859}$
9	6	$\frac{2301952}{143}$
10	7	$\frac{214810624}{19305}$

(J.8)

The OPE coefficients $\tilde{c}_{JJ\mathcal{O}^\pm}$ in the above tables are defined in appendix G.5.

J.1.2 Critical $O(N)$ Models

The spectrum of critical $O(N)$ models at large N is in many ways similar to that one of free theories. Including leading $1/N$ corrections the central charge is given by [84, 85] (also see [86] for further references)

$$C_T = \frac{N}{2} C_T^{\text{free}} \left(1 - \frac{40}{9\pi^2 N} + \dots \right). \quad (\text{J.9})$$

The main difference is the dimension of the lightest parity even scalar. At large N its dimension approaches 2,

$$\Delta_0^+ = 2 - \frac{32}{3\pi^2 N} + \dots \quad (\text{J.10})$$

The dimensions of the parity-odd scalars are less studied. For $N = 2, 3$, the lightest parity odd scalar appearing in the OPE of two currents has the same quantum numbers as (J.2) and is expected to have dimension $\Delta_0^- \approx 7$. For $N \geq 4$ there is also a parity-odd operator in the representation of (J.4). Thus for some generators $J_\mu = \omega_{ij} J_\mu^{ij}$ we still expect $\Delta_0^- \approx 7$ while for others $\Delta_0^- \approx 5$.

For small $N = 2, 3, \dots$ certain dimensions and central charge are known with a good precision from the conformal bootstrap and Monte-Carlo simulations [73, 86–88]. We report some of them in the table below:

N	2	3	4
C_T/C_T^{free}	0.944	1.416	1.892
Δ_ϵ	1.51124(22)	1.5939(10)	1.6649(35)
$\Delta_{\epsilon'}$	3.795(9)	3.782(12)	3.774(12)
Δ_T	1.237(4)	1.211(3)	1.189(2)

(J.11)

Here ϵ, ϵ' are the first and second singlet scalar operators appearing in the OPE $\phi_a \times \phi_b$, while T_{ab} is the leading scalar operator transforming in the tensor traceless representation of $O(N)$. Under a given $U(1) \subset O(N)$, T_{ab} decomposes into neutral and charged components. The neutral ones are allowed to enter the OPE of the conserved current associated with the $U(1)$. This means that

$$\begin{aligned} \Delta_0^+ &= \Delta_\epsilon, & (\Delta_0^+)' &= \Delta_{\epsilon'} & \text{for } O(2) \\ \Delta_0^+ &= \Delta_T, & (\Delta_0^+)' &= \Delta_\epsilon & \text{for } O(N \geq 3) \end{aligned}$$

J.1.3 Free Fermion Theories

In three dimensions, a free Dirac fermion ψ is invariant under a global $U(1)$ symmetry. This theory has the same central charge as a free complex scalar, $C_T = C_T^{\text{free}}$. The lightest parity-odd scalar $\bar{\psi}\psi$ has dimension $\Delta_0^- = 2$, while lightest parity-even scalar $(\bar{\psi}\psi)^2$ has dimension $\Delta_0^+ = 4$. The four point function of the conserved current $J_\mu = \bar{\psi}\sigma_\mu\psi$ can be easily calculated explicitly. The four point function contains two distinct contributions $f_s = f_s^{\text{disc}} - f_s^{\text{con}}/\Upsilon$ where the disconnected piece f_s^{disc} is given by (J.23), while the connected one is given below. Also, $\Upsilon = \text{Tr } \mathbb{1}$ denotes the trace of the identity in γ -matrix algebra in d -dimensions, $\mathbb{1} = (\gamma^1)^2$

($\Upsilon = 2$ in 3 dimensions). Following the same conventions as in (J.5), we have:

$$\begin{aligned}
f_1^{\text{con}} &= u^{-1+\frac{d}{2}} v^{-d/2} \left(-u^{1+\frac{d}{2}} + u^{d/2}(1+v) + (-1+v)(-1+v^{d/2}) - u(1+v^{d/2}) \right), \\
f_2^{\text{con}} &= u^{-1+\frac{d}{2}} v^{-d/2} \left(u + u^{1+\frac{d}{2}} - uv^{d/2} - u^{d/2}(1+v) + (-1+v)(1+v^{d/2}) \right), \\
f_4^{\text{con}} &= -2u^{d/2} v^{\frac{1}{2}-\frac{d}{2}} (-1 + u^{d/2} + v^{d/2}), \\
f_5^{\text{con}} &= 2u^{\frac{1}{2}(-1+d)} v^{1-\frac{d}{2}} (-1 + u^{d/2} + v^{d/2}), \\
f_7^{\text{con}} &= -2u^{-1+\frac{d}{2}} v^{\frac{1}{2}-\frac{d}{2}} (u^{d/2}(1+v) + (-1+v)(-1+v^{d/2})), \\
f_8^{\text{con}} &= 0, \\
f_9^{\text{con}} &= -2u^{\frac{1}{2}(-1+d)} v^{\frac{1}{2}-\frac{d}{2}} (1 + u^{d/2} - v^{d/2}), \\
f_{28}^{\text{con}} &= 4u^{d/2} v^{1-\frac{d}{2}} (-1 + u^{d/2} + v^{d/2}), \\
f_{29}^{\text{con}} &= -4u^{\frac{1}{2}(-1+d)} v^{\frac{3}{2}-\frac{d}{2}} (-1 + u^{d/2} + v^{d/2}), \\
f_{31}^{\text{con}} &= 4u^{-1+\frac{d}{2}} v^{1-\frac{d}{2}} (u^{d/2}(1+v) + (-1+v)(-1+v^{d/2})), \\
f_{34}^{\text{con}} &= 4u^{d/2} v^{1-\frac{d}{2}} (-1 + u^{d/2} + v^{d/2}).
\end{aligned} \tag{J.12}$$

A first few terms in the conformal block decomposition of (J.12) are summarized in the table below.

Δ	ℓ	$(\tilde{\lambda}_{JJ\mathcal{O}^+}^{(1)})^2$	$\tilde{\lambda}_{JJ\mathcal{O}^+}^{(1)} \tilde{\lambda}_{JJ\mathcal{O}^+}^{(2)}$	$(\tilde{\lambda}_{JJ\mathcal{O}^+}^{(2)})^2$
4	0	4	0	0
6	0	$\frac{256}{105}$	0	0
8	0	$\frac{48}{77}$	0	0
10	0	$\frac{131072}{75075}$	0	0
3	2	0	0	192
6	2	$\frac{296}{35}$	$-\frac{2752}{35}$	$\frac{34304}{35}$
8	2	$\frac{676864}{72765}$	$-\frac{446464}{24255}$	$\frac{3948544}{8085}$
10	2	$\frac{182080}{21021}$	$-\frac{1602560}{21021}$	$\frac{24248320}{21021}$
5	4	0	0	2240
6	4	0	0	36864
8	4	$\frac{278912}{7623}$	$-\frac{7470080}{7623}$	$\frac{247930880}{7623}$
10	4	$\frac{98435072}{2147145}$	$-\frac{1516617728}{2147145}$	$\frac{54330294272}{2147145}$
7	6	0	0	16632
8	6	0	0	$\frac{10076160}{11}$
10	6	$\frac{116544}{715}$	$-\frac{6226944}{715}$	$\frac{386924544}{715}$
9	8	0	0	102960
10	8	0	0	$\frac{167903232}{13}$

Δ	ℓ	$(\tilde{\lambda}_{JJ\mathcal{O}^-})^2$
2	0	32
7	0	$\frac{768}{5}$
9	0	$\frac{8192}{91}$
5	2	768
7	2	$\frac{18432}{35}$
9	2	$\frac{7577600}{7007}$
6	3	$\frac{3072}{35}$
8	3	$\frac{4096}{11}$
10	3	$\frac{507904}{1573}$
7	4	$\frac{9728}{3}$
9	4	$\frac{41656320}{11011}$
8	5	$\frac{737280}{539}$
10	5	$\frac{6356992}{1859}$
9	6	$\frac{2301952}{143}$
10	7	$\frac{214810624}{19305}$

The OPE coefficients $\tilde{c}_{JJ\mathcal{O}^\pm}$ in the above tables are defined in appendix G.5.

J.1.4 QED₃

A theory of N_f Dirac fermions ψ^i in 3d coupled to a $U(1)$ gauge field A_μ flows to a non-trivial IR fixed point if N_f is sufficiently large. This theory has global $SU(N_f)$ flavor symmetry, with the currents J^a , with $a = 1, 2, \dots, N_f^2 - 1$. Flavor symmetry might be spontaneously broken for small N_f by chiral condensate. Besides this there is a topological $U(1)$, with the topological current $J^{\text{top}} \propto \star F$. The operators charged under this $U(1)$ are monopole operators. When N_f is odd the theory is not parity-invariant [89]. Accordingly we consider only even N_f such that the effective number of Majorana fermions $N = 2N_f$ is a multiple of four. For large N , the central charge is given by [90]

$$C_T = \frac{N}{2} C_T^{\text{free}} \left(1 + \frac{4192 - 360\pi^2}{45\pi^2 N} + \dots \right) \quad (\text{J.14})$$

For minimal possible value $N = 4$ this gives $C_T \approx 2.72 C_T^{\text{free}}$. This result is valid only if there is no spontaneous symmetry breaking.

Identifying the lightest parity even and odd scalars appearing in the OPE of two currents requires consideration. Since monopole operators are charged under topological $U(1)$ they are excluded from the OPE of both $J^{\text{top}} \times J^{\text{top}}$ and $J^a \times J^a$. First, we consider OPE of two J^{top} which contains only $SU(N_f)$ singlets. For large N , the lightest parity-odd singlet scalar $\bar{\psi}_i \psi^i$ has dimension [91, 92]

$$\Delta_0^- = \Delta_0 = 2 + \frac{256}{3\pi^2 N} + \dots \quad (\text{J.15})$$

while lightest parity-even scalar is a combination of $(\bar{\psi}_i \psi^i)^2$ and $F_{\mu\nu}^2$ of dimension

$$\Delta_0^+ = \Delta'_{0,-} = 4 + \frac{128(2 - \sqrt{7})}{3\pi^2 N} + \dots \quad (\text{J.16})$$

The OPE of two flavor currents include all fields charged in representations appearing in the product of two adjoints. In this case the lightest parity-odd scalar is in adjoint representation of $SU(N_f)$, $(O_{n=1})_j^i = \psi^i \bar{\psi}_j$. At leading order it has dimension [91, 92]

$$\Delta_0^- = \Delta_1 = 2 - \frac{128}{3\pi^2 N} + \dots \quad (\text{J.17})$$

which is smaller than Δ_0 . Similarly, the lightest parity-even operator is $(O_{n=2})_{[kl]}^{[ij]} = \psi^i \psi^j \bar{\psi}_k \bar{\psi}_l$ of dimension

$$\Delta_0^+ = \Delta_2 = 4 - \frac{128}{\pi^2 N} + \dots \quad (\text{J.18})$$

which is smaller than $\Delta'_{0,-} = 4 + \frac{128(2-\sqrt{7})}{3\pi^2 N} + \dots$ and the dimension of the adjoint operator $O'_{1,-}$ made out of four ψ 's, $\Delta'_{1,-} = 4 + \frac{16(25-\sqrt{2317})}{3\pi^2 N} + \dots$

We see that in both cases at large N , $\Delta_0^+ \approx 4$ and $\Delta_0^- \approx 2$, and from this point of view QED₃

is similar to the free fermion theory.

J.1.5 Gross-Neveu Models

One Dirac spinor can be decomposed into two Majorana spinors. A theory of $N \geq 2$ free Majorana fermions has $O(N)$ symmetry, while the dimension of lightest parity even and odd scalars remain the same for all N . Upon adding a parity-odd scalar field ϕ with quartic interaction which couples to Majorana fermions via Yukawa coupling $\bar{\psi}_i \phi \psi^i$, the theory flows into an interacting fixed point characterized by $O(N)$ symmetry. The lowest parity-odd scalar ϕ has dimension [93–95] (also see [96] for further references)

$$\Delta_0^- = 1 - \frac{32}{3\pi^2 N} + \dots \quad (\text{J.19})$$

The lightest parity-even scalar ϕ^2 has dimension

$$\Delta_0^+ = 2 + \frac{32}{3\pi^2 N} + \dots \quad (\text{J.20})$$

while the central charge is given by [97]

$$C_T = \frac{N}{2} C_T^{\text{free}} \left(1 + \frac{8}{9\pi^2 N} + \dots \right) . \quad (\text{J.21})$$

Below we compare C_T and Δ_0^+, Δ_0^- for small N found using leading $1/N$ expansion and Pade extrapolation of ϵ -expansion in $d = 2, 4$ [97, 98]¹ and bootstrap techniques [99].

N	2	3	4
$N/2 + 4/(9\pi^2)$	1.045	1.545	2.045
C_T/C_T^{free} Pade	1.190	1.486	2.029
$1 - 32/(3\pi^2 N)$	0.46	0.64	0.73
Δ_0^- Pade	0.656	0.688	0.753
Δ_0^- Bootstrap	0.660	0.724	0.772
$2 + 32/(3\pi^2 N)$	2.54	2.36	2.27
Δ_0^+ Pade	1.75	2.285	2.148
Δ_0^+ Bootstrap	2.14	2.17	2.25

(J.22)

It is worth noting that, similar to critical bosonic $O(N)$ theories, central charge of Gross-Neveu models even for small N is substantially close to the free theory counterpart.

¹To calculate C_T for small N we follow Pade approximation procedures developed in [97]. Namely we employ Pade_[4,1] or Pade_[1,4] choosing the one which has no poles in the interval $2 < d < 4$. Namely Pade_[1,4] for $N = 3, 4$ and Pade_[4,1] for $N = 2$.

J.1.6 Generalized Free Vector Field

The generalized free vector field (GFVF) is a theory of a conserved current J_μ (of dimension $d-1$) with the standard two-point function and all higher-point correlation functions satisfying Wick theorem. In particular the four-point function of currents $\langle JJJJ \rangle$ includes only the disconnected piece (all other components are zero),

$$f_1 = 1, \quad f_2 = u^{d-1}, \quad f_3 = u^{d-1}v^{1-d}. \quad (\text{J.23})$$

This theory contains no stress-energy tensor, i.e. $C_T^{-1} = 0$. The only operators present in the spectrum are those build of J_μ . In particular the lightest parity even scalar $J^\mu J_\mu$ has dimension $\Delta_0^+ = 4$ and parity-odd scalar given by (J.1) has dimension $\Delta_0^- = 5$. GFVF is dual to $U(1)$ gauge theory in AdS_4 in the limit of zero Newton constant, when only disconnected Witten diagrams contribute. In the table below, we list some OPE coefficients that we obtained from the conformal block expansion of $\langle JJJJ \rangle$ in $d = 3$ dimensions.

Δ	ℓ	$(\tilde{\lambda}_{JJ\mathcal{O}^+}^{(1)})^2$	$\tilde{\lambda}_{JJ\mathcal{O}^+}^{(1)} \tilde{\lambda}_{JJ\mathcal{O}^+}^{(2)}$	$(\tilde{\lambda}_{JJ\mathcal{O}^+}^{(2)})^2$
4	0	$\frac{8}{3}$	0	0
6	0	$\frac{512}{315}$	0	0
8	0	$\frac{512}{385}$	0	0
10	0	$\frac{262144}{225225}$	0	0
4	2	0	0	512
6	2	$\frac{2048}{245}$	$-\frac{11776}{245}$	$\frac{161792}{245}$
8	2	$\frac{1998848}{218295}$	$-\frac{3473408}{72765}$	$\frac{18219008}{24255}$
10	2	$\frac{6103040}{693693}$	$-\frac{33095680}{693693}$	$\frac{626524160}{693693}$
6	4	0	0	$\frac{294912}{7}$
8	4	$\frac{278528}{7623}$	$-\frac{6619136}{7623}$	$\frac{235667456}{7623}$
10	4	$\frac{294649856}{6441435}$	$-\frac{476053504}{585585}$	$\frac{164580294656}{6441435}$
8	6	0	0	$\frac{10485760}{11}$
10	6	$\frac{1048576}{6435}$	$-\frac{17825792}{2145}$	$\frac{29360128}{55}$
10	8	0	0	$\frac{1879048192}{143}$

Δ	ℓ	$(\tilde{\lambda}_{JJ\mathcal{O}^-})^2$
5	0	$\frac{256}{3}$
7	0	$\frac{512}{5}$
9	0	$\frac{131072}{1001}$
5	2	512
7	2	$\frac{24576}{35}$
9	2	$\frac{6553600}{7007}$
6	3	$\frac{24576}{175}$
8	3	$\frac{65536}{231}$
10	3	$\frac{2097152}{4719}$
7	4	$\frac{8192}{3}$
9	4	$\frac{47185920}{11011}$
8	5	$\frac{2621440}{1617}$
10	5	$\frac{16777216}{5577}$
9	6	$\frac{2097152}{143}$
10	7	$\frac{234881024}{19305}$

(J.24)

The OPE coefficients $\tilde{c}_{JJ\mathcal{O}^\pm}$ in the above tables are defined in appendix G.5.

J.2 Relations between parity odd structures

Parity odd conformally invariant three point functions can be construct using the ϵ -tensor. In $d = 3$, there are six parity odd building blocks:

$$\epsilon_{ij} \equiv \epsilon(Z_i, Z_j, P_1, P_2, P_3) \quad \tilde{\epsilon}_{ij} \equiv \epsilon(P_i, P_j, Z_1, Z_2, Z_3) \quad (i, j = 1, 2, 3), \quad (\text{J.25})$$

However, not all of them are independent. To see this we use the following identity

$$0 = \det \begin{pmatrix} (P_1 \cdot \Xi) & (P_2 \cdot \Xi) & (P_3 \cdot \Xi) & (Z_1 \cdot \Xi) & (Z_2 \cdot \Xi) & (Z_3 \cdot \Xi) \\ \left(P_1\right)_A & \left(P_2\right)_A & \left(P_3\right)_A & \left(Z_1\right)_A & \left(Z_2\right)_A & \left(Z_3\right)_A \end{pmatrix} \quad (\text{J.26})$$

where Ξ is an arbitrary 5 dimensional vector. The determinant vanishes automatically because the first row of the matrix is a linear combination of the other 5 rows. By choosing for instance $\Xi = P_1$ one gets

$$-(P_1 \cdot P_2)\epsilon(P_1, P_3, Z_1, Z_2, Z_3) + (P_1 \cdot P_3)\epsilon(P_1, P_2, Z_1, Z_2, Z_3) \quad (\text{J.27})$$

$$-(P_1 \cdot Z_2)\epsilon(P_1, P_2, P_3, Z_1, Z_3) + (P_1 \cdot Z_3)\epsilon(P_1, P_2, P_3, Z_1, Z_2) = 0 \quad (\text{J.28})$$

Similarly one can get two more equations by choosing $\Xi = P_2, P_3$. All together these relations allow to express $\tilde{\epsilon}_{ij}$ in terms of linear combination of ϵ_{ij} only.

In addition, one can find linear relations involving only the three ϵ_{ij} . This follows immediately if we choose Ξ orthogonal to the three P 's. This is achieved with

$$\Xi^A = [Z_1^A(P_1 \cdot X^+) - P_1^A(Z_1 \cdot X^+)](P_2 \cdot P_3) + (X^-)^A(Z_1 \cdot X^-), \quad (\text{J.29})$$

where

$$(X^\pm)^A = P_2^A(P_1 \cdot P_3) \pm P_3^A(P_1 \cdot P_2). \quad (\text{J.30})$$

One can easily check that

$$\Xi \cdot Z_1 = -2(P_1 \cdot P_2)(P_1 \cdot P_3)(P_2 \cdot P_3)V_{1,23}^2 \quad (\text{J.31})$$

$$\Xi \cdot Z_2 = 2(P_1 \cdot P_2)(P_1 \cdot P_3)(P_2 \cdot P_3)(H_{12} + V_{1,23}V_{2,31}) \quad (\text{J.32})$$

$$\Xi \cdot Z_3 = 2(P_1 \cdot P_2)(P_1 \cdot P_3)(P_2 \cdot P_3)(H_{13} + V_{1,23}V_{3,12}) \quad (\text{J.33})$$

and conclude that (J.26) reduces to

$$V_{1,23}^2\epsilon_{23} - (H_{12} + V_{1,23}V_{2,31})\epsilon_{13} - (H_{13} + V_{1,23}V_{3,12})\epsilon_{12} = 0. \quad (\text{J.34})$$

Similarly, we can find another 2 equations by permuting the 3 points. These relations were taken into account in the construction of conformal invariant three point functions in section 5.2.1.

J.3 Basis for four point function

Conformal invariant tensor structures for four point functions are constructed using the building blocks H_{ij} and $V_{i,jk}$. However, not all combinations are linearly independent. In fact, it is sufficient to use the following set:

$$\{H_{12}, H_{13}, H_{14}, H_{23}, H_{24}, H_{34}, V_{1,23}, V_{1,43}, V_{2,34}, V_{2,14}, V_{3,21}, V_{3,41}, V_{4,21}, V_{4,32}\} \quad (\text{J.35})$$

All others can be expressed in terms of a linear combination of the above. For instance the following identity holds:

$$\begin{aligned} & \sqrt{-2(P_i \cdot P_j)(P_i \cdot P_k)(P_j \cdot P_k)(P_i \cdot P_l)} V_{i,jk} + \sqrt{-2(P_i \cdot P_j)(P_i \cdot P_l)(P_j \cdot P_l)(P_i \cdot P_k)} V_{i,kj} \\ & + \sqrt{-2(P_i \cdot P_l)(P_i \cdot P_k)(P_l \cdot P_k)(P_i \cdot P_j)} V_{i,kl} = 0. \end{aligned} \quad (\text{J.36})$$

Out of the above list one can construct 43 tensor structure. These are listed in Table. J.1.

s	Q_s	$1234 \rightarrow 3412$	$1234 \rightarrow 2143$	Indep
1	$H_{12}H_{34}$	1	1	1
2	$H_{13}H_{24}$	2	2	2
3	$H_{14}H_{23}$	3	3	3
4	$H_{12}V_{3,21}V_{4,12}$	27	4	4
5	$H_{12}V_{3,21}V_{4,32}$	26	6	5
6	$H_{12}V_{3,41}V_{4,12}$	25	5	5
7	$H_{12}V_{3,41}V_{4,32}$	24	7	7
8	$H_{13}V_{2,14}V_{4,12}$	11	20	8
9	$H_{13}V_{2,14}V_{4,32}$	9	21	9
10	$H_{13}V_{2,34}V_{4,12}$	10	22	10
11	$H_{13}V_{2,34}V_{4,32}$	8	23	8
12	$H_{14}V_{2,14}V_{3,21}$	19	16	12
13	$H_{14}V_{2,14}V_{3,41}$	17	17	13
14	$H_{14}V_{2,34}V_{3,21}$	18	18	14
15	$H_{14}V_{2,34}V_{3,41}$	16	19	12
16	$H_{23}V_{1,23}V_{4,12}$	15	12	12
17	$H_{23}V_{1,23}V_{4,32}$	13	13	13
18	$H_{23}V_{1,43}V_{4,12}$	14	14	14
19	$H_{23}V_{1,43}V_{4,32}$	12	15	12
20	$H_{24}V_{1,23}V_{3,21}$	23	8	8
21	$H_{24}V_{1,23}V_{3,41}$	21	9	9
22	$H_{24}V_{1,43}V_{3,21}$	22	10	10
23	$H_{24}V_{1,43}V_{3,41}$	20	11	8
24	$H_{34}V_{1,23}V_{2,14}$	7	24	7
25	$H_{34}V_{1,23}V_{2,34}$	6	26	5
26	$H_{34}V_{1,43}V_{2,14}$	5	25	5
27	$H_{34}V_{1,43}V_{2,34}$	4	27	4
28	$V_{1,23}V_{2,14}V_{3,21}V_{4,12}$	43	28	28
29	$V_{1,23}V_{2,14}V_{3,21}V_{4,32}$	39	30	29
30	$V_{1,23}V_{2,14}V_{3,41}V_{4,12}$	35	29	29
31	$V_{1,23}V_{2,14}V_{3,41}V_{4,32}$	31	31	31
32	$V_{1,23}V_{2,34}V_{3,21}V_{4,12}$	42	36	32
33	$V_{1,23}V_{2,34}V_{3,21}V_{4,32}$	38	38	33
34	$V_{1,23}V_{2,34}V_{3,41}V_{4,12}$	34	37	34
35	$V_{1,23}V_{2,34}V_{3,41}V_{4,32}$	30	39	29
36	$V_{1,43}V_{2,14}V_{3,21}V_{4,12}$	41	32	32
37	$V_{1,43}V_{2,14}V_{3,21}V_{4,32}$	37	34	34
38	$V_{1,43}V_{2,14}V_{3,41}V_{4,12}$	33	33	33
39	$V_{1,43}V_{2,14}V_{3,41}V_{4,32}$	29	35	29
40	$V_{1,43}V_{2,34}V_{3,21}V_{4,12}$	40	40	40
41	$V_{1,43}V_{2,34}V_{3,21}V_{4,32}$	36	42	32
42	$V_{1,43}V_{2,34}V_{3,41}V_{4,12}$	32	41	32
43	$V_{1,43}V_{2,34}V_{3,41}V_{4,32}$	28	43	28

Table J.1: The third and fourth column show how the structures map to each other under the special permutations that preserve the cross ratios. The 43 structures split into 19 multiplets under these permutations. In the last column, we show in red the label of the 19 independent functions $f_s(u, v)$ that multiply each multiplet.

J.4 Simplifying crossing

The functions $\tilde{f}_s$ are defined by the following linear map,

$$\tilde{f}_{\tilde{s}}(u, v) = \sum_{s=1}^{43} [\mathbf{M}]_{\tilde{s}s} f_s(u, v) , \quad \tilde{s} = 1, \dots, 19 . \quad (\text{J.37})$$

Here $[\mathbf{M}]$ is a 19×43 matrix defined by

$$\begin{aligned} \tilde{f}_1 &= f_{28} - f_{33} & \tilde{f}_8 &= f_{34} \\ \tilde{f}_2 &= f_{29} - f_{32} & \tilde{f}_9 &= f_{28} + f_{33} \\ \tilde{f}_3 &= f_7 - f_{14} & \tilde{f}_{10} &= f_{29} + f_{32} \\ \tilde{f}_4 &= f_4 - f_{13} & \tilde{f}_{11} &= f_7 + f_{14} \\ \tilde{f}_5 &= f_5 - f_{12} & \tilde{f}_{12} &= f_4 + f_{13} \\ \tilde{f}_6 &= f_9 - f_{10} & \tilde{f}_{13} &= f_5 + f_{12} \\ \tilde{f}_7 &= f_1 - f_3 & \tilde{f}_{14} &= f_9 + f_{10} \\ \tilde{f}_{19} &= f_{31} - f_{40} & \tilde{f}_{15} &= f_8 \\ & & \tilde{f}_{16} &= f_1 + f_3 \\ & & \tilde{f}_{17} &= f_2 \\ & & \tilde{f}_{18} &= f_{31} + f_{40} \end{aligned} \quad (\text{J.38})$$

where we suppressed the arguments (u, v) from all functions $\tilde{f}_s$ and f_s .

J.5 Details about numerical implementation

Let us describe the numerical implementation one step at a time.

1. As explained in appendix G, conformal blocks are naturally computed in a basis with 43 tensor structures. However, in three dimensions there exist two linear relations among the 43 tensor structures, the first task we need to perform is to express the three dimensional crossing equations in terms of the 43 (19 for equal currents) original functions.

In our basis of tensor structures, the two linear relations read:

$$\begin{aligned}
& (u^2 + (-1+v)^2 - 2u(1+v)) Q_2 + (-u^2 - (-1+v)^2 + 2u(1+v)) Q_3 + 4uvQ_8 - 2\sqrt{u}\sqrt{v}(-1+u+v)Q_9 \\
& - 2\sqrt{u}\sqrt{v}(-1+u+v)Q_{10} + 4uvQ_{11} + 2u\sqrt{v}(1-u+v)Q_{12} + 2\sqrt{u}(1+u-v)vQ_{13} \\
& + 2\sqrt{u}(1+u^2-v-u(2+v)) Q_{14} + 2u\sqrt{v}(1-u+v)Q_{15} + 2u\sqrt{v}(1-u+v)Q_{16} + 2\sqrt{u}(1+u-v)vQ_{17} \\
& - 2\sqrt{u}(-1-u^2+v+u(2+v)) Q_{18} + 2u\sqrt{v}(1-u+v)Q_{19} + 4uvQ_{20} - 2\sqrt{u}\sqrt{v}(-1+u+v)Q_{21} \\
& - 2\sqrt{u}\sqrt{v}(-1+u+v)Q_{22} + 4uvQ_{23} - 4u^2vQ_{28} + 4u^{3/2}v^{3/2}Q_{29} + 4u^{3/2}v^{3/2}Q_{30} \\
& - 4u(-1+v)vQ_{31} + 4(-1+u)u^{3/2}\sqrt{v}Q_{32} - 4u(1+u)vQ_{33} - 4(-1+u)uvQ_{34} + 4u^{3/2}v^{3/2}Q_{35} \\
& + 4(-1+u)u^{3/2}\sqrt{v}Q_{36} - 4(-1+u)uvQ_{37} - 4u(1+u)vQ_{38} + 4u^{3/2}v^{3/2}Q_{39} \\
& - 4u(1-2u+u^2-v) Q_{40} + 4(-1+u)u^{3/2}\sqrt{v}Q_{41} + 4(-1+u)u^{3/2}\sqrt{v}Q_{42} - 4u^2vQ_{43} = 0 \\
\\
& (u^2 + (-1+v)^2 - 2u(1+v)) Q_1 + (-u^2 - (-1+v)^2 + 2u(1+v)) Q_3 + 2u(-1+u-v)\sqrt{v}Q_4 \\
& - 2\sqrt{u}(1+u-v)vQ_5 - 2\sqrt{u}(1+u-v)vQ_6 - 2\sqrt{v}((-1+v)^2 - u(1+v)) Q_7 + 2u\sqrt{v}(1-u+v)Q_{12} \\
& + 2\sqrt{u}(1+u-v)vQ_{13} + 2\sqrt{u}(1+u^2-v-u(2+v)) Q_{14} + 2u\sqrt{v}(1-u+v)Q_{15} + 2u\sqrt{v}(1-u+v)Q_{16} \\
& + 2\sqrt{u}(1+u-v)vQ_{17} + 2\sqrt{u}(1+u^2-v-u(2+v)) Q_{18} + 2u\sqrt{v}(1-u+v)Q_{19} \\
& - 2\sqrt{v}((-1+v)^2 - u(1+v)) Q_{24} - 2\sqrt{u}(1+u-v)vQ_{25} - 2\sqrt{u}(1+u-v)vQ_{26} + 2u(-1+u-v)\sqrt{v}Q_{27} \\
& + 4uv(1-u+v)Q_{28} + 4\sqrt{u}(1+u-v)v^{3/2}Q_{29} + 4\sqrt{u}(1+u-v)v^{3/2}Q_{30} + 4v(1-(2+u)v+v^2) Q_{31} \\
& + 4u^{3/2}(-1+u-v)\sqrt{v}Q_{32} - 4u(1+u-v)vQ_{33} + 4uv(-u+v)Q_{34} + 4\sqrt{u}(1+u-v)v^{3/2}Q_{35} \\
& + 4u^{3/2}(-1+u-v)\sqrt{v}Q_{36} - 4u(u-v)vQ_{37} - 4u(1+u-v)vQ_{38} + 4\sqrt{u}(1+u-v)v^{3/2}Q_{39} \\
& - 4u(1+u^2-u(2+v)) Q_{40} + 4u^{3/2}(-1+u-v)\sqrt{v}Q_{41} \\
& + 4u^{3/2}(-1+u-v)\sqrt{v}Q_{42} + 4uv(1-u+v)Q_{43} = 0
\end{aligned} \tag{J.39}$$

In this project we choose to eliminate structures Q_{31} and Q_{40} using this two identities. This is motivated by their invariance under the permutations $1234 \leftrightarrow 3412$ and $1234 \leftrightarrow 2143$, as shown in Table J.1, and by the fact that they do not mix under crossing (see eq. (5.24) and appendix J.4). Also, inverting (J.39) in terms of this pair does not introduce any singularity at the crossing symmetric point $u = v = 1/4$.²

Finally we plug in the expression of Q_{31} and Q_{40} in the tensor structure expansion of the four point function (5.20) to obtain the expression of the 17 independent functions in terms of the original 19:

$$\sum_{s=1}^{43} f_s(u, v) Q_s = \sum_{s=1}^{41} f_s^{3D}(u, v) Q_s, \tag{J.40}$$

Note that, despite the index s on the rhs of the above expression runs from 1 to 41, there are only 17 distinct functions $f_s^{3D}(u, v)$.

2. Next, we pass to the basis that diagonalizes crossing symmetry. This is done by defining

²On the contrary, this choice is not optimal when performing the conformal block decomposition by matching powers of r : this is because it would introduce additional singularities at $r = 0$. For such an exercise it is more convenient to invert eq. (J.39) for Q_{37} and Q_{38} .

the 17 functions $\tilde{f}_s$ shown in (J.38).

3. As described in [61], the problem of finding α satisfying (5.45) can be transformed into a semidefinite program. The form of the functional α is given in (5.47). The first step is to compute the derivatives of the vectors $V_+, V_{\ell+}$ and $V_{\ell-}$ defined in (5.44). To do this, we started directly from the explicit form of the conformal blocks as a power series in the variable r defined in (2.73).
4. Once we take the derivatives and set $r = 3 - 2\sqrt{2}$ and $\eta = 1$, these expressions reduce to rational approximations for conformal blocks in the variable Δ . Keeping only the polynomial numerator in these rational approximations, (5.45) becomes a “polynomial matrix program” (PMP), which can be solved with SDPB [100]. We use **Mathematica** to compute and store tables of derivatives of conformal blocks. Another **Mathematica** program reads these tables, computes the polynomial matrices corresponding to the $\vec{V}$ ’s, and uses the package **SDPB.m** to write the associated PMP to an **xml** file. This **xml** file is then used as input to **SDPB**. Our settings for **SDPB** are given in Table J.2.

As discussed in Sec. 5.2.2.2, the minimal crossing constraints consist in 5 bulk equations, 5 boundary equations and one constraint at a point. Once one considers derivatives of the crossing equations at a given point, the conservation equations (5.27) (and their derivatives at the crossing symmetric point $u = v = 1/4$) simply become a set of linear relations between various derivatives of the functions $\tilde{f}_s$. We explicitly checked that the set of derivatives included in the numerical bootstrap has maximal rank, i.e. there is no linear dependence induced by the conservation equations. Also, we explicitly checked that the system made by the conservation equations and their derivatives at the crossing symmetric point can be used to determine neglected components. Because the functions involved have definite symmetric properties under $z \rightarrow \bar{z}$, the number of non-vanishing derivatives included for a given Λ is:

$$\dim(\vec{\alpha}) = 5 \frac{\lfloor \frac{\Lambda+2}{2} \rfloor (\lfloor \frac{\Lambda+2}{2} \rfloor + 1)}{2} + 5 \lfloor \frac{\Lambda+2}{2} \rfloor + 1. \quad (\text{J.41})$$

The degree of the numerator and denominator is controlled by the order of the conformal blocks expansion in r , or equivalently by the number of poles kept in the recursion relations (4.91-4.92). Contrary to previous conformal bootstrap works [61], we used the obtained expressions as they are, without employing any further approximation. Approximations might be useful to push to higher number of derivatives.

Finally, we must choose which spins to include in the PMP. We have chosen the number of spins to depend on Λ as follows

$$\begin{aligned} S_{\Lambda=11} &= \{0, \dots, 24\} \cup \{29, 30\}, \\ S_{\Lambda=15} &= \{0, \dots, 34\} \cup \{39, 40\}, \\ S_{\Lambda=19} &= \{0, \dots, 40\} \cup \{49, 50\}, \\ S_{\Lambda=23} &= \{0, \dots, 40\} \cup \{44, 45, 49, 50, 59, 60\}. \end{aligned} \quad (\text{J.42})$$

Λ	11	15	19	23
order	30	40	40	50
spins	$S_{\Lambda=11}$	$S_{\Lambda=15}$	$S_{\Lambda=19}$	$S_{\Lambda=23}$
precision	448	576	768	896
findPrimalFeasible	True	True	True	True
findDualFeasible	True	True	True	True
detectPrimalFeasibleJump	True	True	True	True
detectDualFeasibleJump	True	True	True	True
dualityGapThreshold	10^{-30}	10^{-30}	10^{-30}	10^{-70}
primalErrorThreshold	10^{-30}	10^{-30}	10^{-40}	10^{-70}
dualErrorThreshold	10^{-30}	10^{-30}	10^{-40}	10^{-70}
initialMatrixScalePrimal	10^{40}	10^{50}	10^{50}	10^{60}
initialMatrixScaleDual	10^{40}	10^{50}	10^{50}	10^{60}
feasibleCenteringParameter	0.1	0.1	0.1	0.1
infeasibleCenteringParameter	0.3	0.3	0.3	0.3
stepLengthReduction	0.7	0.7	0.7	0.7
choleskyStabilizeThreshold	10^{-40}	10^{-40}	10^{-100}	10^{-120}
maxComplementarity	10^{100}	10^{130}	10^{160}	10^{180}

Table J.2: SDPB parameters for the computations of scaling dimension bounds in this work. For C_T bounds we need to set all of the Boolean parameters in the table to False. In addition to that, we used `dualityGapThreshold` = 10^{-6} , while all the rest of the parameters were kept at the same values as for the dimension bounds.

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