Asymptotic Behavior of Compact Skew-product Semiflows on Hilbert Spaces

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Programa Doutoral em Matemática Aplicada

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Resumo

Considere-se uma família de produtos cruzados de semi-fluxos sobre um fluxo ergódico $\varphi^t : M \to M$ definida num espaço de Hilbert de dimensão infinita separável. Assumindo que M é um espaço de Hausdorff compacto e que φ^t preserva uma medida de probabilidade de Borel μ, regular e ergódica, positiva em conjuntos abertos não vazios, concluímos que existe um subconjunto residual dos produtos cruzados de semi-fluxos (em relação a uma topologia τ) para o qual, para μ quase todo o ponto x, a decomposição de Oseledets-Ruelle ao longo da órbita de x é uniformemente hiperbólica (no espaço projetivo) ou o operador limite de Ruelle ao longo da órbita de x é o operador nulo. Provámos também uma versão L^p desta dicotomia para cociclos com uma topologia do tipo L^p definida no conjunto dos geradores infinitesimais. De facto quebrámos a dicotomia e obtivemos um espectro com todos os expoentes de Lyapunov-Ruelle iguais. Finalmente provámos que os cociclos não uniformemente Anosov são C^0 densos na família dos cociclos parcialmente hiperbólicos com subespaços instáveis não triviais.

MSC 2000: Primary: 37D30; Secondary: 47A10
keywords: Lyapunov exponents; Skew-product infinite dimensional semiflows; Multiplicative ergodic theorem; Dominated splitting.

Abstract

We consider an infinite dimensional separable Hilbert space and its family of skew-product compact semiflows over an ergodic flow $\varphi^t\colon M \to M$. Assuming that M is a compact Hausdorff space and φ^t preserves a Borel regular ergodic probability μ which is positive on non-empty open sets, we conclude that there is a τ-residual subset of skew-product semiflows within which, for μ almost every x, either the Oseledets-Ruelle's decomposition along the orbit of x is uniformly hyperbolic (in the projective space) or else the Ruelle's limit operator over the orbit of x is the null operator. We also prove an L^p version of this dichotomy for cocycles with a L^p topology defined in infinitesimal generators set. In fact, we drop the dichotomy and reach a one-point spectrum under an L^p generic assumption. Finally, we prove that non-uniformly Anosov skew-products are C^0-dense in the family of partially hyperbolic cocycles with non-trivial unstable bundles.

MSC 2000: Primary: 37D30; Secondary: 47A10
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Maria da Glória Ferreira de Carvalho

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Abbreviations and Symbols

The numbers that follow the symbols indicate the chapter or section where their meanings are explained.

X	Chapter 1	$E_i(x)$	2.3
M	Chapter 1	$\wedge^k(\mathscr{H})$	2.4
φ^t	Chapter 1	$\lambda_1^{\wedge^k}$	2.4
$D\varphi_x^t$	Chapter 1	LE_k	2.4
$\lambda^+(x,v)$	Chapter 1	$\Lambda_p(A,m)$	Chapter 3
μ	Chapter 1	$\Gamma_p(A,m)$	Chapter 3
$\mathscr{H}$	1.1	$\Gamma_p^*(A,m)$	Chapter 3
$\mathscr{C}(\mathscr{H})$	1.1	$\Gamma_{p,1}^*$	Chapter 3
π	1.1	$\Gamma_{p,2}^*$	Chapter 3
$A(x)$	1.1	R_θ	Chapter 3
$(A(x))^*$	1.1	$\mathscr{R}^t$	Chapter 3
$F(A)$	1.1	$H(\xi,t)$	Chapter 3
$\log^+(y)$	1.1	E_p	Chapter 3
$C_I^0(X,C(\mathscr{H}))$	1.1	$E_p^\perp$	Chapter 3
$\mathscr{R}$	1.1	$\lfloor A \rfloor$	3.1
Φ	1.2	$\mathbb{R}v_N$	3.1
Φ_A^t	1.2	$\mathscr{G}_{IC}$	Chapter 4
$\mathscr{L}(\mathscr{H},\mathscr{H})$	1.2	$\mathscr{G}$	Chapter 4
$\mathscr{F}^0$	2.1	$\Vert A-B \Vert$	Chapter 4
$\rho_{a,b}$	2.1	$d_p(A,B)$	Chapter 4

$\hat{\Phi}$	2.1	L^p	Chapter 4
ω	2.1	$\mathscr{F}$	4.2
$\mathbb{S}$	2.1	ζ	4.2
$\overline{\mathbb{S}}^{\omega}$	2.1	$\hat{\mu}$	4.2
$C^0([a,b]\times X\times\overline{\mathbb{S}}^{\omega}, X\times\overline{\mathbb{S}}^{\omega})$	2.1	$J_K(A)$	4.2
$D(0,1)$	2.1	$\succ_{\alpha,\ell}$	5.1
f_n	2.1	SF_{E_1,E_2}	5.1
$B(\xi,r)$	2.1	E^u_x	5.2
d	2.1	E^s_x	5.2
ω_d	2.1	E^c_x	5.2
τ	2.1	E^{uc}_x	5.2
$\mathscr{L}(x)$	2.2	$\mathscr{P}$	5.4
$U_i(x)$	2.2	$\pi_{E^u_{x,A}}$	5.4
$V_i(x)$	2.2	$\pi_{E^c_{x,A}}$	5.4
$\mathscr{O}(A)$	2.2		
$\succ_\ell$	2.3		

Other Symbols

C^k-manifold	Chapter 1	manifold with an atlas whose transitions are all k-times continuosly differentiable
C^∞	Chapter 1	a C^k-manifold for all k
TM	Chapter 1	tangent bundle of M
T_xM	Chapter 1	tangent space to M at point x
$\mathbb{R}$	Chapter 1	Real field
$\mathbb{R}^n$	Chapter 1	n-dimensional vector space over $\mathbb{R}$
$\measuredangle$	Chapter 1	angle
Id	1.1	identity matrix
$\emptyset$	2.1	empty set
$<\xi-x_0, f_i>$	2.1	image of f_i at $\xi-x_0$
$\mathscr{H}^*$	2.1	dual of $\mathscr{H}$
$\|\cdot\|$	2.1	norm
$\rho_{a,b}(\mathscr{F}^0)$	2.1	image of $\mathscr{F}^0$ by $\rho_{a,b}$
$\rho^{-1}_{a,b}(y)$	2.1	inverse image
$\mathbb{N}$	2.2	Natural field $\mathbb{N}$
$\oplus$	2.3	direct sum
$SO(D,\mathbb{R})$	5.4	special orthogonal group of matrices $D\times D$ with entries from $\mathbb{R}$
$SL(D,\mathbb{R})$	5.4	special linear group of matrices $D\times D$ with entries from $\mathbb{R}$

Abbreviations

NB	Not Bounded
ND	Not Dominated

Chapter 1

Introduction

Given any smooth dynamical system, the Lyapunov exponents measure the exponential behavior of the tangent map. A positive (or negative) Lyapunov exponent gives us the average exponential rate of divergence (or convergence) of two neighbouring trajectories whereas zero exponents give us the absence of any kind of exponential behaviour. For any smooth diffeomorphism with non-zero Lyapunov exponents we obtain a rich information about geometric properties of the system, namely stable/unstable manifold theory. These geometric tools are the base of most of the central results on dynamical systems nowadays. So it is important to detect when do Lyapunov exponents vanish (see (5; 31)) and also when zero Lyapunov exponents can be removed.

We will start to define the concept of Lyapunov exponents in dynamical system context.

Consider a C^∞, compact, connected and boundaryless Riemannian n-manifold M, a vector field $X : M \to TM$ of class C^s, $s \geqslant 1$ and the flow $\varphi^t : M \to M$ associated with X, namely,

$$\frac{d\varphi^t}{dt}|_{t=s}(x) = X(\varphi^s(x)).$$

For each $x \in M$, the flow $\varphi^t(x)$ has a tangent map $D\varphi^t_x$ which is solution of the non-autonomous linear differential equation

$$\dot{U}(t) = A(x,t) \cdot U(t), \tag{1.0.1}$$

where

$$A(x,t) = DX_{\varphi^t(x)}.$$

The stability of the trajectory $\{\varphi^t(x) : t \in \mathbb{R}\}$ can be described by studying small perturbations of the linear differential equation (1.0.1).

Given $x \in M$ and $v \in T_xM$, the formula

$$\lambda^+(x,v) = \lim_{t\to+\infty} \sup \frac{1}{t} \log \left\| D\varphi_x^t \cdot v \right\| \tag{1.0.2}$$

defines the *Lyapunov exponent* at the point x in the direction of $v \in T_xM \setminus \{0\}$.

For each $x \in M$ and $v \in T_xM$ the function $\lambda^+(x,\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ attains only finitely many values $\lambda_1^+ < \cdots < \lambda_{s^+(x)}$ where $s^+(x) \leqslant n$ and $\lambda^+(x,0) = -\infty$.

In an analogous way we can define the Lyapunov exponent when $t \to -\infty$ by

$$\lambda^-(x,v) = \lim_{t\to-\infty} \sup \frac{1}{t} \log \left\| D\varphi_x^t \cdot v \right\|. \tag{1.0.3}$$

For each $x \in M$ and $v \in T_xM$ the function $\lambda^-(x,\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ attains only finitely many values $\lambda_1^- > \cdots > \lambda_{s^-(x)}^-$ where $s^-(x) \leqslant n$ and $\lambda^-(x,0) = -\infty$.

If $\lambda_s^+ < 0$ then the trivial solution $U(t) \equiv 0$ is asymptotically stable, and even exponentially asymptotically stable (5). Lyapunov introduced also regularity conditions which guarantee that exponential stability remains valid for nonlinear perturbations.

In order to simplify the notation, in what follows, we will drop the superscript $+$ from the forward Lyapunov exponents if it does not cause any confusion.

Definition 1.0.1. *The point $x \in M$ is Lyapunov regular if and only if exists a decomposition*

$$T_xM = \bigoplus_{i=1}^{s(x)} E_i(x) \tag{1.0.4}$$

into subspaces $E_i(x)$, and numbers $\lambda_1(x) < \cdots < \lambda_{s(x)}(x)$ such that:

(i) $E_i(x)$ is invariant under $D\varphi_x^t$, i.e.,

$$D\varphi_x^t E_i(x) = E_i(\varphi_x^t(x))$$

and depends (Borel) measurably on x;

(ii) for $v \in E_i(x) \setminus \{0\}$,

$$\lim_{t\to\pm\infty} \sup \tfrac{1}{t} \log \|D\varphi_x^t \cdot v\| = \lambda_i(x)$$

with uniform convergence on $\{v \in E_i(x) : \|v\| = 1\}$*;*

(iii) for any $v, w \in T_xM \setminus \{0\}$,

$$\lim_{t\to\pm\infty} \sup \tfrac{1}{t} \log \measuredangle(D\varphi_x^t \cdot v, D\varphi_x^t \cdot w) = 0.$$

The decomposition (1.0.4) is called the *Oseledets decomposition* associated with the Lyapunov exponent λ^+. The regularity condition is usually difficult to check in specific situations but the multiplicative ergodic theorem of Oseledets (Theorem 1.0.1) asserts that regularity is "typical" from the measure-theoretical point of view, that is, it holds on almost every flow trajectory, relative to any probability measure invariant under the flow.

Theorem 1.0.1. ***Multiplicative Ergodic Theorem (Oseledets, 1968)*** *If M is a compact Riemannian manifold, $\varphi^t : M \to M$ is a flow associated with a vector field X and μ is a finite measure φ^t-invariant on M, then μ- almost every point $x \in M$ is regular.*

A seminal result for discrete systems is the *Bochi-Mañé* theorem, which provides a C^1-residual set of area-preserving diffeomorphisms on surfaces where either we have Anosov systems (i.e. uniform hyperbolicity in all points) or for Lebesgue almost every point, we have zero Lyapunov exponents. This theorem was announced in the beginning of the 1980's by Mañé in (29) but there was only available a sketch of a proof, (see (30)). A complete proof, due to Bochi, appeared only in (17).

Next, Bochi-Viana in (20) (see also (18)) extended this result to a large class of discrete systems: volume-preserving diffeomorphisms with arbitrary dimension, symplectic maps and also linear cocycles. In this case we no longer obtain a global result. Instead, it is obtained a C^1-residual subset such that for almost every orbit we have a dominated splitting (or partial hyperbolicity in the symplectic case) or else the Lyapunov exponents are zero. For a survey of the theory for the discrete-case see (19).

Bessa started the approach for the continuous-time setting by proving in (6) the dichotomy (dominated-splitting *versus* zero Lyapunov exponents) for 2-dimensional conservative linear differential systems. Then, in (7) he proved the version for three-dimensional divergence-free vector fields without singularities (see (1) for the correspondent statement for vector fields with singularities). For higher dimensions we have available the results by Bessa in (8) and by Bessa and Rocha in (12).

Bessa and Carvalho started the approach of the generalization of previous results on cocycles but for the discrete infinite dimensional setting (9) (see Section 1.1).

Table 1.1: Results already proved on the C^0-generic dichotomy zero exponents versus dominated splitting.

	Discrete-time	Continuous-time
Low dimension	Mañé, 1983, Bochi, 2002 (17)	Bessa, 2006 (6) Bessa and Dias, 2008 (11)
High dimension	Bochi and Viana, 2005 (20) Bochi, 2010 (18)	Bessa, 2008 (8)
Infinite dimension	Bessa and Carvalho, 2008 (9)	Theorem A

Bessa and Stadlbauer proved in (13) similar results for the semigroup of stochastic matrices.

We can also find in (14) identical results using an L^p-topology. Bessa and Vilarinho proved that the class of accessible and saddle-conservative cocycles L^p-densely have a simple spectrum and that for an L^p-residual subset of accessible cocycles we have a one-point spectrum. They also proved versions of previous results for linear differential system, include infinite dimensional discrete cocycles.

Recently, Avila, Crovisier and Wilkinson in (4) obtained a dichotomy for C^1-generic, volume-preserving diffeomorphisms: either all the Lyapunov exponents of almost every point vanish or the volume is ergodic and non-uniformly Anosov (i.e. non-uniformly hyperbolic and the splitting into stable and unstable spaces is dominated). Their theorem is the sharper result concerning these type of dichotomies.

Bessa and Carvalho also proved in (10) that non-uniformly Anosov cocycles (partially hyperbolic cocycles without null Lyapunov exponents) are C^0-dense in the family of partially hyperbolic cocycles with non-trivial unstable bundles (see Definitions 5.2.1 and 5.2.2).

In this thesis we are interested in the continuous-time counterpart of the main theorem in (9) (see Theorem A). The strategy of the proof for the continuous-time case is similar to the one used in the discrete-time case. However the perturbations schemes used are quite different. We also proved a similar result in an L^p-topology, using the ideias of (14) (see Theorem B) and also the continuous counterpart of (10) .

1.1 Infinite Dimension - The Discrete-time Case

Let $\mathscr{H}$ be an infinite dimensional separable Hilbert space and $\mathscr{C}(\mathscr{H})$ the set of linear compact operators acting in $\mathscr{H}$ with the uniform norm. Consider a homeomorphism $f: M \to M$ of a compact Hausdorff space M and μ, an f-invariant Borel regular measure that is positive on non-empty open subsets. Given a family $(A_x)_{x\in M}$ of linear operators in

$\mathscr{C}(\mathscr{H})$ and a continuous vector bundle $\pi : M \times \mathscr{H} \to M$, we define the *associated skew product over f* by

$$\begin{array}{rlcl} F(A): & M \times \mathscr{H} & \to & M \times \mathscr{H} \\ & (x,v) & \to & (f(x), A(x) \cdot v). \end{array}$$

The map F satisfies the equality $\pi \circ F = f \circ \pi$ and, for all $x \in M$, $F_x(A) : \mathscr{H} \to \mathscr{H}$ is linear on the fiber $\mathscr{H} := \pi^{-1}(\{x\})$. We call A a *cocyle*.

A *random product* of a cocycle $A : M \to \mathscr{C}(\mathscr{H})$ associated to the map f is the sequence, indexed by $x \in M$, of linear maps of $\mathscr{H}$ defined, for each $n \in \mathbb{N}_0$, by $A^0(x) = \mathrm{Id}$ and

$$A^n(x) = A(f^{n-1}(x)) \circ \cdots \circ A(f(x)) \circ A(x).$$

In (9) were studied the asymptotic properties of random products for most points x, i.e, the limit

$$\lim_{n\to\infty} ((A(x))^{*^n} A(x)^n)^{\frac{1}{2n}},$$

where $(A(x))^*$ denotes the dual operator of $A(x)$.

Under the integrability hypothesis

$$\int_M \log^+ \|A(x)\| \, d\mu(x) < \infty,$$

where $\log^+(y) = \max\{0, \log(y)\}$, the theorem of Ruelle (33) offers, for μ-almost every point $x \in M$, a description of a complete set of Lyapunov exponents for the above limit of operators and associated invariant directions.

Consider $C^0_I(M, \mathscr{C}(\mathscr{H}))$ the set of integrable compact cocycles. In (9) it was proved that:

Theorem 1.1.1. *There exists a C^0-residual subset $\mathscr{R}$ of $C^0_I(M, \mathscr{C}(\mathscr{H}))$ such that, for $A \in \mathscr{R}$ and μ-almost every $x \in M$, either the limit $\lim_{n\to\infty} ((A(x))^{*^n} A(x)^n)^{\frac{1}{2n}}$ is the null operator or the Oseledets-Ruelle's splitting of A along the orbit of x is dominated (see Definition 2.3.1).*

1.2 Infinite Dimension - The Continuous-time Case

As in previous section let $\mathscr{H}$ be an infinite dimensional separable Hilbert space (we can view $\mathscr{H}$ as the set of square integrable L^2-functions on a measurable space) and $\mathscr{C}(\mathscr{H})$ the space of linear compact operators defined in $\mathscr{H}$. Let M be a compact, connected, Hausdorff manifold endowed with a Borel probability measure μ.

Consider a continuous and ergodic flow $\varphi^t : M \to M$ with respect to the measure μ, i.e, any measurable and φ^t-invariant subset of M has zero or full μ-measure. A linear cocycle based on φ^t is defined by a flow $\Phi^t(p) \in \mathscr{C}(\mathscr{H})$, continuous on space parameter $p \in M$ and on time the parameter t.

The flow Φ^t satisfies the *cocycle* identity:

$$\Phi^{t+s}(p) = \Phi^s(\varphi^t(p)) \circ \Phi^t(p), \tag{1.2.1}$$

for all $t, s \in \mathbb{R}$ and $p \in M$.

Let $\mathscr{L}(\mathscr{H}, \mathscr{H})$ be the set of linear applications from $\mathscr{H}$ to $\mathscr{H}$. We can write $\Phi^t(p)$ as

$$\Phi^t(p) = \mathrm{Id} + \int_0^s A(\varphi^s(p)) \cdot \Phi^s(p)\, \mathrm{d}s, \tag{1.2.2}$$

where $A : M \to \mathscr{L}(\mathscr{H}, \mathscr{H})$ is a measurable map such that $A(\varphi^t(p)) \cdot \Phi^t(p) \in L^1(M, \mu)$.

We can also define the *linear C^0- skew product flow*

$$\begin{array}{rccc} \Phi : & \mathbb{R} \times M \times \mathscr{H} & \to & M \times \mathscr{H} \\ & (t, x, v) & \to & (\varphi^t(x), \Phi^t(x) \cdot v). \end{array} \tag{1.2.3}$$

Once again we are interested in the asymptotic properties of the limit

$$\lim_{t\to\infty} ((\Phi^t(x))^* \Phi^t(x))^{\frac{1}{2t}},$$

where $(\Phi^t(x))^*$ denotes the dual operator of $\Phi^t(x)$.

Let $\mathscr{F}^0$ be set of continuous maps $\Phi : \mathbb{R} \times M \times \mathscr{H} \to \mathscr{H}$ such that $\Phi^t(x)$ is a compact operator acting on $\mathscr{H}$, $\forall t \in \mathbb{R}$ and $\forall x \in M$ and satisfies (1.2.1). We intend to prove the following result:

Theorem A. *There exists a τ-residual*[1] *subset $\mathscr{R}$ of the set of cocyles $\mathscr{F}^0$ such that for $\Phi \in \mathscr{R}$ and μ-almost every $x \in M$, either the limit operator $\lim_{t\to\infty} ((\Phi^t(x))^* \Phi^t(x))^{\frac{1}{2t}}$ is the null*

[1] See the definition of τ-topology in Section 2.1.

operator or the Oseledets-Ruelle's splitting of Φ^t along the φ^t-orbit of x has a dominated splitting.

The idea to prove Theorem A is the following. We take a cocyle which is a continuity point of an upper-semicontinuous function (see Section 2.4) and if, for this skew-product, the Lyapunov exponents of $\Phi^t(x)$ are not all equal, i.e, the limit operator $\lim_{t\to\infty}((\Phi^t(x))^*\Phi^t(x))^{\frac{1}{2t}}$ is not the null operator and there is no dominated splitting along the φ^t-orbit of x, we construct small τ-perturbations which allow us to breaking the continuity, obtaining a contradiction. Finally, as the set $\mathscr{R}$ of points of continuity of upper-semicontinuous functions is a residual set the Theorem A follows. Moreover, as the set of the skew-product flows with τ-topology is a Baire space then $\mathscr{R}$ is dense.

1.3 Examples of Cocycles

The study of Lyapunov exponents has an unusually vast array of interactions with other areas of Mathematics and Physics, such as stochastic processes (random matrices and, more generally, random walks on Lie groups (see (25))), spectral theory, smooth dynamics (see (40)) and in control theory (see (23)).

Simple examples are the continuous quasi-periodic Schrödinger equations, the Mathieu's equations and, more closely related with our study, the dynamical cocycles on infinite dimension.

1.3.1 Schrödinger cocyles

Consider the Schrödinger equation

$$(H_\theta u)(t) = -\frac{d^2u}{dt^2}(t) + \lambda^2 V(t, \theta + \omega t)u(t) = Eu(t), \tag{1.3.1}$$

where $V : \mathbb{T}^2 \to \mathbb{R}$ ($\mathbb{T} = \mathbb{R}/\mathbb{Z}$) is the potential function, E the energy and $\omega \in \mathbb{R}\backslash\mathbb{Q}$ is the frequency. The operator H_θ is called the Schrödinger operator, acting on the Hilbert space $L^2(\mathbb{R})$ of square integrable functions.

We can study (1.3.1) from dynamical systems point of view. With $u_1 := u$ and $u_2 := \dot{u}$ we can write it as the traceless system

$$\begin{pmatrix} \dot{u_1} \\ \dot{u_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \lambda^2 V(t, \theta + \omega t) - E & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{1.3.2}$$

Denote by $F_E(t,\theta)$ the fundamental solution of (1.3.2), i.e.,

$$F_E'(t,\theta)=\begin{pmatrix} 0 & 1 \\ \lambda^2 V(t,\theta+\omega t)-E & 0\end{pmatrix} F_E(t,\theta) \text{ with } F_E(0,\theta)=\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}.$$

We have that $F_E(t,\theta)\in SL(2,\mathbb{R})$ for all t.

We can also study the dynamic of the system (1.3.2) via the skew-product mapping

$$\begin{array}{rccc} G_E: & \mathbb{T}\times\mathbb{R}^2 & \to & \mathbb{T}\times\mathbb{R}^2 \\ & (\theta,v) & \to & (\theta+\omega t, F_E(t,\theta)v). \end{array}$$

The average exponential growth of the fundamental solution is measured by the Lyapunov exponent

$$\gamma(E)=\lim_{t\to\infty}\frac{1}{t}\int_{\mathbb{T}}\log\|F_E(t,\theta)\|\,d\theta\geqslant 0,$$

which exists by subadditivity and which is non-negative since F_E is in $SL(2,\mathbb{R})$. For more details see (15).

1.3.2 Mathieu's equation

The Mathieu's equation is a special case of a linear second order homogeneous differential equation such as occurs in many applications in physics an engineering.

Let $q_1,q_2:\mathbb{R}\to\mathbb{R}$ be T-periodic functions and suppose that q_1 is continuously differentiable and q_2 is continuous. Consider the periodic linear oscillator

$$\ddot{y}+2q_1(t)\dot{y}+q_2(t)y=0. \tag{1.3.3}$$

An example for (1.3.3) is

$$\ddot{y}+2k\dot{y}+(a+\varepsilon\cos(2t))y=0 \text{ with } k>0. \tag{1.3.4}$$

The substitution of the form $x(t) = y(t)\exp(kt)$, $t \in \mathbb{R}$, yields Mathieu's equation (see (24))

$$\ddot{x} + (\delta + \varepsilon\cos(2t))x = 0, \tag{1.3.5}$$

with $\delta := a - k^2$. This is a linear oscillator with periodic restoring force.

With $y_1 := y$ and $y_2 := \dot{y}$, the second order equation (1.3.4) is equivalent to the system

$$\begin{pmatrix} \dot{y_1} \\ \dot{y_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a - \varepsilon\cos(2t) & -2k \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{1.3.6}$$

Similarly, with $x_1 := x$ and $x_2 := \dot{x}$, we can write (1.3.5) as the system

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -(a-k^2) - \varepsilon\cos(2t) & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{1.3.7}$$

The solutions satisfy for $t \in \mathbb{R}$,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{-kt} \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix},$$

hence, for initial values $(x_1,0,x_2,0) = (y_1,0,y_2,0)$, the Lyapunov exponents are related by

$$\lambda(x_1,0,x_2,0) = \lambda(y_1,0,y_2,0) - k.$$

For physical interpretation of Mathieu's equation and more details see (24).

1.3.3 Dynamical Cocycles on Infinite Dimension

Recently Blumenthal and Young, in their notable work (16), generalized the classic results (27), (28) to Banach space mappings f preserving a compactly supported Borel probability measure μ with finitely many positive Lyapunov exponents.

Let $(\mathscr{B}, |\cdot|)$ be a Banach space. Consider (f,μ), where $f : \mathscr{B} \to \mathscr{B}$ is a mapping and μ is an f-invariant Borel probability measure. The following properties are assumed:

(H1) (i) f is C^2 Fréchet differentiable and injective;

(ii) the derivative of f at $x \in \mathscr{B}$, denoted Df_x, is also injective.

(H2) (i) f leaves invariant a compact set $\mathscr{A} \subset \mathscr{B}$, with $f(\mathscr{A}) = \mathscr{A}$;

(ii) μ is supported on $\mathscr{A}$.

(H3) Assume

$$l_\alpha(x) := \lim_{n\to\infty} \frac{1}{n} \log |Df_x^n|_\alpha < 0 \text{ for } \mu\text{-a.e. x},$$

where $|Df_x^n|_\alpha$ is the Kuratowski measure of noncompactness of the set $Df_x^n(B)$, where B is the unit ball in $\mathscr{B}$.

(H4) (f, μ) has no zero Lyapunov exponent.

(H5) the set $\mathscr{A}$ has finite box-counting dimension.

An *SRB measure* can be defined as a measure μ with a positive Lyapunov exponent μ-a.e. and such that the conditional measures of μ on unstable manifolds are in the "Lebesgue measure class" induced on these manifolds (see Definition 2.1, (16)). Blumenthal and Young proved, under conditions (H1)-(H5) and in rought terms, that μ is an *SRB* measure if and only if the entropy of f is equal to the sum of its positive Lyapunov exponents, say Pesin's entropy formula holds.

It is important to point out that the class of mappings to which this result applies includes (but is not limited to) time-t maps of semiflows defined by periodically forced nonlinear dissipative parabolic partial differential equations.

Although the approach used by Blumenthal and Young in their results is similar to that we do in this thesis, they work with the "derivative" cocycle Df_x, which we call *dynamical cocycle*. So, they study the Lyapunov exponents of Df where f is a smooth dynamical system. In this thesis we do not consider Df. Instead, we consider a continuous and ergodic flow $\varphi^t : M \to M$ and choose a cocycle $\Phi_A^t(p)$ based on φ^t acting on $\mathscr{C}(\mathscr{H})$ (see Section 1.2). So when we consider a perturbation B of A (or a perturbation Φ_B^t of Φ_A^t), this B do not need to be associated to a derivative cocycle of a flow near φ^t.

The ideas of Blumental and Young may be fundamental in the attempt to generalize our results for dynamical cocycles.

1.4 Thesis Structure

Besides introduction this work is organized in four chapters. Some preliminaries results, as the definition of the τ-topology, the multiplicative ergodic theorem, the definition of dominated splitting and entropy functions, are presented in Chapter 2. In Chapter 3 it is established the perturbation lemmas which will be used in the proof of Theorem A in Section 3.3. In Chapter 4 it is proved an identical result in a less exigent topology, the L^p infinitesimal generator topology. Finally in Chapter 5 we use a same type of perturbation to remove zero Lypaunov exponents in a topology sharper than τ, inspired in the result of (10).

Chapter 2

Preliminaries Results

2.1 Topologies

In previous section we denoted the set of C^0 compact cocycles by $\mathscr{F}^0 \subset C^0(\mathbb{R} \times M \times \mathscr{H}, \mathscr{H})$. Given any $a, b \in \mathbb{R}$ with $a < b$ consider the restriction map

$$\begin{array}{rcl} \rho_{a,b}: \ \mathscr{F}^0 & \to & C^0([a,b] \times M \times \overline{\mathbb{S}}^{\omega}, \overline{\mathbb{S}}^{\omega}) \\ \Phi & \to & \hat{\Phi}: \ [a,b] \times M \times \overline{\mathbb{S}}^{\omega} \ \to \ \overline{\mathbb{S}}^{\omega}, \end{array}$$

such that

$$\begin{array}{rcl} \hat{\Phi}(t,x,v) & = & \frac{\Phi^t(x)\cdot v}{\|\Phi^t(x)\cdot v\|} \ \text{ if } \ \|\Phi^t(x)\cdot v\| > 1, \\ \hat{\Phi}(t,x,v) & = & \Phi^t(x)\cdot v \ \text{ if } \ \|\Phi^t(x)\cdot v\| \leqslant 1, \end{array}$$

where $\overline{\mathbb{S}}^{\omega}$ is the weak closure of the unit sphere on $\mathscr{H}$.

First, we will prove that $\overline{\mathbb{S}}^{\omega}$ is compact for the weak topology ω and that ω is metrizable inside $\overline{\mathbb{S}}^{\omega}$ (see (34)).

Lemma 2.1.1. *Let $\mathbb{S} = \{\xi \in \mathscr{H} : \|\xi\| = 1\}$ be the unit sphere of $\mathscr{H}$, where $\|\cdot\|$ is the norm induced by an inner product in $\mathscr{H}$. The weak closure $\overline{\mathbb{S}}^{\omega}$ of $\mathbb{S}$ is the closed unit ball $D(0,1) = \{\xi \in \mathscr{H} : \|\xi\| \leqslant 1\}$.*

Proof. As the closed unit ball $D(0,1)$ is a norm closed convex subset of $\mathscr{H}$, it is also weakly closed and thus it contains the weak closure $\overline{\mathbb{S}}^{\omega}$ of $\mathbb{S}$. Suppose that $\overline{\mathbb{S}}^{\omega}$ is not equal to $D(0,1)$. Then there exists an element $\xi_0 \in D(0,1)$ such that ξ_0 is not in $\overline{\mathbb{S}}^{\omega}$. In this case we can find a weakly open neighbourhood V of ξ_0 such that $V \cap \overline{\mathbb{S}}^{\omega} = \emptyset$. By definition of weak topology ω on $\mathscr{H}$, we can find finitely many $f_1, \cdots, f_k \in \mathscr{H}^\star$, and a (sufficiently small) $\varepsilon > 0$ such that

$$\xi_o \in D(\xi_0, f_1, \cdots f_k, \varepsilon) = \xi_0 + D(0, f_1, \cdots, f_k, \varepsilon) \subseteq V.$$ [1]

In this case, we must have $\xi_0 + D(0, f_1, \cdots, f_k, \varepsilon) \cap \overline{\mathbb{S}}^{\omega} = \emptyset$.

Now since $\mathscr{H}$ is infinite dimensional, $D(0, f_1, \cdots, f_k, \varepsilon)$ must contain an infinite dimensional subspace of $\mathscr{H}$. So we can find a non-zero element $\eta_0 \in D(0, f_1, \cdots, f_k, \varepsilon)$ such that

$$t\eta_0 \in D(0, f_1, \cdots, f_k, \varepsilon),$$

for all $t \in \mathbb{R}$. Notice that $N : t \in \mathbb{R} \to \|\xi_0 + t\eta_0\| \in [0, \infty)$ defines a continuous function with $N(0) = \|\xi_0\| < 1$ (since $\xi_0 \notin \mathbb{S}$), and $N(t) \to \infty$ when $t \to \infty$. By Intermediate Value Theorem, we can choose some $t_0 > 0$ such that $\|\xi_0 + t_0\eta_0\| = 1$. This implies that $\xi_0 + t_0\eta_0 \in \xi_0 + D(0, f_1, \cdots, f_k, \varepsilon) \cap \mathbb{S}$, which is a contradiction. So we must have $\overline{\mathbb{S}}^{\omega} = D(0,1)$. □

Proposition 2.1.2. *The weak closure $\overline{\mathbb{S}}^{\omega}$ of $\mathbb{S}$ is compact for the weak topology.*

Proof. By Banach-Alaoglu theorem the closed unit ball in $\mathscr{H}^\star$ defined by

$$D(0,1)^\star = \{\lambda \in \mathscr{H}^\star : |\lambda(\xi)| \leqslant 1 \ , \ \|\xi\| \leqslant 1\}$$

is compact with respect to the weak star topology (denoted by $\omega^\star$) in $\mathscr{H}^\star$. Since we are dealing with a separable Hilbert space $\mathscr{H}$, which is reflexive, then we know that $D(0,1)$ is also compact for ω in $\mathscr{H}$. As, by Lemma 2.1.1, $\overline{\mathbb{S}}^{\omega} = D(0,1)$ then $\overline{\mathbb{S}}^{\omega}$ is compact for ω. □

We need now to prove that ω is metrizable inside $\overline{\mathbb{S}}^{\omega}$ from a metric d. We recall that a family $\mathscr{L}$ of functionals $\lambda : \mathscr{H} \to \mathbb{R}$ *separates points* in a set $K \subset \mathscr{H}$ if for any distinct $\xi, \nu \in K$, there exists $\lambda \in \mathscr{L}$ such that $\lambda(\xi) \neq \lambda(\nu)$. Clearly, when we have $\mathscr{L} = \mathscr{H}^\star$ we get that the family $\mathscr{L}$ separate points in $K = \mathscr{H}$.

Lemma 2.1.3. *The weak topology ω is metrizable inside $\overline{\mathbb{S}}^{\omega} \subset \mathscr{H}$, i.e., ω derives from a metric d.*

[1] Recall that $D(\xi_0, f_1, \cdots f_k, \varepsilon) = \{\xi \in \mathscr{H} : | < \xi - \xi_0, f_i > | < \varepsilon, i \in \{1, \cdots, k\}\}$.

Proof. As $\mathscr{H}$ is separable there exists a dense sequence $\xi_n \subset \mathscr{H}$. Making the usual isomorphism between $\mathscr{H}$ and $\mathscr{H}^\star$ established by the Riesz representation theorem (i.e. $\lambda_\xi(\nu) = \langle \nu, \xi \rangle$ identifying λ with ξ) we can conclude that $\lambda_{\xi_n} \in \mathscr{H}^\star$ is a dense family of functionals which separates points in $\mathscr{H}$ and, in particular, separates points in any subset of $\mathscr{H}$. Indeed, if by contradiction there exist distinct $\xi, \nu \in \mathscr{H}$, such that for all λ_{ξ_n} we have $\lambda_{\xi_n}(\xi) = \lambda_{\xi_n}(\nu)$, that is, $\langle \xi, \xi_n \rangle = \langle \nu, \xi_n \rangle$ for all n, hence $\lambda_\xi(\xi_n) = \lambda_\nu(\xi_n)$ for all n. Then $\lambda_\xi = \lambda_\nu$ (i.e. $\xi = \nu$) because they coincide in a dense subset of $\mathscr{H}$ and are continuous linear functionals.

The proof of the lemma is finished once we prove the following claim.

Claim 2.1.1. *If exists a family of continuous functions $f_n \colon \overline{\mathbb{S}}^\omega \to \mathbb{R}$ which separate points in $\overline{\mathbb{S}}^\omega$, then there exists a metric $d \colon \overline{\mathbb{S}}^\omega \times \overline{\mathbb{S}}^\omega \to [0, \infty)$ which generates ω.*

Suppose, without loss of generality, that $\|f_n\|_\infty \leqslant 1$ for all n, and let ω_d be the topology induced on $\overline{\mathbb{S}}^\omega$ by the metric

$$d(\xi, \nu) = \sum_{n \in \mathbb{N}} \frac{1}{2^n} |f_n(\xi) - f_n(\nu)|, \tag{2.1.1}$$

with ξ, $\nu \in \overline{\mathbb{S}}^\omega$.

This is indeed a metric since $\{f_n\}$ separates points in $\overline{\mathbb{S}}^\omega$. Since each f_n is ω-continuous and the series (2.1.1) converges uniformly on $\overline{\mathbb{S}}^\omega \times \overline{\mathbb{S}}^\omega$, d is a ω-continuous function on $\overline{\mathbb{S}}^\omega \times \overline{\mathbb{S}}^\omega$. The balls

$$B(\xi, r) = \left\{ \nu \in \overline{\mathbb{S}}^\omega : d(\xi, \nu) < r \right\}$$

are therefore ω-open. Thus $\omega_d \subset \omega$.

To prove that $\omega_d = \omega$ let $C \in \overline{\mathbb{S}}^\omega$ be ω-closed. Since $\overline{\mathbb{S}}^\omega$ is ω-compact, so is C. Since $\omega_d \subset \omega$, it follows that C is ω_d-compact (every ω_d open cover of C is a ω-open cover). As compact sets in metric spaces are closed then C is ω_d-closed. So the claim is proved.

Since $\overline{\mathbb{S}}^\omega$ is a compact metric space then it is complete. $\square$

Note that the domain $[a,b] \times M \times \overline{\mathbb{S}}^\omega$ and the image $\overline{\mathbb{S}}^\omega$ are topological spaces endowed with the product topology, given by the absolute-value norm in $[a,b]$, the distance function in M and the ω topology in $\overline{\mathbb{S}}^\omega$. So we consider the compact-open topology on $C^0([a,b] \times M \times \overline{\mathbb{S}}^\omega, \times \overline{\mathbb{S}}^\omega)$.

Now we topologize $\mathscr{F}^0$ with the initial topology (the one which has less open sets and makes $\rho_{a,b}$ continuous), which we will denote by τ. As in (32) it is independent of a and b.

As each component of $[a,b] \times M \times \overline{\mathbb{S}}^{\omega}$ is compact with a countable base and $\overline{\mathbb{S}}^{\omega}$ is a complete metric space, we can apply Theorem 4.1 of (26) to conclude that $C^0([a,b] \times M \times \overline{\mathbb{S}}^{\omega}, \overline{\mathbb{S}}^{\omega})$ has a complete metric, which is essential to prove the next lemma.

Lemma 2.1.4. *The image of $\mathscr{F}^0$ by $\rho_{a,b}$ is a closed set of $C^0([a,b] \times M \times \overline{\mathbb{S}}^{\omega}, \overline{\mathbb{S}}^{\omega})$ for the compact-open topology.*

Proof. Consider a Cauchy sequence $\hat{\Phi}_n$ of $\rho_{a,b}(\mathscr{F}^0)$. As $C^0([a,b] \times M \times \overline{\mathbb{S}}^{\omega}, \overline{\mathbb{S}}^{\omega})$ is a complete metric space we have that $\hat{\Phi}_n$ converges for some $\hat{\Phi} \in C^0([a,b] \times M \times \overline{\mathbb{S}}^{\omega}, \overline{\mathbb{S}}^{\omega})$. If $\hat{\Phi} \in \rho_{a,b}(\mathscr{F}^0)$ we can conclude that $\rho_{a,b}(\mathscr{F}^0)$ is a complete subspace of $C^0([a,b] \times M \times \overline{\mathbb{S}}^{\omega}, M \times \overline{\mathbb{S}}^{\omega})$ and consequently it is closed.

For $t \in [a,b]$ consider the notation $\hat{\Phi}^t(x,v) = \hat{\Phi}(t,x,v)$. First we need to verify that $\hat{\Phi}$ satisfies the cocycle identity $\hat{\Phi}^{t+s}(x,v) = \hat{\Phi}^t \circ \hat{\Phi}^s(x,v)$, for $t \in [a,b]$.

In fact,

$$\begin{aligned} \hat{\Phi}^t \circ \hat{\Phi}^s(x,v) &= \Phi^t\left(\frac{\Phi^s(x)\cdot v}{\|\Phi^s(x)\cdot v\|}\right) \\ &= \frac{\Phi^t(\varphi^s(x)) \cdot \frac{\Phi^s(x)\cdot v}{\|\Phi^s(x)\cdot v\|}}{\left\|\Phi^t(\varphi^s(x)) \cdot \frac{\Phi^s(x)\cdot v}{\|\Phi^s(x)\cdot v\|}\right\|}. \end{aligned}$$

As Φ^t is linear on variable v then

$$\left\|\Phi^t(\varphi^s(x)) \cdot \frac{\Phi^s(x)\cdot v}{\|\Phi^s(x)\cdot v\|}\right\| = \frac{1}{\|\Phi^s(x)\cdot v\|} \left\|\Phi^t(\varphi^s(x)) \cdot \Phi^s(x)\cdot v\right\|.$$

So

$$\begin{aligned} \hat{\Phi}^t \circ \hat{\Phi}^s(x,v) &= \frac{\Phi^t(\varphi^s(x)) \cdot \Phi^s(x)\cdot v}{\|\Phi^t(\varphi^s(x)) \cdot \Phi^s(x)\cdot v\|} \\ &= \frac{\Phi^{t+s}(x)\cdot v}{\|\Phi^{t+s}(x)\cdot v\|}) \\ &= \hat{\Phi}^{t+s}(x,v). \end{aligned}$$

As this property is valid for every $x \in M$, we can consider for the same interval $[a,b]$ other points in the orbit of x with the guarantee that the property remains valid and prove that it is valid along all the orbit of x.

□

As $[a,b] \times M \times \overline{\mathbb{S}}^{\omega}$ is compact the compact-open and the strong topology on $C^0([a,b] \times M \times \overline{\mathbb{S}}^{\omega}, \overline{\mathbb{S}}^{\omega})$ are the same. As, by Lemma 2.1.4, $\rho_{a,b}(\mathscr{F}^0)$ is a closed subset of $C^0([a,b] \times M \times \overline{\mathbb{S}}^{\omega}, \overline{\mathbb{S}}^{\omega})$ we can apply Theorem 4.4 of (26) and obtain that $\rho_{a,b}(\mathscr{F}^0)$ is a Baire space for the compact-open topology.

We want to prove that $\mathscr{F}^0$ is also a Baire space for the τ-topology.

Lemma 2.1.5. *Given two topological spaces X and Y and a map $\rho : X \to Y$, such that Y is endowed with the compact-open topology and X with the initial topology, the open and dense subsets of X are pre-images, by ρ, of open and dense sets of $\rho(X)$.*

Proof. Consider the sets $U_n \subset Y$ such that $U_n \cap \rho(X) \neq \varnothing$ and $V_n = \rho^{-1}(U_n \cap \rho(X))$ are open and dense sets. We will prove that $U_n \cap \rho(X)$ are open and dense.

To prove that each $U_n \cap \rho(X)$ is open it is necessary to prove that

$$\forall y \in U_n \cap \rho(X),\ \exists \delta > 0 \text{ such that } B(y,\delta) \subseteq U_n \cap \rho(X).$$

Consider $y \in U_n \cap \rho(X)$ (see Figure 2.1). We have that $\rho^{-1}(y) \in \rho^{-1}(U_n \cap \rho(X)) = V_n$. As V_n is open then there exists $\varepsilon > 0$ such that $B(\rho^{-1}(y),\varepsilon) \subseteq V_n$.

As ρ is continuous and we are using the initial topology on X, $B(\rho^{-1}(y),\varepsilon) = \rho^{-1}(A)$ for some open set $A \subseteq Y$ such that $y \in A$. As $\rho(B(\rho^{-1}(y),\varepsilon)) \subseteq \rho(V_n) = U_n \cap \rho(X)$ then $A \subseteq U_n \cap \rho(X)$. As A is open there exists $\delta > 0$ such that $B(y,\delta) \subseteq A \subseteq U_n \cap \rho(X)$.

Now as y is arbitrary we can conclude that $U_n \cap \rho(X)$ is an open set.

To prove that each $U_n \cap \rho(X)$ is dense in $\rho(X)$ it is necessary to prove that

$$\forall y \in \rho(X)\ \forall \varepsilon > 0 \text{ we have that } B(y,\varepsilon) \cap [U_n \cap \rho(X)] \neq \varnothing.$$

Let $w \in \rho(X)$ and consider $B(w,\varepsilon)$, for some $\varepsilon > 0$ (see Figure 2.2). As ρ is continuous and we are using the initial topology on X, we have that there exists $\delta > 0$ such that $B(\rho^{-1}(w),\delta) \subseteq \rho^{-1}(B(w,\varepsilon))$. As V_n is dense in X, $B(\rho^{-1}(w),\delta) \cap V_n \neq \varnothing$. Consider a point $z \in B(\rho^{-1}(w),\delta) \cap V_n$. We have that $\rho(z) \in \rho(B(\rho^{-1}(w),\delta)) \subseteq B(w,\varepsilon)$ and $\rho(z) \in U_n \cap \rho(X)$ because $z \in V_n$. So $B(w,\varepsilon) \cap [U_n \cap \rho(X)] \neq \varnothing$.

Now as w and ε are arbitrary we can conclude that $U_n \cap \rho(X)$ is a dense set.

□

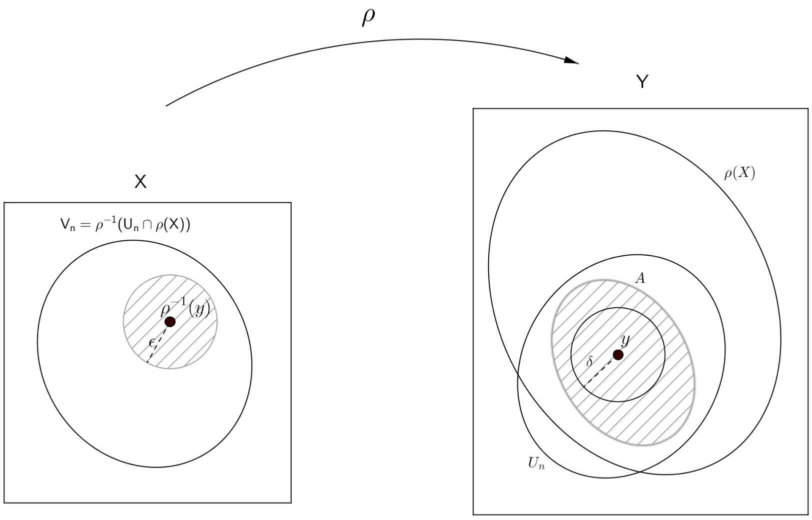

Figure 2.1: Illustration to Lemma 2.1.5 - Openness.

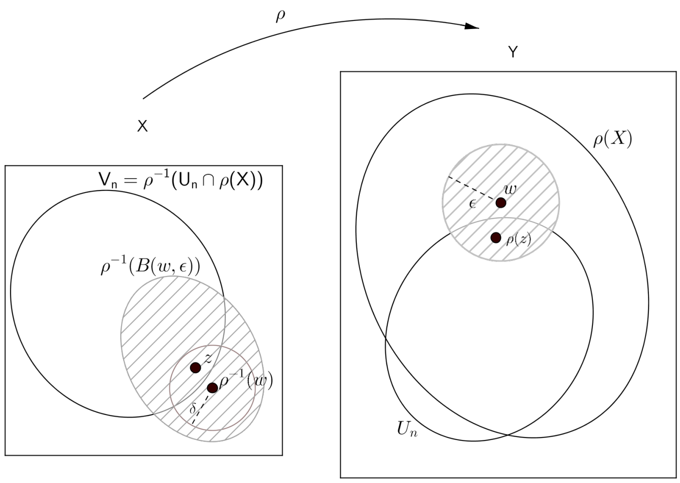

Figure 2.2: Illustration to Lemma 2.1.5 - Denseness.

Proposition 2.1.6. *The space $\mathscr{F}^0$ with τ-topology is a Baire space.*

Proof. As $\rho_{a,b}\left(\mathscr{F}^0\right)$ is a Baire space, using the continuity of ρ and Lemma 2.1.5, we conclude the $\mathscr{F}^0$ is a Baire space for the τ-topology.

□

2.2 The Multiplicative Ergodic Theorem

The following result of Ruelle (Corollary 2.2 of (33)) gives a Lyapunov-spectral decomposition for the limit

$$\lim_{t\to\infty}\left[(\Phi^t(x))^*\Phi^t(x)\right]^{\frac{1}{2t}},$$

under the general integrability condition:

$$\int_M \log^+ \|\Phi^1(x)\|\, d\mu(x) < \infty. \tag{2.2.1}$$

Theorem 2.2.1. *Let $\varphi^t : M \to M$ be a continuous flow, μ a φ^t-invariant Borel probability and $\Phi : \mathbb{R}\times M\times\mathscr{H}\to\mathscr{H}$. If the cocycle Φ^t is a compact operator (for each x), then, for μ-almost every $x\in M$, we have the following properties:*

(a) The limit $\lim_{t\to\infty}[(\Phi^t(x))^*\Phi^t(x)]^{\frac{1}{2t}}$ exists and is a compact operator $\mathscr{L}(x)$.

*(b) Let $e^{\lambda_1(x)} > e^{\lambda_2(x)} > \ldots$ be the nonzero eigenvalues of $\mathscr{L}(x)$ and $U_1(x)$, $U_2(x)$, ... the associated eigenspaces whose dimensions are denoted by $n_i(x)$. The sequence of real functions $\lambda_i(x)$, called **Lyapunov exponents** of Φ^t, where $1\leqslant i\leqslant j(x)$ and $j(x)\in\mathbb{N}\cup\{\infty\}$, satisfies:*

(b.1) The functions $\lambda_i(x)$, $j(x)$ and $n_i(x)$ are φ^t-invariant and depend in a measurable way on x.

(b.2) If $V_i(x)$ is the orthogonal complement of $U_1(x)\oplus U_2(x)\oplus\ldots\oplus U_{i-1}(x)$, for $i<j(x)+1$, and $V_{j(x)+1}(x) = Ker(\mathscr{L}(x))$, then:

(i) $\lim_{t\to\infty}\frac{1}{t}\log\|\Phi^t(x)\cdot u\| = \lambda_i(x)$ if $u\in V_i(x)\backslash V_{i+1}(x)$ and $i<j(x)+1$;

(ii) $\lim_{t\to\infty}\frac{1}{t}\log\|\Phi^t(x)\cdot u\| = -\infty$ if $u\in V_{j(x)+1}(x)$.

As we are assuming that μ is ergodic, the maps $j(x)$, $n_i(x)$ and $\lambda_i(x)$ are cons-tant μ-almost everywhere. We will denote by $\mathscr{O}(\Phi)$ the full measure set of points given by this theorem. Since μ is positive on non-empty open subsets, $\mathscr{O}(\Phi)$ is dense in M. Furthermore, by Lemma 3.3 of (9), if $\lambda_i \neq -\infty$, then $U_i(x)$ has finite dimension.

2.3 Dominated Splittings

Definition 2.3.1. *Given φ^t and Φ^t as above and an φ^t-invariant set $\mathscr{K}$, we say that a splitting $E_1(x) \oplus E_2(x) = \mathscr{H}$ is ℓ-dominated in $\mathscr{K}$, denoting by $E_1 >_\ell E_2$, if*

(C1) $\Phi^t(E_i(x)) \subset E_i(\varphi^t(x))$ *for every* $x \in \mathscr{K}$.

(C2) The dimension of $E_i(x)$ is constant in $\mathscr{K}$ for $i = 1, 2$.

(C3) There are $\theta_{\mathscr{K}} > 0$ and $\ell \in \mathbb{N}$ such that, for any $x \in \mathscr{K}$ and every pair of unit vectors $u \in E_2(x)$ and $v \in E_1(x)$, one has

(C3.1) $\left\|\Phi^1(x) \cdot v\right\| \geqslant \theta_{\mathscr{K}}$;

(C3.2) $\frac{\left\|\Phi^\ell(x) \cdot u\right\|}{\left\|\Phi^\ell(x) \cdot v\right\|} \leqslant \frac{1}{2}$.

This definition corresponds to hyperbolicity in an infinite dimensional projective space (see (22)). Condition (C3.2) is a standard hypothesis of the classical concept of domination (21), whereas (C3.1) is an essential to guarantee that the norm of Φ^1 in $\mathscr{K}$ is bounded away form zero. In fact, for finite dimensional automorphisms, domination implies that the angle between any two subbundles of a dominated splitting is uniformly bounded away from zero. Due too the lack of compactness of $\mathscr{O}(A)$ and the fact that we are dealing with operators acting on an infinite dimensional space we have that $\Phi_A^t(x)$ is not invertible and its norm may not be uniformly bounded away from zero, unless we relate, as we have done through the condition (C3.1), domination with nonzero norms.

Given a φ^t-invariant set $\mathscr{K}$ contained on $\mathscr{O}(\Phi)$, we say that the Oseledets-Ruelle's decomposition is ℓ-dominated in $\mathscr{K}$ if we may detach in it a direct sum of two subspaces, $E_1(x) \oplus E_2(x) = \mathscr{H}(x)$, such that $E_1(x)$ is associated to a finite number of the first Lyapunov exponents, say $\lambda_1, \lambda_2, \cdots, \lambda_k$, the subspace $E_2(x)$ is associated to the remaining ones and $E_1 >_\ell E_2$. As in (9) we will consider the decompositions where $E_1(x)$ is the Lyapunov subspace associated to the first k finite Lyapunov exponents $\lambda_1 > \lambda_2 > \cdots > \lambda_k > -\infty$.

We can use the arguments described in (9) (see Lemma 3.4 and Proposition 3.5) to prove that the operator $\Phi^t(x) : E_1(x) \to E_2(\varphi^t(x))$ is invertible and $[\Phi^t(\varphi^t(x))]^{-1}$ is compact, and also that if the Oseledets-Ruelle's splitting $E_1(x) \oplus E_2(x) = \mathscr{H}$ is ℓ-dominated over an invariant set $\mathscr{K} \subset \mathscr{O}(\Phi)$, it may be extended continuously to an ℓ-dominated splitting over the closure of $\mathscr{K}$.

2.4 Entropy Functions

Given a cocycle $\Phi^t : \mathscr{H} \to \mathscr{H}$ and a positive integer k, let $\wedge^k(\mathscr{H})$ be the k^{th} exterior power of $\mathscr{H}$, *i.e.*, the infinite dimensional space generated by k vectors of the form $e_1 \wedge e_2 \wedge \cdots \wedge e_k$ with $e_i \in \mathscr{H}$. Consider the operator $\wedge^k(\Phi^t)$, defined by

$$\begin{array}{cccc} \wedge^k(\Phi^t): & \wedge^k(\mathscr{H}) & \to & \wedge^k(\mathscr{H}) \\ & e_1 \wedge e_2 \wedge \cdots \wedge e_k & \mapsto & \Phi^t(e_1) \wedge \Phi^t(e_2) \wedge \cdots \wedge \Phi^t(e_k). \end{array}$$

By Lemma 3.10 of (9) as Φ^t is a compact and integrable operator, then $\wedge^k(\Phi^t)$ also is.

We can apply to it Ruelle's theorem and conclude that, for μ-almost every x,

$$\lim_{t \to +\infty} \frac{1}{t} \log \left\| \wedge^k(\Phi^t(x)) \right\| = \lambda_1^{\wedge^k}(x).$$

This is the largest Lyapunov exponent given by the dynamics of the operator $\wedge^k(\Phi^t)$ at x. Moreover, for μ-almost every x, we have

$$\lambda_1^{\wedge^k}(x) = \sum_{i=1}^{k} \lambda_i(x).$$

In fact, as μ is ergodic, for μ-almost every x, we have

$$\lambda_1^{\wedge^k} = \sum_{i=1}^{k} \lambda_i = \lim_{t \to +\infty} \frac{1}{t} \log \left\| \wedge^k(\Phi^t(x)) \right\|.$$

Given $k \in \mathbb{N}$ define the k^{th}-*entropy function* by

$$\begin{array}{cccc} LE_k: & \mathscr{F}^0 & \to & \mathbb{R} \cup \{-\infty\} \\ & \Phi & \mapsto & \sum_{i=1}^{k} \lambda_i(\Phi^t). \end{array}$$

In the next proposition we will prove that LE_k is an upper semicontinuous function, which is fundamental for the construction of the proof of Theorem A.

Proposition 2.4.1. *LE_k is upper semicontinuous.*

Proof. Since Oseledet-Ruelle's theorem is an asymptotic result and $\Phi^r(x)$, for a fixed r, is a bounded operator, we can replace $\Phi^t(x) = \Phi^r(\varphi^n(x))\Phi^n(x)$ by the last integer time-n

map, $\Phi^n(x)$ and consider

$$\lambda_1^{\wedge^p} = \lim_{n\to+\infty}\frac{1}{n}\log\left\|\wedge^k(\Phi^n(x))\right\|.$$

Consider, for a cocyle Φ and for each $n\in\mathbb{N}$, the sequence $a_n = \log\left\|\wedge^k(\Phi^n)\right\|$. Using the same arguments used in (9) we can prove that

(i) $\lambda_1^{\wedge^k} = \lim\limits_{n\to+\infty}\frac{a_n}{n}$.

(ii) $(a_n)_{n\in\mathbb{N}}$ is sub-additive.

(iii) $\lim\limits_{n\to+\infty}\frac{a_n}{n} = \inf\limits_{n\in\mathbb{N}}\frac{a_n}{n}$.

Note that we can write Φ^n as

$$(\Phi^1)^n = \Phi^1(\varphi^{n-1}(x))\circ\cdots\Phi^1(\varphi^1(x))\circ\Phi^1(x).$$

By the construction of the τ-topology if $\Phi,\Psi\in\mathscr{F}^0$ are τ-close and $t\in[a,b]$ then $\hat{\Phi}$ and $\hat{\Psi}$ are close for the compact-open topology, that is, $\frac{\Phi^t(x)\cdot v}{\|\Phi^t(x)\cdot v\|}$ and $\frac{\Psi^t(x)\cdot v}{\|\Psi^t(x)\cdot v\|}$ are ω-close. So for $t\in[0,1]$ the continuity of the map $\Phi\to a_n(\Phi)$ is ensured by the continuity of the operator, of the logarithmic, of the operator $\wedge^k$ and of the norm $\|\cdot\|$. Now we can prove that LE_k is upper semicontinuous using the fact that it is the infimum of a sequence of continuous functions with values on the extended real line.

□

Chapter 3

Perturbation Lemmas

Let $\Lambda_p(\Phi,m)$ be the set of points x such that, the orbit $\varphi^t(x)$ has an Oseledets-Ruelle's decomposition of index p for Φ^t which is m-dominated. Denote by $\Gamma_p(\Phi,m) = M \backslash \Lambda_p(\Phi,m)$ and by $\Gamma_p^*(\Phi,m)$ the set of points in $\mathscr{O}(\Phi) \cap \Gamma_p(\Phi,m)$ which are non-periodic and satisfies $\lambda_p > \lambda_{p+1}$.

Notice that if x belongs to $\Gamma_p^*(\Phi,m)$ for some m, then the m-domination on $\mathscr{K} = \{\text{orbit of } x\}$ of the Oseledets-Ruelle's splitting may fail by two (possible coexisting) events:

(NB) The norm of operator Φ^1 restricted to E_1 takes values arbitrarily small along the orbit of M.

That is, for all $\theta > 0$ there are $N = N_{\theta,x} \in \mathbb{N}$ and a unit vector $v_N \in E_1(\varphi^N(x))$ such that

$$\left\|\Phi^1(\varphi^N(x)) \cdot v_N\right\| < \theta.$$

We call $\Gamma_{p,1}^*$ the set of points $x \in \Gamma_p^*(\Phi,m)$ where this happens.

(ND) The dynamics on the subspace E_1 does not m-dominate the one on E_2.

This means that there are $n \in \mathbb{N}$ and unit vectors $v_n \in E_1(\varphi^n(x))$ and $u_n \in E_2(\varphi^n(x))$ such that

$$\frac{\|\Phi^m(\varphi^n(x)) \cdot u_n\|}{\|\Phi^m(\varphi^n(x)) \cdot v_n\|} \geqslant \frac{1}{2}.$$

The points $x \in \Gamma_p^*(\Phi, m)$ where property (ND) is valid but not (NB) will be denoted by $\Gamma_{p,2}^*$.

The next lemma is the basic perturbation tool which will be used in Proposition 3.0.2 to interchange Oseledets-Ruelle's directions.

Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denote the rotation of angle θ in $\mathbb{R}^2$ which in canonic coordinates can be written as $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$.

Lemma 3.0.1. *Given an integrable compact cocycle* $\Phi^t \in \mathscr{C}(\mathscr{H})$ *and* $\varepsilon > 0$*, there exists an angle* $\xi > 0$*, such that for all* $p \in M$ *(non-periodic or with period larger than 1) and all two-dimensional subspace* $E_p \subset \mathscr{H}$ *with rank* $\left(\Phi^1(p)|_{E_p}\right) = 2$*, there exists a measurable integrable cocycle* Ψ^t_ξ *such that:*

(a) Ψ_ξ *is* ε*-close (with respect to* τ*) to* Φ*;*

(b) Ψ^t_ξ *is supported in* $\varphi^t(p)$ *for* $t \in [0,1]$*;*

(c) $\Psi^t_\xi(p) \cdot u = \Phi^t(p) \cdot u$*, for all* $u \in E_p^\perp$*;*

(d) $\Psi^1_\xi(p) \cdot u = \Phi^1(p) \cdot R_\xi \cdot u$*, for all* $u \in E_p$*, where* R_ξ *is the rotation of angle* ξ *in* E_p*.*

Proof. Consider the direct sum $\mathscr{H}_p = E_p \oplus E_p^\perp$. Let $\eta \in (0,1)$ and $G : \mathbb{R} \to \mathbb{R}$ be the *bump-function* defined by $G(t) = 0$ for $t < 0$, $G(t) = 1$ for $t \in [\eta, 1-\eta]$ and $G(t) = 1$ for $t \geqslant 1$. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(t) = \int_0^t G(s)\,ds$.

Define, in a matricial notation,

$$\Phi^t(p) = \begin{pmatrix} \delta^t(p) & 0 \\ 0 & \gamma^t(p) \end{pmatrix} \quad \text{and} \quad \delta^t(p) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix},$$

where $\delta^t(p) : E_p \to E_{\varphi^t(p)}$ and $\gamma^t(p) : E_p^\perp \to E_{\varphi^t(p)}^\perp$.

For each $u \in \mathscr{H}_p$ consider the decomposition $u = v + w$, where $v \in E_p$ and $w \in E_p^\perp$. For $t \in \mathbb{R}$ define

$$\Omega^t_\xi(y)\cdot u = \Phi^{\tau(y)}(p)\cdot R^t_p\cdot \Phi^{-\tau(y)}(y)\cdot v + w,$$

where $\tau : M \to \mathbb{R}$ is such that $\tau(y) = s$ if and only if $\varphi^s(p) = y$ and R^t_p is the rotation of angle $\xi g(t)$ in $\mathbb{R}^2$ defined on E_p.

Take $\Psi^t_\xi(p) = \Phi^t(p)\cdot\Omega^t(p)$. First we need to verify that Ψ^t_ξ is an integrable cocyle, that is, satisfies the following properties:

(i) $\Psi^0_\xi(y) = \mathrm{Id}$;

(ii) $\Psi^{t+s}_\xi(y) = \Psi^t_\xi(\varphi^s(y))\cdot\Psi^s_\xi(y)$.

Concerning with (i), $\Psi^0_\xi(y) = \Phi^0(y)\cdot\Omega^0(y) = \mathrm{Id}$, since $\Omega^0(y) = \Phi^{\tau(y)}(y)\cdot R^0_y\cdot\Phi^{-\tau(y)}(y)$ and $R^0_y = \mathrm{Id}$.

Concerning with (ii) we have that

$$\begin{aligned}
\Psi^t_\xi(\varphi^s(y))\cdot\Psi^s_\xi(y) &= \Phi^t(\varphi^s(y))\cdot\Omega^t(\varphi^s(y))\cdot\Phi^s(y)\cdot\Omega^s(y)\\
&= \Phi^t(\varphi^s(y))\cdot\Phi^{\tau(\varphi^s(y))}(p)\cdot R^t_p\cdot\Phi^{-\tau(\varphi^s(y))}(\varphi^s(y))\cdot\\
&\quad\cdot\Phi^s(y)\cdot\Phi^{\tau(y)}(p)\cdot R^s_p\cdot\Phi^{-\tau(y)}(y).
\end{aligned}$$

As $\Phi^{-\tau(\varphi^s(y))}(\varphi^s(y)\cdot\Phi^s(y)\cdot\Phi^{\tau(y)}(p) = \mathrm{Id}$ and $\Phi^{\tau(\varphi^s(y))}(p) = \Phi^s(y)$, we can write

$$\begin{aligned}
\Psi^t_\xi(\varphi^s(y))\cdot\Psi^s_\xi(y) &= \Phi^t(\varphi^s(y))\cdot\Phi^s(y)\cdot R^t_p\cdot R^s_p\cdot\Phi^{-\tau(y)}(y)\\
&= \Phi^{t+s}(y)\cdot R^t_p\cdot R^s_p\cdot\Phi^{-\tau(y)}(y).
\end{aligned}$$

Claim 3.0.1. *The rotation R^t_p satisfies $R^{t+s}_p = R^t_p\cdot R^s_p$.*

As $g(t) = \int_0^t G(s)\,\mathrm{ds}$ we have that $g(t+s) = g(t)+g(s)$ and so $R^{t+s}_p = R_{\xi g(t+s)} = R_{\xi(g(t)+g(s))}$. Computing the product $R_{\xi g(t)}\cdot R_{\xi g(s)}$ is easy to prove that $R_{\xi(g(t)+g(s))} = R_{\xi g(t)}\cdot R_{\xi g(s)}$ and the claim is proved.

So,

$$\begin{aligned}
\Psi^t_\xi(\varphi^s(y))\cdot\Psi^s_\xi(y) &= \Phi^{t+s}(y)\cdot R^{t+s}_p\cdot\Phi^{-\tau(y)}(y)\\
&= \Psi^{t+s}_\xi(y).
\end{aligned}$$

For $t = 1$ and $u \in E_p$, as $\tau(p) = 0$ and $R_p^1 = R_\xi$, we obtain $\Psi_\xi^1(p) \cdot u = \Phi^1(p) \cdot R_\xi \cdot u$. If $u \in E_p^\perp$ clearly $\Psi_\xi^t(p) \cdot u = \Phi_A^t(p) \cdot u$, so (c) and (d) are proved.

As $g'(t) = 0$ for $t \notin]0,1[$, Ψ_ξ^t is supported in $\varphi^t(p)$ for $t \in [0,1]$ and (b) is proved. Since $t \in [0,1]$ given any size of perturbation allowed by $\varepsilon > 0$ we take ξ sufficiently small to guarantee that

$$\left\|(\Psi^t(p) - \Phi^t(p)) \cdot u\right\| \leqslant \varepsilon.$$

The corresponding unitary vectors, $\frac{\Psi^t(p)\cdot u}{\|\Psi^t(p)\cdot u\|}$ and $\frac{\Phi^t(p)\cdot u}{\|\Phi^t(p)\cdot u\|}$ will also satisfy the same property. On the other hand, we are using the norm derived from the inner product on $\mathscr{H}$, but it is stronger then the weak topology.

Now, using the continuity of the application $\rho_{a,b}$ defined on section 2.1, we take ξ sufficiently small to guarantee that Ψ and Φ are ε-close with respect to the τ topology.

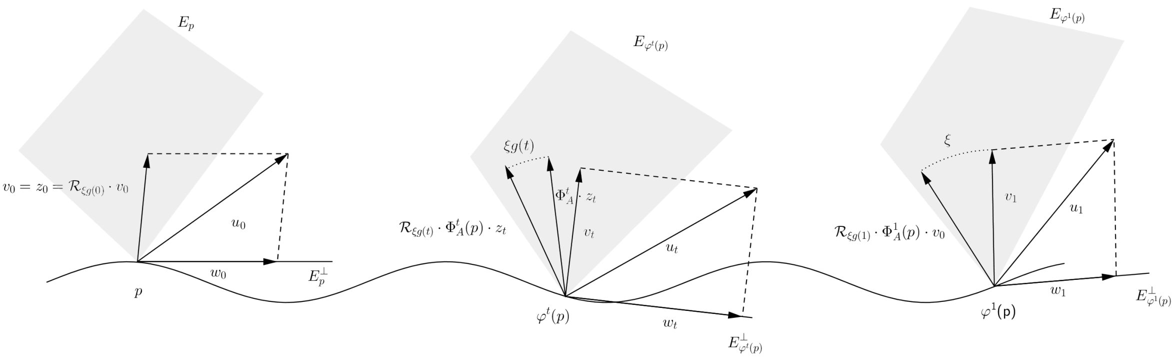

Figure 3.1: Illustration to Lemma 3.0.1.

The following proposition is analogous to Proposition 4.2 of (9) and tell us how to interchange directions. The main idea, coming from Proposition 7.1 of (20) is to use the absence of domination to concatenate several rotations given by Lemma 3.0.1.

Proposition 3.0.2. *Consider a cocycle Φ^t, $\varepsilon > 0$ and $x \in M$ a non-periodic point endowed with a splitting $\mathscr{H} = E_x \oplus F_x$ such that the restriction of $\Phi^t(x)$ to E_x is invertible and for some $m(\varepsilon, \Phi^t) = m \in \mathbb{N}$ large enough, we have*

$$\frac{\|\Phi^m(x)|_F\|}{\left\|(\Phi^{-1}|_E)^m(x)\right\|^{-1}} \geqslant \frac{1}{2}.$$

Then, for each $j = 0, \cdots, m-1$ there exists an integrable compact operator

$$L_j^t : \mathscr{H}_{\varphi^j(x)} \to \mathscr{H}_{\varphi^{j+1}(x)}$$

with $\left\|L_j^t - \Phi^t(\varphi^j(x))\right\| < \varepsilon$ for $t \in [0,1]$ and such that $L_{m-1}^t \circ \cdots \circ L_0^t(v) = w$ for some non-zero vectors $v \in E$ and $w \in \Phi^m(x)(F)$.

Our goal is to apply this type of perturbation to the set of point where domination fails.

3.1 Perturbation (p, NB)

Consider a point $x \in \Gamma_{p,1}^*$. The next lemma gives us a perturbation strategy that produce a cocycle Ψ^t which is the null operator (in time one) along the direction inside $E_1(x)$, restricted to which the norm of Φ^t is very close to zero, remaining equal to Φ^t at all other directions. In the discrete case the perturbation is quite simple, consisting in the replacement of the original cocycle by the null operator at the nth-iterate of x along the referred direction inside $E_1(x)$.

Lemma 3.1.1. *Given $\varepsilon > 0$, for μ-almost every $x \in \Gamma_{p,1}^*$ there exists $N = N_{\varepsilon,x}$ and a integrable cocycle Ψ such that,*

(a) Ψ *is ε-close (with respect to τ) to Φ;*

(b) $\| \wedge^p (\Psi^1(\varphi^N(x))\| = 0.$

Proof. We may assume that $\mu(\Gamma_{p,1}^*) > 0$, otherwise there is nothing to prove. Therefore $\mu(\mathscr{O}(\Phi) \cap \Gamma_{p,1}^*) = \mu(\Gamma_{p,1}^*)$.

Consider Φ^t in the integral form

$$\Phi^t(p) = \mathrm{Id} + \int_0^s A(\varphi^s(p)) \cdot \Phi^s(p)\, \mathrm{ds}. \tag{3.1.1}$$

We will use the notation Φ_A^t to associate Φ^t with the corresponding A.

As Φ_A^t is a compact operator defined on a compact manifold M we can use the *logarithmic norm* (see (36; 37)) to obtain a majorant of the maximum growth of A:

$$\lfloor A \rfloor = \sup_{\|v\|=1} \inf_{h>0} \frac{\|v + hA \cdot v\| - 1}{h},$$

Take $L := \lfloor A \rfloor$. Consider $\theta = \theta(A, \varepsilon)$, such that $\theta < \varepsilon \exp(-3L)$.

If $x \in \mathscr{O}(\Phi) \cap \Gamma^*_{p,1}$, we can take N as in (NB) and choose a unit vector $v_N \in E_1(\varphi^N(x))$ such that

$$\left\| \Phi_A^1(\varphi^N(x)) \cdot v_N \right\| < \theta. \tag{3.1.2}$$

Take $p = \varphi^N(x)$. We want to define a perturbation $P(\varphi^t(p))$, for $t \in [0,1]$, such that $\Phi_{A+P}^t(p) \cdot v = \Phi_A^t(p) \cdot v$ for $v \in \langle v_N \rangle^\perp$ and $\Phi_{A+P}^1(p) \cdot v = 0$ for v collinear with v_N.

Consider the direct sum $\mathscr{H}_{\varphi^t(p)} = \Phi_A^t(p)\mathbb{R}v_N \oplus \langle \Phi_A^t(p)\mathbb{R}v_N \rangle^\perp$, where $\mathbb{R}v_N$ is the space generated by the vector v_N.

For each $v_t \in \mathscr{H}_{\varphi^t(p)}$ consider the decomposition $v_t = u_t + w_t$, where $u_t \in \Phi_A^t(p)\mathbb{R}v_N$ and $w_t \in \langle \Phi_A^t(p)\mathbb{R}v_N \rangle^\perp$.

Define in matricial notation

$$\Phi_A^t(p) = \begin{pmatrix} \psi^t(p) & 0 \\ 0 & \delta^t(p) \end{pmatrix},$$

where $\psi^t(p) : \mathbb{R}v_N \to \Phi_A^t(p)\mathbb{R}v_N$ and $\delta^t(p) : \bigoplus_{i \neq N} U_i(p) \to \bigoplus_{i \neq N} U_i(\varphi^t(p))$, where U_i are the spaces given by Ruelle's theorem (see Section 2.2).

Since the space $\Phi_A^t(p)\mathbb{R}v_N$ has dimension one and we want to perturb $A(\varphi^t(p))$ restricted to $\Phi_A^t(p)\mathbb{R}v_N$, we will reduce our problem to one-dimensional case and replace the $A(\varphi^t(p))$ in equation (3.1.1) by the corresponding real function $\alpha(\varphi^t(p))$.

Consider

$$\Phi_\alpha^t(p) = Id + \int_0^t \alpha(\varphi^s(p)) \cdot \Phi_\alpha^s(p)\, \mathrm{d}s. \tag{3.1.3}$$

We can use Lusin's theorem to obtain a function $\tilde{\alpha}(\varphi^t(p))$ which is continuous for $t \in [0,1]$ arbitrarily close to α, then applying Gronwall's lemma and get

$$\Phi^t_{\tilde{\alpha}}(p) = \exp\left(\int_0^t \tilde{\alpha}(\varphi^s(p))\,\mathrm{ds}\right). \tag{3.1.4}$$

As $\Phi^t_{\tilde{\alpha}}$ satisfies (3.1.2) then $\int_0^1 \tilde{\alpha}(\varphi^s(p))\,\mathrm{ds} < \log\theta$.

By mean value theorem of integral there exists $c \in (0,1)$ such that $\int_0^1 \tilde{\alpha}(\varphi^s(p)) \cdot \mathrm{ds} = \tilde{\alpha}(\varphi^c(p)) =: K < \log(\theta)$.

Consider $\eta(\tilde{\alpha},\varepsilon,c) > 0$ such that $\eta < \frac{\ln(\varepsilon)-M}{L} - 2$, with $M = \tilde{\alpha}(\varphi^{c-\eta}(p))$. For $t \in [0,1]$ define the perturbation $\beta(\varphi^t(p))$ such that

$$\beta(\varphi^t(p)) = \begin{cases} \tilde{\alpha}(\varphi^t(p)) & \text{if} \quad t \in [0,c-\eta] \cup [c,1], \\ \frac{\eta}{c-t}\tilde{\alpha}(\varphi^{c-\eta}(p)) & \text{if} \quad t \in (c-\eta,c). \end{cases} \tag{3.1.5}$$

Consider $P(\varphi^t(p)) = \beta(\varphi^t(p)) - \tilde{\alpha}(\varphi^t(p))$ and $\Phi^t_{\tilde{\alpha}+P}$, solution of the linear variational equation

$$\dot{u}(t) = (\tilde{\alpha}(\varphi^t(p)) + P(\varphi^t(p))) \cdot u(t)$$

and an unitary vector v.

We have that

$$\begin{aligned} \left|\Phi^1_{\tilde{\alpha}+P}(p) \cdot v\right| &= \left|\exp\left(\textstyle\int_0^1 (\tilde{\alpha}+P)(\varphi^s(p))\,\mathrm{ds}\right) \cdot v\right|. \\ &= \left|\exp\left(\textstyle\int_0^1 \beta(\varphi^s(p))\,\mathrm{ds}\right) \cdot v\right|. \end{aligned}$$

As $\int_0^1 \beta(\varphi^s(p))\,\mathrm{ds} = -\infty$ then $\left|\Phi^1_{\tilde{\alpha}+P}(p) \cdot v\right| = 0$.

We want now to evaluate $\left|\left(\Phi^1_{\tilde{\alpha}+P}(\varphi^t(p)) - \Phi^1_{\tilde{\alpha}}(\varphi^t(p))\right) \cdot v\right|$, for $t \in [c-\eta-1,c]$ and an unitary vector v.

We will consider three sub-intervals, $[c-\eta-1,c-1]$, $[c-1,c-\eta]$ and $[c-\eta,c]$.

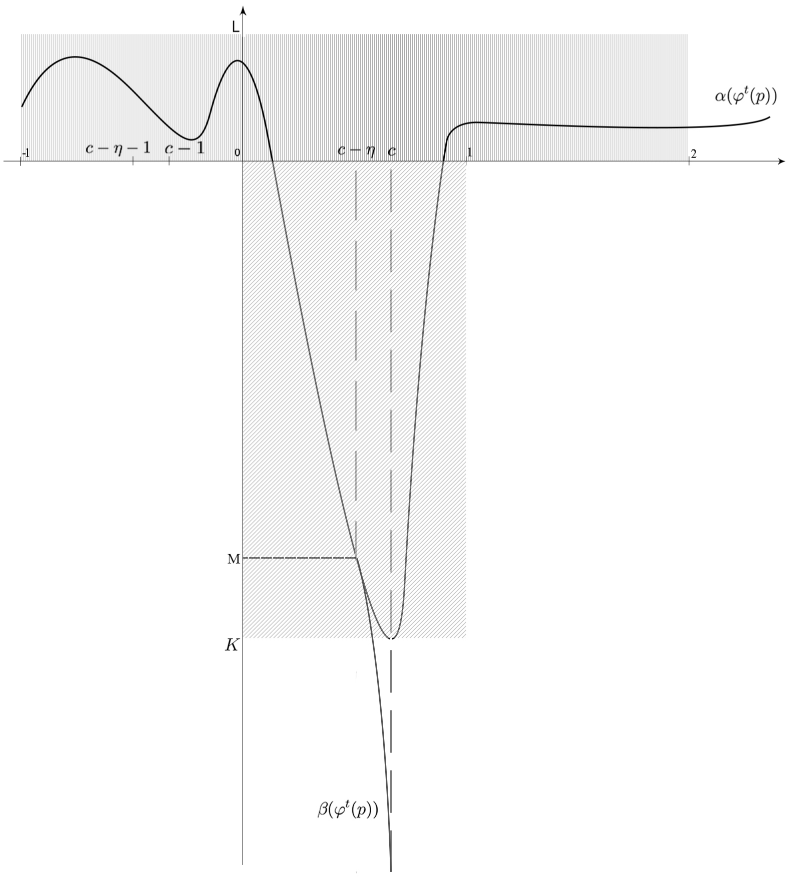

Figure 3.2: Illustration to Lemma 3.1.1.

Take $w=\varphi^{c-1+t}(p)$ with $t\in[0,1-\eta]$, that is, consider a point w in the orbit of p between $\varphi^{c-1}(p)$ and $\varphi^{c-\eta}(p)$. As $\int_0^1 P(\varphi^s(w))\,\mathrm{ds}=-\infty$ then

$$\begin{aligned}
\left|\left(\Phi^1_{\tilde\alpha+P}(w)-\Phi^1_{\tilde\alpha}(w)\right)\cdot v\right| &= \left|\left(\exp\left(\int_0^1(\tilde\alpha+P)(\varphi^s(w))\,\mathrm{ds}\right)-\exp\left(\int_0^1\tilde\alpha(\varphi^s(w))\,\mathrm{ds}\right)\right)\cdot v\right|\\
&= \left|\exp\left(\int_0^1\tilde\alpha(\varphi^s(w))\,\mathrm{ds}\right)\left(\exp\left(\int_0^1 P(\varphi^s(w))\,\mathrm{ds}\right)-1\right)\cdot v\right|\\
&= \left|\exp\left(\int_0^1\tilde\alpha(\varphi^s(w))\cdot v\,\mathrm{ds}\right)\right|=\left|\Phi^1_{\tilde\alpha}(w)\cdot v\right|.
\end{aligned}$$

Consider now $w=\varphi^{c-\eta-1+t}(p)$ with $t\in[0,\eta]$, that is, w is a point in the orbit of p between $\varphi^{c-\eta-1}(p)$ and $\varphi^{c-1}(p)$.

We can write the integral $\int_0^1\beta(\varphi^s(w))\,\mathrm{ds}$ as $\int_0^{1-t}\beta(\varphi^s(w))\,\mathrm{ds}+\int_{1-t}^1\beta(\varphi^s(w))\,\mathrm{ds}$. Note that $t\in[0,\eta]$ is fixed and $\varphi^{1-t}(w)=\varphi^{1-t}(\varphi^{c-\eta-1+t}(p))=\varphi^{c-\eta}(p)$. Doing this we have that

$$\begin{aligned}
\exp\left(\int_0^1 P(\varphi^s(w))\,\mathrm{ds}\right) &= \exp\left(\int_0^{1-t} P(\varphi^s(w))\,\mathrm{ds} + \int_{1-t}^1 P(\varphi^s(w))\,\mathrm{ds}\right)\\
&= \exp\left(0 + \int_{1-t}^1 (\beta - \tilde{\alpha})(\varphi^s(w))\,\mathrm{ds}\right)\\
&= \exp\left(\int_{1-t}^1 \beta(\varphi^s(w))\,\mathrm{ds} - \int_{1-t}^1 \tilde{\alpha}(\varphi^s(w))\,\mathrm{ds}\right)\\
&= \frac{\exp\left(\int_{1-t}^1 \beta(\varphi^s(w))\,\mathrm{ds}\right)}{\exp\left(\int_{1-t}^1 \tilde{\alpha}(\varphi^s(w))\,\mathrm{ds}\right)}.
\end{aligned}$$

If $\int_{1-t}^1 \beta(\varphi^s(w))\,\mathrm{ds} \leqslant \int_{1-t}^1 \tilde{\alpha}(\varphi^s(w))\,\mathrm{ds}$ then $\frac{\exp\left(\int_{1-t}^1 \beta(\varphi^s(w))\,\mathrm{ds}\right)}{\exp\left(\int_{1-t}^1 \tilde{\alpha}(\varphi^s(w))\,\mathrm{ds}\right)} \leqslant 1$ for all $t \in [0,\eta]$, if not we consider another kind of asymptote in (3.1.5) such that $\int_{1-t}^1 \beta(\varphi^s(w))\,\mathrm{ds} \leqslant \int_{1-t}^1 \tilde{\alpha}(\varphi^s(w))\,\mathrm{ds}$. So,

$$\begin{aligned}
\left|\left(\Phi^1_{\tilde{\alpha}+P}(w) - \Phi^1_{\tilde{\alpha}}(w)\right)\cdot v\right| &= \left|\left(\exp\left(\int_0^1 (\tilde{\alpha}+P)(\varphi^s(w))\,\mathrm{ds}\right) - \exp\left(\int_0^1 \tilde{\alpha}(\varphi^s(w))\,\mathrm{ds}\right)\right)\cdot v\right|\\
&= \left|\exp\left(\int_0^1 \tilde{\alpha}(\varphi^s(w))\,\mathrm{ds}\right)\left(\exp\left(\int_0^1 P(\varphi^s(w))\,\mathrm{ds}\right) - 1\right)\cdot v\right|\\
&\leqslant \left|\left(\Phi^1_{\tilde{\alpha}}(w)\right)\cdot v\right|.
\end{aligned}$$

Finally if we consider $w = \varphi^{c-\eta+t}(p)$ with $t \in [0,\eta]$, that is, w between $\varphi^{c-\eta}(p)$ and $\varphi^c(p)$, $\int_0^1 P(\varphi^s(w))\,\mathrm{ds} = -\infty$, once again, $\left|\left(\Phi^1_{\tilde{\alpha}+P}(w) - \Phi^1_{\tilde{\alpha}}(w)\right)\cdot v\right| = \left|\left(\Phi^1_{\tilde{\alpha}}(w)\right)\cdot v\right|$.

We need now to evaluate $\left|\Phi^1_{\tilde{\alpha}}(w)\cdot v\right|$ for $w = \varphi^t(p)$ and $t \in [c-\eta-1, c]$.

As we consider $\theta < \varepsilon\exp(-3L)$ then $\exp(K+3L) < \varepsilon$, that is $|K| \gg 3L$. So, when we consider a point w between $\varphi^{c-1}(p)$ and $\varphi^c(p)$, the norm $\left|\Phi^1_{\tilde{\alpha}}(w)\cdot v\right| < \varepsilon$ (see Figure 3.2).

Consider now a point w between $\varphi^{c-\eta-1}(p)$ and $\varphi^{c-1}(p)$. As we choose η such that $\eta < \frac{\ln(\varepsilon)-M}{L} - 2$ we have that $\exp(M + L(2+\eta)) < \varepsilon$. So we can also conclude that $\left|\Phi^1_{\tilde{\alpha}}(w)\cdot v\right| < \varepsilon$.

Consider now the perturbation

$$\mathrm{H}(\varphi^t(p)) = \begin{pmatrix} P(\varphi^t(p)) & 0 \\ 0 & [0] \end{pmatrix},$$

take $B(\varphi^t(p)) = A(\varphi^t(p)) + H(\varphi^t(p))$ and consider $\Psi^t = \Phi^t_B$.

By the arguments used above, we can conclude that

$$\left\|(\Psi^t(p) - \Phi^t(p)) \cdot u\right\| \leqslant \varepsilon.$$

The corresponding unitary vectors, $\frac{\Psi^t(p)\cdot u}{\|\Psi^t(p)\cdot u\|}$ and $\frac{\Phi^t(p)\cdot u}{\|\Phi^t(p)\cdot u\|}$ will also satisfy the same property. On the other hand, we are using the norm derived from the inner product on $\mathscr{H}$, but it is stronger then the weak topology.

Now, using the continuity of the application $\rho_{a,b}$ defined on section 2.1, we take θ sufficiently small to guarantee that Ψ and Φ are ε-close with respect to the τ topology.

To prove that $\| \wedge^p (\Psi^1(\varphi^N(x))\| = 0$ consider p vectors of the form $e_1 \wedge e_2 \wedge \cdots \wedge e_p$ with $e_i \in E_i(\varphi^N(x))$ and e_1 collinear with v_N. As $\Psi^1(\varphi^N(x) \cdot v_N = 0$ then $\| \wedge^p (\Psi^1(\varphi^N(x))\| = 0$. □

3.2 Perturbation (p, ND)

Consider a point $x \in \Gamma^*_{p,2}$. If among the $p+1$ first Lyapunov exponents of Φ^t the value $-\infty$ is not present, we can use the argument in (20) to alter the norm of $\wedge^p$. In the case $\lambda_{p+1} = -\infty$, we may take advantage of the fact that, in subbundle E associated to the Lyapunov exponents λ_j for $j > p+1$ the norm $\|\Phi^t(x)|_E\|$ is close to zero for n large enough.

Lemma 3.2.1. *Consider $\varepsilon > 0$. If $m \in \mathbb{N}$ is large enough, then there exists a measurable function $\mathscr{N} : \Gamma^*_{p,2} \to \mathbb{N}$ such that, for $\mu-$almost every $x \in \Gamma^*_{p,2}$, and every $n \geqslant \mathscr{N}(x)$ we may find integrable compact operators $L_0, \cdots, L_{n-1}$ satisfying*

(a) $\|\wedge^p(L_{n-1} \circ \cdots \circ L_0)\| \leqslant e^{n\left(\lambda_1 + \cdots + \lambda_{p-1} + \frac{\lambda_p + \lambda_{p+1}}{2} + \varepsilon\right)}$ *if* $\lambda_{p+1} \neq -\infty$;

(b) $\|\wedge^p(L_{n-1} \circ \cdots \circ L_0)\| \leqslant e^{-n\varepsilon^{-1}}$ *if* $\lambda_{p+1} = -\infty$.

Proof. To prove (a) consider $\Gamma^*_{p,2}$, subset of $\Gamma^*_p(\Phi, m)$ where m is large enough as demanded in Proposition 3.0.2. We may assume that $\mu(\Gamma^*_{p,2}) > 0$, otherwise there is nothing to prove. By definition of $\Gamma^*_p(\Phi, m)$ we have $\lambda_p \neq -\infty$. Thus by Lemma 3.12 of (17) we conclude that for μ-almost every $x \in \Gamma^*_{p,2}$, there exists $\mathcal{N}_1(x)$ such that, for all $n \geqslant \mathcal{N}_1(x)$ and $s \approx \frac{n}{2}$ then the iterate $y = \varphi^s(x)$ satisfies

$$\frac{\|\Phi^m(y)|_{E_2}\|}{\|(\Phi^m(y)|_{E_1})^{-1}(y)\|^{-1}} \geqslant \frac{1}{2}.$$

We can apply Proposition 3.0.2 to such a generic $x \in \Gamma^*_{p,2}$ because, by hypothesis, x is not periodic and $\mathcal{H} = E_{1,x} \oplus E_{2,x}$, where $E_{1,x}$ has dimension p and corresponds to the infinite dimensional space spanned by the Lyapunov exponents $\lambda_1, \cdots, \lambda_p$ (which may be not all distinct but whose multiplicities add up to p) and $E_{2,x}$ is associated to the infinite dimensional vector space spanned by the remaining ones. Therefore, we consider, for $i = 1, \cdots, s, s+m, \cdots, n$ and for $t \in [0,1]$, the operators $L_i^t = \Phi^t(\varphi^i(x))$ and for the iterates $\varphi^i(y)$ with $i = 0, \cdots, m-1$, we take L_i^t as given by Proposition 3.0.2.

We need now to evaluate the norm of $\wedge^p(L_{n-1} \circ \cdots \circ L_0)$. Take U_x the subspace associated to the largest Lyapunov exponent of the p^{th}-exterior product, say $\lambda_1^{\wedge^p} = \sum_{i=1}^p \lambda_i$. The subspace U_x is one-dimensional because $\lambda_p > \lambda_{p+1}$. Denote by S_x the vector space related to the remaining Lyapunov exponents, which sum up to $\lambda_2^{\wedge^p}$. To the splitting $\wedge^p(\mathcal{H}) = U \oplus S$ we may apply Lemma 4.4 of (20) and Proposition 3.0.2 to deduce that

$$\wedge^p(L_{m-1} \circ \cdots \circ L_0)(y) : \wedge^p(\mathcal{H}_y) \to \wedge^p(\mathcal{H}_{\varphi^m(y)})$$

satisfies

$$\wedge^p(L_{m-1} \circ \cdots \circ L_0)(y)(U_y) \subset S_{\varphi^m(y)}. \tag{3.2.1}$$

If A_1 denotes the action of $\wedge^p(\Phi^t)$ between x and $y = \varphi^s(x)$ and A_2 denotes the action of $\wedge^p(\Phi^t)$ between $\varphi^{s+m}(x)$ and $\varphi^n(x)$, we can consider a suitable (Oseledets-Ruelle's) basis with respect to which A_1, A_2 and $B := L_{m-1} \circ \cdots \circ L_0(y)$ are written as simple 4-block "matrices"

$$A_1 = \begin{pmatrix} A_1^{uu} & 0 \\ 0 & A_1^{ss} \end{pmatrix}, \quad B = \begin{pmatrix} B^{uu} & B^{us} \\ B^{su} & B^{ss} \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} A_2^{uu} & 0 \\ 0 & A_2^{ss} \end{pmatrix}$$

where, for $i = 1, 2$, $A_i^{uu} \in \mathbb{R}$ and A_i^{ss} is an infinite dimensional operator. It follows from (3.2.1) that $B^{uu} = 0$ and so

$$\wedge^p(L_{m-1}\circ\cdots\circ L_0)=\begin{pmatrix}0 & A_2^{uu}B^{us}A_1^{ss}\\ A_2^{ss}B^{su}A_1^{uu} & A_2^{ss}B^{ss}A_1^{ss}\end{pmatrix}.$$

Now, following the arguments in Lemmas 4.5, 4.6 and 4.7 of (14) we conclude that

$$\log\|\wedge^p(L_{m-1}\circ\cdots\circ L_0)\|<n\left(\frac{\lambda_1^{\wedge p}+\lambda_2^{\wedge p}}{2}+\varepsilon\right).$$

Recalling that

$$\begin{aligned}\lambda_1^{\wedge p} &= \lambda_1+\lambda_2+\cdots+\lambda_{p-1}+\lambda_p \text{ and}\\ \lambda_2^{\wedge p} &= \lambda_1+\lambda_2+\cdots+\lambda_{p-1}+\lambda_{p+1},\end{aligned}$$

then

$$\|\wedge^p(L_{n-1}\circ\cdots\circ L_0)\|\leqslant e^{n\left(\lambda_1+\cdots+\lambda_{p-1}+\frac{\lambda_p+\lambda_{p+1}}{2}+\varepsilon\right)}.$$

To prove (b) we proceed as in (a). Since $\lambda_{p+1}=-\infty$, we have $\lambda_2^{\wedge p}=-\infty$ and so the operator A_i^{ss} $(i=1,2)$ is arbitrarily close to the null one for large choices of n. Moreover, all entries A_1^{uu}, A_2^{us}, A_2^{su} and A_2^{ss} are bounded. Then it suffices to consider a large n, larger than $\mathcal{N}_1(x)$ and m, to reach inequality (b).

□

Doing this perturbation at μ-almost every point of $\Gamma_{p,2}^*$ and following the argument in Proposition 7.3. of (20), we deduce that

Corollary 3.2.2. *Let Φ^t be a cocycle in $\mathscr{C}(\mathscr{H})$, $\varepsilon>0$ and $\delta>0$. Then there exist $m\in\mathbb{N}$, $p\in\mathbb{N}$ and a continuous cocycle $\Psi^t\in\mathscr{C}(\mathscr{H})$ with Ψ^t δ-close (with respect to τ) to Φ^t, equal to Φ^t outside the open set $\Gamma_{p,2}^*$ and satisfying*

$$\lambda_1^{\wedge p}(\Psi^t)<\begin{cases}\left[\lambda_1^{\wedge p-1}(\Phi^t)+\frac{\lambda_p(\Phi^t)+\lambda_{p+1}(\Phi^t)}{2}\right]+\varepsilon & \text{if } \lambda_{p+1}(\Phi^t)\neq-\infty,\\ -\varepsilon^{-1} & \text{if } \lambda_{p+1}(\Phi^t)=-\infty.\end{cases}$$

It is important to refer that the proof of Corollary 4.2.2 uses Lusin's theorem. In fact the perturbation may produce a cocycle Ψ^t that is measurable and not continuous. The Lusin theorem ensures the existence of a continuous integrable cocycle $\tilde{\Psi}^t$ which is equal to Ψ^t outside the open set $\Gamma_p(A,m)$ and the set $E = \left\{x \in M : \tilde{\Psi}^t(x) \neq \Psi^t(x)\right\}$ has measure as small as we want.

3.3 Proof of Theorem A

Consider the p^{th}-*entropy function* defined by

$$\begin{array}{rccl} LE_p : & \mathscr{F}^0 & \to & \mathbb{R} \cup \{-\infty\} \\ & \Phi & \to & \sum_{i=1}^{p} \lambda_i(\Phi^t) \end{array}$$

where $(\lambda_i(\Phi^t))_{i=1,\cdots,\infty}$ are the Lyapunov exponents of the operator $\Phi^t(x)$, for every x in $\mathscr{O}(\Phi)$. This map is upper semicontinuous, so it has a residual subset of points of continuity in the Baire set $\mathscr{F}^0$. Take Φ in this generic subset, consider a point x in the Oseledets-Ruelle's domain $\mathscr{O}(\Phi)$ and denote by $\mathscr{K}$ the orbit of x.

If the Lyapunov exponents of $\Phi^t(x)$ are all equal, then the proof is complete. Otherwise, if $p \in \mathbb{N}$ is such that $\lambda_p > \lambda_{p+1}$, we pursue as follows:

If x is periodic we follow the arguments in (9) to prove that the Oseledets-Ruelle's splitting along the orbit of x is m-dominated for some m.

If x is non-periodic and the Oseledets-Ruelle's splitting along the orbit of x is m-dominated for some m, the proofs ends.

If x is non-periodic and belongs to $\Gamma_p^*(\Phi,m)$ for all m and one of these subsets, say $\Gamma_p^*(\Phi,m_0)$, has positive μ measure, then the m_0-domination on $\mathscr{K}$ of the Oseledets-Ruelle's splitting may fail because x is in one of the corresponding sets $\Gamma_{p,1}^*$ or $\Gamma_{p,2}^*$. If $x \in \Gamma_{p,1}^*$, given ε, by Lemma 3.1.1 there is a cocycle Ψ which is τ-close to Φ but $LE_p(\Psi) = -\infty$ while $LE_p(\Phi)$ is finite. If $x \in \Gamma_{p,2}^*$, by Corollary 4.2.2, given ε there is a cocycle Ψ which is τ-close to Φ but

- $|LE_p(\Phi) - LE_p(\Psi)| > \varepsilon$, in the case $\lambda_{p+1}(\Phi^t) \neq -\infty$
- $LE_p(\Psi) = -\infty$ while $LE_p(\Phi)$ is finite, when $\lambda_{p+1}(\Phi^t) = -\infty$.

In both cases the continuity at Φ of map LE_p is contradicted.

Chapter 4

The L^p-case

As in previous chapter let $\mathscr{H}$ be an infinite dimensional separable Hilbert space and $\mathscr{C}(\mathscr{H})$ the space of linear compact operators defined in $\mathscr{H}$. Let M be a compact, connected, Hausdorff manifold endowed with a Borel probability measure μ.

Consider a continuous and ergodic flow $\varphi^t : M \to M$ with respect to the measure μ, and a compact cocycle $\Phi^t(p) : \mathscr{H} \to \mathscr{H}$ based on φ^t, differentiable on time parameter $t \in \mathbb{R}$ and continuous on space parameter $p \in M$, acting on $\mathscr{C}(\mathscr{H})$.

Let $\mathscr{L}(\mathscr{H},\mathscr{H})$ be the set of linear applications from $\mathscr{H}$ to $\mathscr{H}$. If we define a map $A : M \to \mathscr{L}(\mathscr{H},\mathscr{H})$ in a point $p \in M$ by

$$A(p) = \frac{d}{ds}\Phi^s(p)|_{s=0},$$

and along the orbit $\varphi^t(p)$ by

$$A(\varphi^t(p)) = \frac{d}{ds}\Phi^s(p)|_{s=t} \circ \Phi^{-t}(\varphi^t(p)), \tag{4.0.1}$$

then $\Phi^t(p)$ will be the solution of the linear variational equation

$$\frac{d}{ds}u(s)|_{s=t} = A(\varphi^t(p))u(t). \tag{4.0.2}$$

We will call A the *infinitesimal generator* of (4.0.2). Given a cocycle Φ^t we can induce the associated A by using (4.0.1) and given A we can recover the cocycle by solving the linear variational equation (4.0.2), from which we get Φ^t. We will use the notation Φ^t_A to associate Φ^t with its infinitesimal generator A.

In this chapter we will define an L^p topology in the set of infinitesimal generators, which is coarser then the C^0 topology and, as a consequence, interchanging directions will be easier. In Corollary 4.2.2 it was considered the subset $\Gamma_p^*(\Phi, m)$ of points without a m-dominated splitting of index p. Considering an L^p topology the dominated splitting is no more an obstruction to cause a decay on the Lyapunov exponents, so we will perform the perturbations in a full measure set M. On the other hand, using an L^p topology we can change Oseledets' directions with one single perturbation instead of several perturbations as was done in Lemma 3.2.1. As we will not consider the set $\Gamma_p^*(\Phi, m)$, we don't need to separate the two cases (NB) and (ND), and use always the strategy used in (ND) case. Note that in Chapter 3 when a point $x \in \Gamma_{p,1}^*$ we performed a perturbation of the infinitesimal generator that is not close to the original one if we consider an infinitesimal generator topology and for that reason we used the τ-topology. Now we will only use the (ND)-strategy so we must use a topology defined in the set of infinitesimal generators which is stronger than an L^p topology defined in the set of cocycles.

Consider the set $\mathscr{G}$ of the measurable maps $A : M \to \mathscr{L}(\mathscr{H}, \mathscr{H})$ and the set $\mathscr{G}_{IC}$ of maps $A \in \mathscr{G}$ such that Φ_A^t satisfies the integrability condition

$$\int_M \log^+ \|\Phi_A^1(x)\| \, d\mu(x) < \infty, \tag{4.0.3}$$

where $\log^+(y) = \max\{0, \log(y)\}$. For $A, B \in \mathscr{G}$ set

$$\|A - B\|_p := \begin{cases} \left(\int_M \|A(x) - B(x)\|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}}, \\ \infty \text{ in case that the above integral does not converge,} \end{cases}$$

if $1 \leqslant p < \infty$, and for $p = \infty$ put

$$\|A - B\|_p := \text{ess sup}_{x \in M} \|A(x) - B(x)\|,$$

where $\|\cdot\|$ is the norm operator. We will call $\|\cdot\|_p$ the *L^p-norm*.

Set

$$d_p(A, B) = \begin{cases} \frac{\|A-B\|_p}{1+\|A-B\|_p} & \text{if } \|A - B\|_p < \infty, \\ 1 & \text{if } \|A - B\|_p = \infty. \end{cases} \tag{4.0.4}$$

The following lemma is analogous to Lemma 2.1 of (3).

Lemma 4.0.1. *Let $1 \leqslant p \leqslant \infty$ and consider d_p as defined above, then*

(i) d_p *is a metric on* $\mathscr{G}$*, hence on* $\mathscr{G}_{IC}$*;*

(ii) If $A \in \mathscr{G}_{IC}$ *and* $B \in \mathscr{G}$ *with* $d_p(A,B) < 1$*, then* $B \in \mathscr{G}_{IC}$*. In particularly,* $\mathscr{G}_{IC}$ *is both* d_p*-closed and* d_p*-open in* $\mathscr{G}$*;*

(iii) $(\mathscr{G}, d_p)$*, hence* $(\mathscr{G}_{IC}, d_p)$*, is complete.*

Proof. The proof of (i) is elementary.

To prove (ii) suppose first that $p = 1$, let $A \in \mathscr{G}_{IC}$ and $B \in \mathscr{G}$ with $d_1(A,B) < 1$. Then $\|A - B\|_1 < \infty$. As $\log^+(x+y) \leqslant \log^+(x) + y$ for any positive x, y, then for any $w \in M$

$$\log^+ \left\|\Phi_B^1(w)\right\| \leqslant \log^+ \left\|\Phi_A^1(w)\right\| + \left\|\Phi_B^1(w) - \Phi_A^1(w)\right\|.$$

Therefore $\int_M \log^+ \left\|\Phi_B^1(w)\right\| d\mu(w) < \infty$, and hence $B \in \mathscr{G}_{IC}$.

Suppose $1 < p \leqslant \infty$, $A \in \mathscr{G}_{IC}$ and $B \in \mathscr{G}$ with $d_p(A,B) < 1$. Then $\|A - B\|_p < \infty$. This implies $\|A - B\|_1 < \infty$, hence $d_1(A,B) < 1$. Thus $B \in \mathscr{G}_{IC}$.

To prove (iii) let $1 \leqslant p < \infty$ and $\{A_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\mathscr{G}, d_p)$. By the completeness of the classical L^p-spaces there exists $A(\cdot)$ from $\mathscr{G}$ such that

$$\int_M \|A_n(x) - A(x)\|^p \, d\mu(x) \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore $d_p(A_n, A) \underset{n\to\infty}{\longrightarrow} 0$. By (ii), $A \in \mathscr{G}_{IC}$. Thus $(\mathscr{G}_{IC}, d_p)$ is complete.
Similarly, $(\mathscr{G}_{IC}, d_\infty)$ is complete. □

We will call the topology induced by 4.0.4 as the L^p *infinitesimal generator topology*. As by Lemma 4.0.1 $(\mathscr{G}_{IC}, d_p)$ is complete for $1 \leqslant p \leqslant \infty$ we can conclude that $(\mathscr{G}_{IC}, d_p)$ is a Baire space.

Theorem B. *There exists an* L^p*-residual subset* $\mathscr{R}$ *of the set of integrable compact cocycles* $\mathscr{G}_{IC}$ *such that, for* $A \in \mathscr{R}$ *and* μ*-almost every* $x \in M$

$$\lim_{t\to\infty} \left((\Phi^t(x))^* \Phi^t(x)\right)^{\frac{1}{2t}} = [0],$$

where $[0]$ *stands for the null operator.*

4.1 Entropy Functions

Let us consider the following function LE_k, analogous to the entropy function defined in Section 2.4,

$$\begin{array}{rccc} LE_k: & \mathscr{G}_{IC} & \longrightarrow & \mathbb{R} \cup \{-\infty\} \\ & A & \longmapsto & \sum_{i=1}^{k} \lambda_i(\Phi_A^t). \end{array}$$

Recall that, as μ is ergodic and by Proposition 2.4.1, for μ-almost every $x \in M$ and $A \in \mathscr{G}_{IC}$ we have

$$LE_k(A) = \lambda_1^{\wedge^k} = \lim_{t \to +\infty} \frac{1}{t} \log \| \wedge^k (\Phi_A^t(x)) \| = \inf_{n \in \mathbb{N}} \frac{1}{n} \log \| \wedge^k (\Phi_A^n(x)) \|. \tag{4.1.1}$$

We can prove that, for each k the function LE_k is upper semicontinuous when we endow $\mathscr{G}_{IC}$ with the L^p infinitesimal generator topology, $1 \leqslant p < \infty$, using the same strategy of the proof of Proposition 3.11 of (14). The continuity of the L^p-norm is ensured by Lemma 3.2 of (14).

4.2 Perturbation

The next result is the basic perturbation tool which allows us to interchange Oseledets' directions, using the ideas of Lemma 3.0.1. The only novelty is the perturbation flowbox following the proof of Lemma 3.4 of (14).

Lemma 4.2.1. *Given $1 \leqslant p < \infty$, a continuous differential system $A \in \mathscr{G}_{IC}$, $\varepsilon > 0$ and $x \in M$, for all $\theta \in \left[0, \frac{\pi}{2}\right]$, there exists $r > 0$ (depending on ε) such that for all $\sigma \in (0,1)$, all $y \in B(x, \sigma r)$ (the ball transversal to φ^t at x) and a two-dimensional subspace $E_x \subset \mathscr{H}$, such that rank $\left(\Phi_A^1(x)|_{E_x}\right) = 2$, there exists a continuous integrable cocycle B_θ such that:*

(a) $d_p(A, B_\theta) < \varepsilon$;

(b) $B_\theta(y) = A(y) + H(y)$ *for all* $y \in B(x, \sigma r)$, *where* H *is a perturbation supported in the flowbox* $\mathscr{F} := \{\varphi^t(y) : t \in [0,1], y \in B(x,r)\}$, *such that* $\|H\|_p < \varepsilon$, *and* $B_\theta(z) = A(z)$ *if* $z \notin \mathscr{F}$;

(c) $\Phi_{B_\theta}^t(y) \cdot u = \Phi_A^t(y) \cdot u$, $\forall u \in E_x^\perp$ *and for all* $y \in M$;

(d) $\Phi^1_{B_\theta}(y)\cdot u = \Phi^1_A(y)\cdot R_\theta\cdot u$, *for all* $u\in E_x$ *and* $y\in B(x,\sigma r)$, *where* R_θ *is the rotation of angle* θ *in* E_x.

Proof. For a given small $r>0$ we take the closed ball centered in x and radius r transversal to the flow direction and denoted it by $B(x,r)$. We fix $\sigma\in(0,1)$. Let also $\zeta:\mathbb{R}\to[0,1]$ be a C^∞ function such that $\zeta(t)=0$ for $t\leqslant\sigma$ and $\zeta(t)=1$ for $t\geqslant 1$.

Consider $R_\theta = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}$.

For each $u\in\mathscr{H}_p$ consider the decomposition $u=v+w$, where $v\in E_p$ and $w\in E_p^\perp$. For $t\in\mathbb{R}$ define

$$\mathscr{R}^t\cdot u = R_{\xi g(t)}\cdot v + w.$$

Note that, since $\mathscr{R}^t$ is constant on $E_p^\perp$, i.e. $\mathscr{R}^t\cdot w = w$, we have that

$$(\mathscr{R}^t)'\cdot u = \xi g'(t)\begin{pmatrix}-\sin(\xi g(t)) & -\cos(\xi g(t))\\ \cos(\xi g(t)) & -\sin(\xi g(t))\end{pmatrix}\cdot v,$$

and $(\mathscr{R}^t)^{-1}\cdot v_t = R_{-\xi g(t)}\cdot v_t$.

We know that $u(t)=\Phi^t_A(p)$ is a solution of the linear variational equation

$$\dot u(t) = A(\varphi^t(p))\cdot u(t).$$

Take $\Phi^t_A(p)\cdot\mathscr{R}^t$ and compute the time derivative using the derivative of the product of operators:

$$\begin{aligned}(\Phi^t_A(p)\cdot\mathscr{R}^t)' &= (\Phi^t_A(p))'\mathscr{R}^t + \Phi^t_A(p)(\mathscr{R}^t)'\\ &= A(\varphi^t(p))\Phi^t_A(p)\mathscr{R}^t + \Phi^t_A(p)(\mathscr{R}^t)'.\end{aligned} \tag{4.2.1}$$

Let $\Upsilon^t:\mathscr{H}_{\varphi^t(p)}\to\mathscr{H}_p$ be a map such that for each $u_t\in\mathscr{H}_{\varphi^t(p)}$, with the decomposition $u_t=v_t+w_t$, $v_t\in E_{\varphi^t(p)}$ and $w_t\in E^\perp_{\varphi^t(p)}$, $\Upsilon^t(\varphi^t(p))\cdot u_t = (\Phi^t_A(\varphi^t(p))^{-1}\cdot v_t + w_t$.

Since $(\mathscr{R}^t)'\cdot u = R'_{\xi g(t)}\cdot v$ we can write

$$\Phi^t_A(p)(\mathscr{R}^t)' = \Phi^t_A(p)(\mathscr{R}^t)'(\mathscr{R}^t)^{-1}\Upsilon^t(\varphi^t(p))\Phi^t_A(p)\mathscr{R}^t,$$

and, using (4.2.1),

$$\begin{aligned}
(\Phi^t_A(p)\cdot\mathscr{R}^t)' &= A(\varphi^t(p))\Phi^t_A(p)\mathscr{R}^t+\\
&\quad +\Phi^t_A(p)(\mathscr{R}^t)'(\mathscr{R}^t)^{-1}\Upsilon^t(\varphi^t(p))\Phi^t_A(p)\mathscr{R}^t\\
&= \left[A(\varphi^t(p))+\Phi^t_A(p)(\mathscr{R}^t)'(\mathscr{R}^t)^{-1}\Upsilon^t(\varphi^t(p))\right]\cdot\left(\Phi^t_A(p)\mathscr{R}^t\right).
\end{aligned}$$

Define $B_\xi(\varphi^t(p)) = A(\varphi^t(p)) + H(\varphi^t(p))$, where

$$H(\varphi^t(p)) = H(\xi,t) = \Phi^t_A(p)(\mathscr{R}^t)'(\mathscr{R}^t)^{-1}\Upsilon^t(\varphi^t(p)).$$

Let $L>0$ be sufficiently large in order to get $\left\|(\mathscr{R}^t)'(\mathscr{R}^t)^{-1}\right\| < L$ and take $K := \max_{z\in M}\left\{\left\|\Phi^t_A(z))\right\|, \|\Upsilon^t(\varphi^t(z))\|\right\}$ for $t\in[0,1]$.

Consider the flowbox $\mathscr{F} := \{\varphi^t(y) : t\in[0,1], y\in B(x,r)\}$ and define the continuous linear differential system

$$B(z) = \begin{cases} A(z), & \text{if } z\notin\mathscr{F},\\ A(z)+\left(1-\zeta\left(\frac{\|x-y\|}{r}\right)\right)H(z), & \text{if } z=\varphi^t(y)\in\mathscr{F}.\end{cases}$$

To prove (a) we need to evaluate $\|H\|_p$. As in (14) we consider Rokhlin's theorem on disintegration of the measure μ into a measure $\hat{\mu}$ in the transversal section and the length in the flow section, say $\mu = \hat{\mu}\times dt$. Take $r<0$ such that

$$\hat{\mu}(B(x,r)) < \left(\frac{\varepsilon}{K^2L}\right)^p.$$

Then, we have

$$
\begin{aligned}
\|H\|_p &= \left(\int_{\mathscr{F}} \|H(z)\|^p \, d\mu(z)\right)^{\frac{1}{p}} \\
&= \left(\int_0^1 \int_{B(x,r)} \|H(\varphi^t(y))\|^p \, d\hat{\mu}(y) dt\right)^{\frac{1}{p}} \\
&= \left(\int_0^1 \int_{B(x,r)} \left\|\Phi_A^t(y)(\mathscr{R}^t)'(\mathscr{R}^t)^{-1}\Upsilon^t(\varphi^t(y))\right\|^p \, d\hat{\mu}(y) dt\right)^{\frac{1}{p}} \\
&\leqslant (K^2 L)(\hat{\mu}(B(x,r)))^{\frac{1}{p}} < \varepsilon.
\end{aligned}
$$

So $d_p(A,B) < \varepsilon$.

As in Lemma 3.0.1 we have that the perturbed system B generates the linear flow $\Phi_B^t = \Phi_{A+H}^t$ which is the same as $\Phi_A^t \cdot \mathscr{R}^t$ by unicity of solutions with same initial conditions. So for $t = 1$, $y \in B(x, \sigma r)$ and $u \in E_p$ we obtain $\Phi_{B_\theta}^1(y) \cdot u = \Phi_A^1(y) \cdot R_\theta \cdot u$.

If $u \in E_x^\perp$ clearly $\Phi_{B_\theta}^t(y) \cdot u = \Phi_A^t(y) \cdot u$ for all $y \in M$, so (c) and (d) are proved. □

Doing this perturbation we can deduce that

Proposition 4.2.2. *Let Φ_A^t be a cocycle in $\mathscr{C}(\mathscr{H})$, $\varepsilon > 0$ and $\delta > 0$. Then there exist $m \in \mathbb{N}$, $p \in \mathbb{N}$ and a continuous cocycle $\Phi_B^t \in \mathscr{C}(\mathscr{H})$ with $d_p(A,B) < \varepsilon$ satisfying*

$$
\lambda_1^{\wedge p}(\Phi_B^t) < \begin{cases} \left[\lambda_1^{\wedge p-1}(\Phi_A^t) + \frac{\lambda_p(\Phi_A^t)+\lambda_{p+1}(\Phi_A^t)}{2}\right] + \varepsilon & \text{if } \lambda_{p+1}(\Phi_A^t) \neq -\infty, \\ -\varepsilon^{-1} & \text{if } \lambda_{p+1}(\Phi_A^t) = -\infty. \end{cases}
$$

4.3 Proof of Theorem B

As LE_k is upper semicontinuous it has a residual subset of points of continuity in the Baire set $\mathscr{G}_{IC}$. Let $A \in \mathscr{G}_{IC}$ be a continuity point of the functions LE_k for all k.

If the Lyapunov exponents of $\Phi_A^t(x)$ are all equal, then the proof is complete. Otherwise, if $p \in \mathbb{N}$ is such that $\lambda_k > \lambda_{k+1}$, we use the perturbation given by Lemma 4.2.1 to mix Oseledet's directions and obtain a cocyle $B \in \mathscr{G}_{IC}$ such that $d_p(A,B) < \varepsilon$ but

- $|LE_p(A) - LE_p(B)| > \varepsilon$, in the case $\lambda_{p+1}(\Phi_A^t) \neq -\infty$
- $LE_p(B) = -\infty$ while $LE_p(A)$ is finite, when $\lambda_{p+1}(\Phi_A^t) = -\infty$.

In both cases the continuity at A of map LE_k is contradicted.

Chapter 5

Removing Zero Lyapunov Exponents

In Chapter 3.3 we proved that there exists a τ-residual subset $\mathscr{R}$ of the set of cocycles $\mathscr{F}^0$ such that for $\Phi \in \mathscr{R}$ and μ- almost every $x \in M$, either the limit operator

$$\lim_{t\to\infty} \left[(\Phi^t(x))^*\Phi^t(x)\right]^{\frac{1}{2t}}$$

is the null operator or the Oseledets-Ruelle's splitting of Φ^t along the φ^t-orbit of x has a dominated splitting. However, a domination splitting does not prevent the existence of zero Lyapunov exponents.

In (10) the authors established a perturbation scheme, in context of discrete-time infinite dimensional cocycles, to increase the *sum* of central Lypaunov exponents, while keeping the sum of all the center-unstable Lyapunov exponents invariant. After that they perform another C^0-perturbation such that *each* Lyapunov exponent of the central direction became different from zero.

In this chapter we intend to prove the continuous counterpart of Theorem 1 of (10).

Theorem C. *Non-uniformly Anosov cocycles (see Definitions 5.2.1 and 5.2.2) are C^0-dense in the set of partially hyperbolic cocycles with nontrivial unstable bundles.*

5.1 Definitions

In this section we will use the same definition of dominated splitting used in Section 2.3, replacing the condition (C3.2) by

$$(C3.2^*)\ \frac{\|\Phi_A^\ell(x)\cdot u\|}{\|\Phi_A^\ell(x)\cdot v\|} \leqslant \alpha,$$

where $\alpha \in]0,1[$. We will use the notation

$$E_1 \succ_{\ell,\alpha} E_2$$

to refer that the direct sum $E_1(x) \oplus E_2(x)$ is (ℓ,α)-*dominated.*

Definition 5.1.1. *An (ℓ,α)-dominated direct sum $E_1 \oplus E_2$ on X is said to be the finest if, after a nontrivial decomposition of any of these two subspaces, the sum is no longer dominated.*

Given a dominated sum $E_1 \oplus E_2$ on M, we may find a finest one and it is unique. But the continuation of a finest dominated sum is not necessarily finest for the perturbed cocycle. Still, the set of cocycles to whom this happens is a closed meagre set, that is, its complement, which we denote by SF_{E_1,E_2}, is open and dense.

Definition 5.1.2. *Given a positive integer ℓ and a number $\alpha \in]0,1[$, a splitting $E_1(x) \oplus \cdots \oplus E_k(x) = \mathscr{H}$, defined on a φ^t-invariant set K, is (ℓ,α)-dominated if, for any $x \in K$ and every $1 \leqslant i < j \leqslant k$, we have $E_i \succ_{\ell,\alpha} E_j(x)$.*

As in Chapter 2 the splitting we are interested in is the Oseledets-Ruelle's decomposition associated to the Lyapunov exponents. Hence, conditions (C1) and (C2) are fulfilled *a priori*. In what follows we will always address to this specific decomposition.

Definition 5.1.3. *The Oseledets-Ruelle's decomposition in $\mathscr{O}(A)$ is said to be (ℓ,α)-dominated if there are a positive integer ℓ and a number $\alpha \in]0,1[$ such that, for each $x \in \mathscr{O}(A)$, we may rewrite this decomposition as a direct sum of k subspaces, say $E_1(x) \oplus \cdots \oplus E_k(x)$, so that:*

(D1.) For all $i \in \{1,2,\cdots,k\}$, the dimension of $E_i(x)$ is independent of $x \in \mathscr{O}(A)$.

(D2.) For all $x \in \mathscr{O}(A)$ and every $i < j$, we have $E_i \succ_{\ell,\alpha} E_j(x)$.

By definition, a dominated Oseledets-Ruelle's decomposition regards the order of the Lyapunov exponents: for instance, E_1 is associated with a finite number of the first (biggest) Lyapunov exponents.

5.2 Partial Hyperbolicity

Definition 5.2.1. *A cocycle Φ_A^t with an extended Oseledets-Ruelle's decomposition is said to be partially hyperbolic if, for any $x \in M$, this splitting may be rewritten as a direct sum of three subspaces, say $E_x^u \oplus E_x^c \oplus E_x^s = \mathscr{H}$, and there are $\ell \in \mathbb{N}$, $\alpha \in]0,1[$ and $\beta \in]0,1[$ such that:*

PH1. E_x^u *contains only Oseledets' subspaces associated to positive Lyapunov exponents.*

PH2. E_x^c *contains all the Oseledets' subspaces associated to zero Lyapunov exponents.*

PH2. E_x^s *contains only Oseledets' subspaces associated to negative Lyapunov exponents and includes all the Oseledets' subspaces determined by Lyapunov exponents equal to* $-\infty$.

PH4. $E_x^u \succ_{\ell,\alpha} E_x^c$ *and* $E_x^c \succ_{\ell,\beta} E_x^s$.

The dynamics along E_x^u (respectively E_x^s) is strongly expanding (respectively contracting). The weaker forms of expansion or contraction are gathered inside E_x^c. From Lemma 3.3 of (9), we know that, if Φ_A^t is a partial hyperbolic cocycle, then E_x^u and $E_x^{cu} := E_x^u \oplus E_x^c$ are finite dimensional. We say that a C^0 partial hyperbolic cocycle has *nontrivial unstable bundle* if, for any $x \in M$, the unstable space E_x^u does not reduce to $\{0\}$.

Definition 5.2.2. *If, besides being partially hyperbolic, no Lyapunov exponent is zero, the cocycle is said to be non-uniformly Anosov.*

5.3 Center-unstable metric entropy

For μ-almost every $x \in \mathscr{O}(A)$, we have

$$\lim_{t\to\infty} \log\left|\det(\Phi_A^t(x))|_{E_x^u}\right| = \sum_{i=1}^{d} n_{i,A}\lambda_A^i$$

and

$$\lim_{t\to\infty} \log\left|\det(\Phi_A^t(x))|_{E_x^{cu}}\right| = \sum_{i=1}^{D} n_{i,A}\lambda_A^i,$$

where $n_{i,A}$ is the dimension of $U^i(x)$, λ_A^i represents Lyapunov exponents of Φ_A^t in E_x^u or E_x^c, and d and D are the dimensions of E_x^u and E_x^{cu}, respectively (see (10)). To simplify the notation, $\mathscr{O}(A)$ will also stand for this full μ measure set.

We will call

$$\sum_{i=1}^{D} n_{i,A}\lambda_A^i$$

center-unstable metric entropy of the partially hyperbolic cocycle Φ_A^t, with respect to φ^t and the probability measure μ.

5.4 Perturbations Lemmas

Proposition 5.4.1. *Let Φ_A^t be an integrable, compact and partially hyperbolic cocycle with a nontrivial unstable bundle and such that $\sum n_{c,A}\lambda_A^c = 0$, where λ_A^c denotes Lyapunov exponents of Φ_A^t in E^c. Then for any $\varepsilon > 0$, there exists an integrable compact cocycle Φ_B^t such that:*

- *Φ_B^t is partially hyperbolic;*
- *B is $\varepsilon - C^0$- close to A;*
- *The center-unstable metric entropy of Φ_B^t is equal to that of Φ_A^t;*
- *$\sum n_{c,B}\lambda_B^c > 0$.*

The perturbation need for this result will diminish the strength of the expansion along E^u while keeping the center-unstable metric entropy invariant, so the sum of all the central Lyapunov exponents will have to increase. If $D - d > 1$, this sum may still have terms equal to zero. To remove them, we need another perturbation that makes all these summands essentially equal without destroying the positiveness of the sum.

Proposition 5.4.2. *Let Φ_B^t be an integrable, compact and partially hyperbolic cocycle with a nontrivial unstable bundle and such that $\sum n_{c,B}\lambda_B^c \neq 0$. Then for any $\varepsilon > 0$, there exists am integrable, compact and non-uniformly Anosov cocyle Φ_C^t such that C is ε-C^0-close to B.*

The next lemma is analogous to Lemma 5.1 of (10) and is the basic tool to perturb a partial hyperbolic cocycle in order to increase the sum of Lyapunov exponents along the central directions without changing the center-unstable metric entropy.

Let $B(x,\sigma r)$ denote the ball transversal to φ^t at x. Recall that, as μ is ergodic and positive on nonempty open sets, there is a residual subset of M whose elements have dense orbits by φ.

Lemma 5.4.3. *Consider $A \in \mathscr{C}(\mathscr{H})$ which is partially hyperbolic with $D = dim(E^{cu})$ and $\sum n_{c,A}\lambda_A^c = 0$. Fix a point $p \in \mathscr{O}(A)$ with dense orbit, an $r > 0$ and an $\varepsilon > 0$ small enough. There exists an angle $\xi > 0$ and $B_\xi \in \mathscr{C}(\mathscr{H})$ such that for all $\sigma \in (0,1)$:*

(a) $B_\xi(x) = A(x) + H(x)$ *for all* $x \in B(p,\sigma r)$, *where H is a perturbation supported in the flowbox* $\mathscr{F} := \{\varphi^t(y) : t \in [0,1], y \in B(p,r)\}$ *and* $B_\xi(z) = A(z)$ *if* $z \notin \mathscr{F}$;

(b) $\Phi_{B_\xi}^t(x)\cdot v_x^s = \Phi_A^t(x)\cdot v_x^s$, $\forall v_x^s \in E_{x,A}^s$ *and* $\forall x \in M$;

(c) $\Phi^1_{B_\xi}(x)\cdot v^{cu}_x = \Phi^1_A(x)\cdot R_x\cdot v^{cu}_x$, $\forall v^{cu}_x \in E^{cu}_{x,A}$ *and* $\forall x\in B(p,\sigma r)$, *where* R_x *belongs to the space* $SO(D,\mathbb{R})$ *of rotations in* $SL(D,\mathbb{R})$*;*

(d) $\|A-B_\xi\| \leqslant \varepsilon$*;*

(e) $\sum n_{c,B}\lambda^c_B + \sum n_{u,B}\lambda^u_B = \sum n_{u,A}\lambda^u_A$*;*

(f) $\sum n_{c,B}\lambda^c_B > 0$.

Proof. Given $x\in M$, we may write each vector $v_x\in\mathscr{H}_x=\mathscr{H}$ in a unique way as $v_x = v^s_x + v^{cu}_x$, where $v^s_x\in E^s_{x,A}$ and $v^{cu}_x\in E^{cu}_{x,A}$.

We will now proceed as in Lemmas 3.0.1 and 4.2.1 to perform a perturbation in $E^{cu}_{x,A}$. Consider a two dimensional subspace $F^{cu}_{x,A}\subset E^{cu}_{x,A}$ with $\operatorname{rank}\left(\Phi^1_A(x)|_{F^{cu}_{x,A}}\right)=2$.

Let $\eta\in(0,1)$ and $g:\mathbb{R}\to\mathbb{R}$ be the *bump-function* defined by $g(t)=0$ for $t<0$, $g(t)=t$ for $t\in[\eta,1-\eta]$ and $g(t)=1$ for $t\geqslant 1$.

For each $v^{cu}_x\in E^{cu}_{x,A}$ consider the decomposition $v^{cu}_x = u^{cu}_x + w^{cu}_x$, with $u^{cu}_x\in F^{cu}_{x,A}$ and $w^{cu}_x\in\left(F^{cu}_{x,A}\right)^{\perp}$. For $t\in\mathbb{R}$ consider

$$R_{\xi g(t)} = \begin{pmatrix} \cos(\xi g(t)) & -\sin(\xi g(t)) \\ \sin(\xi g(t) & \cos(\xi g(t)) \end{pmatrix},$$

and define for $v_x\in\mathscr{H}_x$

$$\mathscr{R}^t\cdot v_x = v^s_x + R^t_x\cdot v^{cu}_x,$$

where

$$R^t_x\cdot v^{cu}_x = R_{\xi g(t)}u^{cu}_x + w^{cu}_x.$$

For a given small $r>0$ we take the closed ball $B(p,r)$, centred in p and radius r transversal to the flow direction. Let also $\zeta:\mathbb{R}\to[0,1]$ be a C^∞ function such that $\zeta(t)=0$ for $t\leqslant\sigma$ and $\zeta(t)=1$ for $t\geqslant 1$.

Now we proceed as in Lemma 3.0.1 to obtain $H(\varphi^t(x))$, that is

$$H(\varphi^t(x)) = H(\xi,t) = \Phi^t_A(x)(\mathscr{R}^t)'(\mathscr{R}^t)^{-1}\Upsilon^t(\varphi^t(x)),$$

where $\Upsilon^t:\mathscr{H}_{\varphi^t(x)}\to\mathscr{H}_x$ is a map such that $\Upsilon^t(\varphi^t(x))\cdot v_x = v^s_x + (\Phi^t_A(x))^{-1}\cdot u^{cu}_x + w^{cu}_x$, for every $v_x\in\mathscr{H}_{\varphi^t(x)}$.

Consider the flowbox $\mathscr{F} := \{\varphi^t(x) : t \in [0,1], x \in B(p,r)\}$ and define the continuous linear differential system

$$B_\xi(z) = \begin{cases} A(z), & \text{if } z \notin \mathscr{F}, \\ A(z) + \left(1 - \zeta\left(\frac{\|p-x\|}{r}\right)\right) H(z), & \text{if } z = \varphi^t(x) \in \mathscr{F}. \end{cases}$$

By construction, B_ξ is a compact operator and $E^s_{x,A}$ and $E^{cu}_{x,A}$ era B_ξ-invariant for all $x \in M$. And as, for all $x \in M$, the map $\mathscr{R}^t_x$ is a rotation acting in $E^{cu}_{x,A}$, we have $E^s_{x,A} = E^s_{x,B_\xi}$ and $E^{cu}_{x,A} = E^{cu}_{x,B_\xi}$. However, for any point x whose orbit visits $\mathscr{F}$, the dynamics dynamics along $E^c_{x,A}$ and $E^u_{x,A}$ changes after the perturbation.

As in Lemma 3.0.1 we have that the perturbed system B_ξ generates the linear flow $\Phi^t_{B_\xi} = \Phi^t_{A+H}$ which is the same as $\Phi^t_A \cdot \mathscr{R}^t$ by unicity of solutions with same initial conditions. So for $t = 1$ and $v_x \in E^{cu}_{x,a}$ we obtain $\Phi^1_{B_\xi}(x) \cdot v_x = \Phi^1_A(x) \cdot R^1_x \cdot v_x = R_\xi u^{cu}_x + w^{cu}_x$.

If $v_x \in E^s_{x,A}$ clearly $\Phi^t_{B_\xi}(x) \cdot v_x = \Phi^t_A(x) \cdot v_x$, so the first three properties are proved.

Concerning with the fourth property we have that

$$\begin{aligned} \|A - B\| &= \sup_{z=\varphi^t(x) \in \mathscr{F}} \left\| \left(1 - \zeta\left(\frac{\|p-x\|}{r}\right)\right) H(z) \right\| \\ &\leqslant \|H\|. \end{aligned}$$

Since $t \in [0,1]$ and all the terms in the definition of $H(\xi,t)$ are uniformly bounded for all $x \in M$, given any size of perturbation allowed by $\varepsilon > 0$ we take ξ sufficiently small o guarantee that $\|A - B\| < \varepsilon$.

We also have, by Corollary 5.2 of (10) that the splitting $\mathscr{H} = E^s_B \oplus E^{cu}_B = E^s_A \oplus E^{cu}_A$ is (ℓ,ν)-dominated for Φ^t_B with $\nu = \frac{1+\alpha}{2}$.

Since Oseledets theorem is an asymptotic result and $\Phi^r_B(x)$, for a fixed r, is a bounded operator, we can replace $\Phi^t_B(x) = \Phi^r_B(\varphi^n(x))\Phi^n(x)$ by the last integer time-n map, $\Phi^n_B(x)$ and consider

$$\lim_{t\to\infty} \frac{1}{t} \log|\det \Phi^t_B(z)|_{E^{cu}_{z,B}}| = \lim_{n\to\infty} \frac{1}{n} \log|\det \Phi^n_B(z)|_{E^{cu}_{z,B}}|.$$

Concerning the fifth property: as the map $\mathscr{R}^t$ belongs to $SO(D,\mathbb{R})$ and $E^{cu}_{x,B} = E^{cu}_{x,A}$, for any $z \in \mathscr{O}(A) \cap \mathscr{O}(B)$ we have

$$\begin{aligned}
\sum n_{c,B}\lambda_B^c + \sum n_{u,B}\lambda_B^u &= \lim_{n\to\infty}\frac{1}{n}\log|\det\Phi_B^1(\varphi^{n-1}(z))|_{E^{cu}_{\varphi^{n-1}(z),B}} \\
&\quad \times\cdots\times|\det\Phi_B^1(z)|_{E^{cu}_{z,B}}| \\
&= \lim_{n\to\infty}\frac{1}{n}\log|\det\Phi_A^1(\varphi^{n-1}(z))|_{E^{cu}_{\varphi^{n-1}(z),A}} \\
&\quad \times\cdots\times|\det\Phi_A^1(z)|_{E^{cu}_{z,A}}| \\
&= \lim_{n\to\infty}\frac{1}{n}\log|\det\Phi_A^n(z)|_{E^{cu}_{z,A}}| = \sum n_{u,A}\lambda_A^u.
\end{aligned}$$

To prove (f) we can use the same arguments used in (10). The idea is to show that for some $0<\Delta<1$ we have that $\sum n_{u,B}\lambda_B^u \simeq \sum n_{u,A}\lambda_A^u + d\log(\Delta)$ and then use the invariance of the center-unstable metric entropy.

Let us resume the proof.

Consider $d=1$. As μ is ergodic to evaluate the unstable Lyapunov exponent of $\Phi_B^t(x)$ we only need to determine the average growth of $\Phi_B^t(x)\cdot v^u$ for any $x\in\mathscr{O}(A)\cap\mathscr{O}(B)$ and a unit vector $v^u\in E^u_{x,A}$.

As p is not periodic, we may consider r small enough such that $\mathscr{F}$ is a flowbox. Take $x\in\mathscr{F}\cap\mathscr{O}(A)\cap\mathscr{O}(B)\cap\mathscr{P}$, where $\mathscr{P}$ is the full μ measure subset of $\mathscr{F}$ given by Poincaré's recurrence theorem. For a unit vector $v^u\in E^u_{x,A}$ we have that

$$\Phi_B^1(x)\cdot v^u = \Phi_A^1(x)\cdot\pi_{E^u_{x,A}}\cdot R_x^1\cdot v^u + \Phi_A^1(x)\cdot\pi_{E^c_{x,A}}\cdot R_x^1\cdot v^u,$$

where $\pi_{E^u_{x,A}}$ is the projection on $E^c_{x,A}$ parallel to the bundle $E^c_{x,A}$ (and analogous definition for $\pi_{E^u_{x,A}}$). As in (10) we have that in general, while the orbit keeps out of $\mathscr{F}$, we have

$$\Phi_B^j(x)(v^u) = \Phi_A^j(x)\cdot\pi_{E^u_{x,A}}\cdot R_x^1\cdot v^u + \Phi_A^j(x)\cdot\pi_{E^c_{x,A}}\cdot R_x^1\cdot v^u,$$

and, by the domination of E^u over E^c under the action of A, there are constants $C>0$ and $\beta\in]0,1[$ such that, for any $m\in\mathbb{N}$ and any unit vectors $w_1\in E^u$ and $w_2\in E^c$, we have

$$\|\Phi_A^m(w_2)\| \leqslant C\beta^m\|\Phi_A^m(w_1)\|. \tag{5.4.1}$$

Thus, the first component of $\Phi_B^j(x)\cdot v^u$ dominates the second and contributes to λ_B^u with an approximate rate

$$\frac{1}{j}\log\left\|\Phi_B^j(x)\cdot v^u\right\| \sim \lambda_A^u + \log(\Delta) < \lambda_A^u, \tag{5.4.2}$$

where $\Delta \in]0,1[$ is an upper bound of the set $\{\cos(\xi) : z \in M\}$ and ξ_z stands for the small angle of the rotation displaced by the action of $\mathscr{R}^t$. We take r small in order to guarantee, by Kac's theorem (see a version of Kac's theorem for flows in (39)), that the expected first return to $\mathscr{F}$ of the orbit of x amounts to the rather big fraction $\frac{1}{\mu(\mathscr{F})}$, and so the contribution of the piece of orbit out $\mathscr{F}$ for the estimation 5.4.2 is the prevalent one.

We can also obtain an average as in (5.4.2) if we consider the iterate $N+1$, where N is an iterate correponding to a return to $\mathscr{F}$ of the orbit of x, which happens with probability μ (see 5.1.1 of (10) for details).

The case $d > 1$ can be treated with an identical approach. (see 5.1.2 of (10))

□

If $1-\Delta$ is small enough, we ensure that there is still a uniform gap between the Lyapunov exponents that correspond to different bundles of the Oseledets-Ruelle's decomposition of Φ^t_B. As the decomposition $E^{cu}_B = E^u_B \oplus E^c_B$ is finest, we can deduce that

Corollary 5.4.4. *Φ^t_B is partially hyperbolic with $E^u_B = E^u_A$, $E^c_B = E^c_A$ and $E^s_B = E^s_A$.*

5.5 Proof of Theorem C

The proof of Theorem C follows exactly the same steps of the proof of Theorem 1 of (10). Let Φ^t_A be a integrable compact and partially hyperbolic cocyle with an extended Oseledets-Ruelle's decomposition $E^u \oplus E^c \oplus E^s = \mathscr{H}$ such that, for any $x \in \mathscr{O}(A)$, the space $E^u_{x,A}$ is nontrivial and has dimension $d > 0$. Denote by D the dimension of $E^{cu}_{x,A}$.

For any $k \in \{1,2,\cdots,D\}$, consider the map:

$$LE_k : C \in C^0_I(X,\mathscr{C}(\mathscr{H})) \to \lambda^C_1 + \cdots + \lambda^C_k,$$

where λ^C_j is the j^{th} Lyapunov exponent of the cocycle Φ^t_C. As LE_k is upper semicontinuous, for any k (Sec. 3.5 of (9)), and is defined on a Baire space (Sec. 3.1 of (9)), it has a residual set $\mathscr{R}_k$ of continuity points which is dense in $C^0_I(X,\mathscr{C}(\mathscr{H}))$. Therefore, there is a partially hyperbolic cocycle A_0 inside the residual subset

$$\mathscr{R}_1 \cap \cdots \mathscr{R}_D \cap SF_{E^u,E^c}$$

close enough to A and whose norm is positive. Thus, given $\varepsilon > 0$, there is a neighbourhood $\mathscr{U}$ of A_0 such that:

- $\forall C \in \mathscr{U}$, Φ^t_C is partially hyperbolic, $\dim(E^u_C) = d$ and $\dim(E^{cu}_C) = D$;
- $\forall C_1, C_2 \in \mathscr{U}$ $\forall k \in \{1,\cdots,D\}$ $|LE_k(C_1) - LE_k(C_2)| < \varepsilon$.

If $(LE_D - LE_d)(A_0) = 0$ we can apply Lemma 5.4.3 to A_0 to get a cocycle $B \in \mathscr{U}$ whose sum of central Lyapunov exponents is positive. If $(LE_D - LE_d)(A_0) \neq 0$ we take $B = A_0$.

If either $D = d$ or the Lyapunov exponents corresponding to E_B^c are all equal, there is nothing to be proved. If both conditions fail, take two distinct Lyapunov exponents, say $\lambda_k > \lambda_{k+1}$, in E_B^c. As the sum $E_B^{cu} = E_B^u \oplus E_B^c$ is dominated and finest, and both λ_k and λ_{k+1} belong to E_B^c, there is no dominated sum $E_B^{cu} = V_1 \oplus V_2$ with $\dim(V_1) = k$. As $\lambda_{k+1} \neq -\infty$, by Corollary 4.2.2 of Section 3.3 there is a cocycle $C \in \mathscr{U}$ close to B such that

$$LE_k(C) < LE_k(B) - \frac{\lambda_k(B) - \lambda_{k+1}(B)}{2} + \varepsilon.$$

Thus

$$\lambda_k(B) - \lambda_{k+1}(B) < 2|LE_k(C) - LE_k(B)| + 2\varepsilon < 4\varepsilon$$

which means that the central Lyapunov exponents are all close to each other. Hence each one is approximately equal to $\frac{LE_D - LE_d}{D-d}$, and so doesn't vanish.

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