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Applications of Mathematics and
Game Theory to Industrial Organization



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**Applications of Mathematics and
Game Theory to Industrial Organization**

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Signature:

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Their love, endless support and encouragement

gave me strength to make this work.

Abstract

In this PhD Thesis, several concepts of distinct areas of Mathematics, such as Dynamical Systems, Game Theory, Optimization, Statistics, Stochastic Optimal Control and Dynamic Programming are applied and developed.

In Chapter 2, we introduce a dichotomous decision model, where individuals can make the decision yes or no and can influence the decisions of others. We characterize all decisions that form Nash equilibria. Taking into account the way individuals influence the decisions of others, we construct the decision tilings where the axes reflect the personal preferences of the individuals for making the decision yes or no. These tilings characterize geometrically all the pure and mixed Nash equilibria. There are 289 combinatorial classes of decision tilings, described by the decision Bussola, which demonstrates the high complexity of making decision. We show, in these tilings, that Nash equilibria form degenerated hystereses with respect to the replicator dynamics, with the property that the pure Nash equilibria are asymptotically stable and the strict mixed equilibria are unstable. These hystereses can help to explain the sudden appearance of social, political and economic crises. We observe the existence of limit cycles for the replicator dynamics associated to situations where the individuals keep

changing their decisions along time, but exhibiting a periodic repetition in their decisions. We introduce the notion of altruist and individualist leaders and study the way that the leader can affect the individuals to make the decision that the leader pretends. We introduce a resort pricing model, where different types of tourists choose between different resorts. We study the influence of the resort prices on the choices of the different types of tourists. In particular, the dependence of the resort prices on the characteristics of the tourists. We characterize the cohesive strategies of the tourists that are Nash equilibria. We find the prices that lead to the bankruptcy of the resorts and, in particular, their dependence on the characteristics of the tourists. We introduce a more general framework for the decision model, where individuals will have to make decisions according to their preferences that also depend upon the other individuals' decisions. In this model, there are N types $t \in \{t_1, t_2, \dots, t_N\}$ of individuals and M possible decisions $d \in \{d_1, d_2, \dots, d_M\}$ that individuals can make. We fully characterize the space of all parameters where the Nash equilibria are cohesive, i.e. individuals with similar preferences make the same decisions. We prove that small changes in the parameters can imply the appearance or disappearance of cohesive Nash equilibria. These abrupt changes in the structure of the cohesive Nash equilibria can be used to characterize economical, political or social revolutions.

In Chapter 3, we consider a family of stochastic optimal control problems with the property that the objective functional depends on multiple random time horizons, assumed to be independent and identically distributed continuous random variables. The state variable follows a stochastic differ-

ential equation driven by a standard multi-dimensional Brownian motion. We resort to the concept of order statistics to restate the stochastic optimal control problem as one with a fixed planning horizon. For this class of stochastic optimal control problems, we derive a sequence of dynamic programming principles and the corresponding Hamilton-Jacobi-Bellman equations. We conclude with an application to a consumption-investment problem.

Resumo

Nesta Tese de Doutorado são aplicados diversos conceitos de áreas distintas da Matemática, tais como Sistemas Dinâmicos, Teoria dos Jogos, Otimização, Estatística, controle estocástico de otimização e programação dinâmica. Estes conceitos são aplicados e desenvolvidos.

No Capítulo 2, apresenta-se um modelo de decisão dicotômica, onde os indivíduos podem tomar a decisão sim ou não, e desta forma influenciar as decisões dos outros. Neste capítulo caracterizam-se todas as decisões que formam equilíbrios de Nash. Tendo em conta a forma como os indivíduos influenciam as decisões dos outros, são construídas as pavimentações de decisão onde os eixos refletem as preferências pessoais dos indivíduos na tomada de decisão. Essas pavimentações caracterizam geometricamente todos os equilíbrios de Nash puros e mistos. Existem 289 classes combinatórias de padrões de decisão, descritos pela Bússola de decisão, facto que demonstra a alta complexidade da tomada de decisão. Evidencia-se, nas referidas pavimentações, que os equilíbrios Nash formam histereses degeneradas relativamente à dinâmica do replicador, com a propriedade de que os equilíbrios de Nash puros são assintoticamente estáveis, e os equilíbrios mistos estritos são instáveis. Estas histereses podem ajudar a explicar o súbito apareci-

mento de crises sociais, políticas e económicas. Observa-se a existência de ciclos limite para a dinâmica do replicador, associados a situações onde os indivíduos mudam as suas decisões ao longo do tempo, exibindo contudo uma repetição periódica nas suas decisões. Introduce-se o conceito de líder altruísta e de líder individualista, e estuda-se a forma como o líder pode afetar os indivíduos na tomada de decisão que o próprio pretende. Apresenta-se um modelo incluindo preços associados a uma estância turística, onde diferentes tipos de turistas podem escolher entre diferentes estâncias turísticas. É estudada a influência dos preços das estâncias turísticas sobre as escolhas dos diferentes tipos de turistas. Caracterizam-se as estratégias coesas dos turistas que formam equilíbrios de Nash. São estimados os preços que conduzem à falência as estâncias turísticas e, em particular, a dependência desta falência nas características dos turistas. Apresenta-se um modelo de decisão onde os indivíduos tomam decisões de acordo com as suas preferências, as quais dependem também das decisões dos outros indivíduos. Neste modelo, existem N tipos $t \in \{t_1, t_2, \dots, t_N\}$ de indivíduos e M possíveis decisões $d \in \{d_1, d_2, \dots, d_M\}$ que os indivíduos podem tomar. É caracterizado o espaço de todos os parâmetros em que os equilíbrios de Nash são coesos, ou seja, indivíduos com preferências semelhantes tomam a mesma decisão. Prova-se que pequenas mudanças nos parâmetros podem implicar o aparecimento e desaparecimento de equilíbrios de Nash coesos. Estas mudanças abruptas na estrutura da coesão do equilíbrio de Nash podem ser utilizadas para caracterizar revoluções económicas, políticas ou sociais.

No capítulo 3, consideramos uma família de problemas de controlo óptimo estocástico com uma funcional objetivo dependente de vários horizontes

com tempos aleatórios, representados por variáveis aleatórias contínuas independentes e identicamente distribuídas. A variável de estado segue uma equação diferencial estocástica gerada por um movimento Browniano multi-dimensional. Recorremos ao conceito de estatísticas de ordem para transformar o problema de controlo óptimo estocástico inicial, num outro com horizonte determinístico fixo. Para esta classe de problemas de controlo óptimo estocástico, construímos um princípio de programação dinâmica e derivamos a equação de Hamilton-Jacobi-Bellman correspondente. Concluimos com uma aplicação a um problema de consumo-investimento.

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Chapter 1

Introduction

In this PhD Thesis, several concepts of distinct areas of Mathematics, such as Dynamical Systems, Game Theory, Optimization, Statistics, Stochastic Optimal Control and Dynamic Programming are applied and developed.

In Chapter 2, we introduce a dichotomous decision model, where individuals can make the decision yes or no and can influence the decisions of others. The main goal in Planned Behavior or Reasoned Action theories, as developed in the works of Ajzen (see [1]) and Baker (see [4]), is to understand and predict the way individuals turn intentions into behaviors. Almeida-Cruz-Ferreira-Pinto (see [2, 3]) developed a game theoretical model for reasoned action, inspired by the works of J. Cownley and M. Wooders (see [17]). Here, we study the Pinto's dichotomous decision model (see [32, 38]), which is a simplified version of the Almeida-Cruz-Ferreira-Pinto decision model. In this model, there are just two types $t \in \{t_1, t_2\}$ of individuals and two possible decisions d that individuals can make. In this case, they have to choose between yes or no, i.e. $d \in \{Y, N\}$. The

yes-no decision model incorporates, in the coordinates of the preference decision matrix how much an individual with type t_1 or with type t_2 likes or dislikes, to make a decision $d \in \{Y, N\}$. In the coordinates of the preference neighbors and non-neighbors matrices, the preference that individuals with a certain type t_i have for other individuals, with the same or a different type t_j , to make the same decision or the opposite decision as theirs (see [16, 37]). The preference decision matrix and the neighbors and non-neighbors matrices can be very complex to find explicitly in real cases because they encode, for instance, information from economic, educational, political, psychological and social variables. However, if we have a qualitative or rough knowledge of these matrices, we can obtain relevant information on how individuals make decisions and why to make decisions can be so complex. We characterize all the pure Nash equilibria and we show that the pure Nash equilibria are, in general, asymptotically stable with respect to the replicator dynamics. The pure Nash equilibria are either cohesive, i.e. all individuals with the same preferences make the same decision, or disparate, i.e. there are individuals with the same preferences that make opposite decisions. The disparate pure Nash equilibria can correspond to conflicting decisions that divide a community. We characterize all the strict mixed Nash equilibria and we prove that the strict mixed Nash equilibria are, in general, unstable. Fixing the parameters of the preference neighbors matrices we construct tilings in the plane, where the horizontal axis represent the relative preference of individuals with type t_1 to make the decision yes or no, the vertical axis represent the relative preference of individuals with type t_2 to make the decision yes or no and the pure and

mixed Nash equilibria form the tiles. We prove that the tilings give geometrically a full characterization of the pure and mixed Nash equilibria. We say that individuals have a positive (resp. negative) influence on each other if they have gains (resp. losses) in their utility when making the same decision. If all the individuals have a positive influence over individuals with the same type, then there are no disparate Nash equilibria. However, if individuals of the same type have negative influence on each other, then there are disparate Nash equilibria which are asymptotically stable. The stable manifolds of the strict mixed Nash equilibria can be locally characterized by appropriate symmetries of the model. They are the main reason for certain decision strategies to persist for long periods of time before breaking down and converge to quite different decision strategies. They also explain, partially, the complexity of the non-intuitive successive reversal of the individuals' decisions along time before converging to a stable equilibrium, i.e. a high number of individuals have to keep modifying their decisions through the transient dynamics before reaching the equilibrium. Furthermore, we observe the existence of stable periodic orbits for the replicator dynamics, i.e. the individuals' decisions keep changing along time exhibiting a periodic pattern. The replicator dynamics equilibria form hysteresis that provide insight into how small changes in economic, educational, political, psychological or social variables can reverse abruptly individual and collective decisions. These changes in the collective decisions can lead to serious political, social or economic transformations in society. In section 2.6, we study the bifurcations and the combinatorial equivalent tilings in the decision model. We observe that there are 289 combinatorial classes

of decision tilings, described by the decision Bussola, which demonstrates the high complexity of making decision, and 64 of them are being structurally stable and 225 combinatorial classes of bifurcation decision tilings. In section 2.7 and following [3], we introduce the notion of altruist and individualist leaders. The altruist and individualist leaders offer, respectively, advantages or disadvantages to the individuals of both types. The leader is biased if the leader offers advantages to individuals with a certain type and disadvantages to individuals with the other type. The individuals can increase or decrease the advantages or disadvantages offered by the leader depending upon their own abilities characterized by their type. We study the way that the leader can affect the individuals (potential followers) to make the decision that the leader pretends. In section 2.8, we introduce a resort pricing model where different types of tourists choose between different resorts. The distribution of different types of tourists reaching a destination affects both the demand and supply side of the tourism industry. From the demand perspective, the choice of a particular destination will depend greatly on the beliefs of the agent about which kind of tourists will share the resort with him/her (see [20]). On the supply side, resorts try to sell their destination based on reputation, and a large factor that determines the character and reputation of a resort is the type of tourists who frequent that resort (see [30]). Brida *et al.* [15, 16] presented a tourism model where the choice of a resort by a tourist depends not only on the product offered by the resort, but also depends on the characteristics of the other tourists present in the resort. In order to explore the effect that the type of resident tourist has on other potential tourists selecting the same resort,

they introduced a game theoretical model and described some relevant Nash equilibria. Soeiro *et al.* [42] presented a game theoretical model to study the effects of tourists relations on the market shares and characterized all possible tourist allocations that form Nash equilibria. Here, we study the influence of resort prices on the tourist's choice of a resort. We characterize the prices that lead to the bankruptcy of the resorts and, in particular, their dependence on the characteristics of the tourists. In section 2.9, we study a modified version of the game theoretical model presented in [32] and we call the decision model. In this model, there are N types $t \in \{t_1, t_2, \dots, t_N\}$ of individuals and M possible decisions $d \in \{d_1, d_2, \dots, d_M\}$ that individuals can make. We consider that individuals with the same utility functions belong to a same group. We say that a group is cohesive if every individual has a gain in his utility when other individuals of the same group make the same decision as his. Almeida *et al.* [2] prove that all the individuals of a cohesive group make the same decision at a pure Nash equilibria. We say that a pure strategy is cohesive if all the individuals belonging to a same group will make the same decision. Hence, by Almeida *et al.* [2], if all groups are cohesive, then all pure Nash are cohesive strategies. Here, we find the cohesive thresholds that characterize the space of all parameters where the cohesive strategies are Nash equilibria. This chapter is essentially based on the research articles [1, 2, 4, 15, 17, 30], the book chapters [3, 16, 20] and the books [36, 39].

In Chapter 3, we consider a family of stochastic optimal control problems with multiple random time horizons. Optimal control theory has its roots in the late 1950s, resulting of an extension of the Calculus of Variations,

which itself dates back to the 1600s. A key motivation for the development of the modern optimal control theory was the study of differential games from 1945 onwards by a group in the Rand Corporation which included R. Bellman, J. P. LaSalle, D. Blackwell, R. Isaacs, W. H. Fleming, and L. D. Berkovitz. Another group, based in the former Soviet Union and organized around Pontryagin, independently developed an alternative approach to address optimal control problems. From the pioneering work of these two groups, there are nowadays two main techniques which can be used to address optimal control problems: the dynamic programming principle and the Pontryagin maximum principle. The dynamic programming principle was initiated by Bellman in the 1950s. Bellman's first work on dynamic programming [5] was published in 1952, and one year later this publication was expanded to a monograph [6]. In 1954, Bellman found that the dynamic programming technique could be used to address calculus of variations problems or optimal control problems whose state variable evolution is given by ordinary differential equations [7]. Using this technique Bellman derived a nonlinear partial differential equation, that is now referred to as Hamilton-Jacobi-Bellman (HJB) equation. The earliest paper mentioning stochastic control was probably the work of Bellman [8] in 1958. However, Itô stochastic differential equations were not yet considered. In 1961, Florentin [23, 24] considered systems of diffusion type and used Bellman's dynamic programming approach to derive a partial differential equation associated with a continuous-time controlled Markov process. In 1962, Kushner [28] studied the stochastic optimal control problem with the state variable dynamics given by an Itô stochastic differential equation. There

has been intensive research on the use of dynamic programming to address stochastic optimal control problems over the last few decades. There are several monographs dealing with this subject which provide an up to date overview of the subject, such as the textbooks by Fleming and Rishel [21], Fleming and Soner [22] and Yong and Zhou [43]. An alternative approach to optimal control, developed by Pontryagin and his group, started to be developed in the 1950s. Pontryagin started the investigation of optimal processes of regulation [41] in 1952. Boltyanski, Gamkrelidze and Pontryagin [13] formulated what is now known as the Pontryagin's maximum principle in 1956, with technical details given in 1961 [14]. In 1972, Kushner [29] applied variational methods to continuous parameter stochastic optimization problems in order to derive a maximum principle for linear programming. The maximum principle provides an alternative approach to the design of optimal controls, giving also necessary conditions for optimality. Several other developments are due to Haussmann [25] and Bismut [9–11]. More recently, a lot more work has been developed by Li's group, with their results collected in the excellent monograph by Yong and Zhou [43]. In this paper, we deal with a class of stochastic optimal control problems with multiple random time horizons. The state variable follows a stochastic differential equation driven by a standard multi-dimensional Brownian motion. We assume that the objective functional depends on a finite sequence of non-negative continuous random variables $\tau_1, \tau_2, \dots, \tau_K$ that are assumed to be independent and identically distributed. Moreover, we assume that these random variables are completely independent of the filtration generated by the Brownian motion. Let T be a finite deterministic time horizon. The

objective functional has two key components: a first one given by the sum of integrals of (not necessarily identical) Lagrangian functions L_i over random time intervals $[0, \min\{\tau_i, T\}]$, and a second component given by the sum of (eventually distinct) terminal costs Ψ_i evaluated at random instants of time $\min\{\tau_i, T\}$. More precisely, the objective functional to be optimized is the expected value of the sum of K terms, each one of the form

$$\int_0^{\xi_i} L_i(t, x(t), u(t)) dt + \Psi_i(\xi_i, x(\xi_i)) ,$$

where $x(\cdot)$ denotes the state variable, $u(\cdot)$ denotes the control variable and $\xi_i = \min\{\tau_i, T\}$. We use dynamic programming techniques to address the stochastic optimal control problem under consideration. However, before being able to formulate a dynamic programming principle, one needs to overcome two major difficulties related with the dependence of the objective functional on the finite sequence of random variables $\tau_1, \tau_2, \dots, \tau_K$ representing the multiple random time horizons of the optimal control problem under consideration. The first such difficulty has to do with the precise order for the values of the random variables τ_i , which is also of a stochastic nature. To deal with it, we use order statistics and consider all possible permutations for the order of the random variables $\tau_1, \tau_2, \dots, \tau_K$ at any given instant of time. A second difficulty is related with the comparison between the value of each one of the random variables τ_i and the fixed deterministic horizon T . We introduce a family of conditional probability functions representing the probability that a given order statistic has value smaller or equal than the fixed horizon T conditional on the knowledge that the

value of some preceding order statistic is larger than a given instant of time $t \in [0, T)$. Indeed, the influence of these conditional probability functions is clear from the distinctive nature of the dynamic programming principle stated here and the corresponding Hamilton-Jacobi-Belman equation. Yet another distinguishing feature of the problem under consideration is that its solution may be seen as the juxtaposition of the solutions of a finite number of different stochastic optimal control problems, all belonging to the same family. This is due to the nature of the objective functional under consideration in this paper and its dependency on the random variables $\tau_1, \dots, \tau_K$. Our motivation to address the problem described above comes from its apparent relevance for mathematical finance. See, for instance, the papers [12, 19, 40] where a stochastic optimal control problem with a random time horizon is considered in the context of mathematical finance. This paper ultimate aim is to study this class of problems from a general point of view, and to set up this framework for eventual applications to other relevant fields. This chapter is based on the research articles [12, 19, 40] and on the books [21, 22, 26, 43].

Chapter 2

Human decisions: a game theoretical model

We introduce a dichotomous decision model, where individuals can make the decision yes or no and can influence the decisions of others. We characterize all decisions that form Nash equilibria. Taking into account the way individuals influence the decisions of others, we construct the decision tilings where the axes reflect the personal preferences of the individuals for making the decision yes or no. These tilings characterize geometrically all the pure and mixed Nash equilibria. There are 289 combinatorial classes of decision tilings, described by the decision Bussola, which demonstrates the high complexity of making decision. We show, in these tilings, that Nash equilibria form degenerated hystereses with respect to the replicator dynamics, with the property that the pure Nash equilibria are asymptotically stable and the strict mixed equilibria are unstable. These hystereses can help to explain the sudden appearance of social, political and economic crises. We observe

the existence of limit cycles for the replicator dynamics associated to situations where the individuals keep changing their decisions along time, but exhibiting a periodic repetition in their decisions. We introduce the notion of altruist and individualist leaders and study the way that the leader can affect the individuals to make the decision that the leader pretends. We introduce a resort pricing model, where different types of tourists choose between different resorts. We study the influence of the resort prices on the choices of the different types of tourists. In particular, the dependence of the resort prices on the characteristics of the tourists. We characterize the cohesive strategies of the tourists that are Nash equilibria. We find the prices that lead to the bankruptcy of the resorts and, in particular, their dependence on the characteristics of the tourists. We introduce a more general framework for the decision model, where individuals will have to make decisions according to their preferences that also depend upon the other individuals' decisions. In this model, there are N types $t \in \{t_1, t_2, \dots, t_N\}$ of individuals and M possible decisions $d \in \{d_1, d_2, \dots, d_M\}$ that individuals can make. We fully characterize the space of all parameters where the Nash equilibria are cohesive, i.e. individuals with similar preferences make the same decisions. We prove that small changes in the parameters can imply the appearance or disappearance of cohesive Nash equilibria. These abrupt changes in the structure of the cohesive Nash equilibria can be used to characterize economical, political or social revolutions. In this chapter, we survey in part, the work presented in [32, 34, 37, 38].

2.1 Yes-no decision model

The *yes-no decision model* has two types $\mathbf{T} = \{t_1, t_2\}$ of individuals. Let $I_1 = \{1, \dots, n_1\}$ be the set of all individuals with type t_1 , and let $I_2 = \{1, \dots, n_2\}$ be the set of all individuals with type t_2 . Let $I = I_1 \sqcup I_2$. The individual $i \in \mathbf{I}$ has to make one decision $d \in \mathbf{D} = \{Y, N\}$ ¹. Let $\mathcal{L}$ be the *preference decision matrix* whose *coordinates* γ_p^d indicate how much an individual with type t_p likes or dislikes, to make decision d

$$\mathcal{L} = \begin{pmatrix} \gamma_1^Y & \gamma_1^N \\ \gamma_2^Y & \gamma_2^N \end{pmatrix}.$$

The preference decision matrix indicates for each type of individuals the decision that the individuals prefer, i.e. the taste type of the individuals (see [2, 15, 17, 36, 38]). Let $\mathcal{N}_d$ be the *preference neighbors matrix* whose *coordinates* β_{pq}^d indicate how much an individual with type t_p who decides d likes or dislikes that an individual with type t_q also makes decision d

$$\mathcal{N}_d = \begin{pmatrix} \beta_{11}^d & \beta_{12}^d \\ \beta_{21}^d & \beta_{22}^d \end{pmatrix}.$$

Let $\overline{\mathcal{N}}_d$ be the *preference non-neighbors matrix* whose *coordinates* $\overline{\beta}_{pq}^d$ indicate how much an individual with type t_p who decide d , likes or dislikes that an individual with type t_q makes decision $d' \neq d$

$$\overline{\mathcal{N}}_d = \begin{pmatrix} \overline{\beta}_{11}^d & \overline{\beta}_{12}^d \\ \overline{\beta}_{21}^d & \overline{\beta}_{22}^d \end{pmatrix}.$$

¹Similarly, we can consider that there is a single individual with type t_p that has to make n_p decisions, or we can, also, consider a mixed model using these two possibilities.

The preference neighbors and non-neighbors matrices indicate, for each type of individuals whose decision is d , whom they prefer, or do not prefer, to be with in each decision, i.e. the crowding type of the individuals (see [2, 15, 17, 36, 38]).

We describe the (pure) decision of the individuals by a (*pure*) *strategy map* $S : \mathbf{I} \rightarrow \mathbf{D}$ that associates to each individual $i \in \mathbf{I}$ its decision $S(i) \in \mathbf{D}$. Let $\mathbf{S}$ be the space of all strategies S . Given a strategy S , let $\mathcal{O}_S$ be the *strategic decision matrix* whose coordinates $l_p^d = l_p^d(S)$ indicate the number of individuals with type t_p , who make decision d

$$\mathcal{O}_S = \begin{pmatrix} l_1^Y & l_1^N \\ l_2^Y & l_2^N \end{pmatrix}.$$

The *strategic decision vector* associated to a strategy S is the vector $(l_1, l_2) = (l_1^Y(S), l_2^Y(S))$. Hence, l_1 (resp. $n_1 - l_1$) is the number of individuals with type t_1 who make the decision Y (resp. N). Similarly, l_2 (resp. $n_2 - l_2$) is the number of individuals with type t_2 who make the decision Y (resp. N).

The set $\mathbf{O}$ of all possible *strategic decision vectors* is

$$\mathbf{O} = \{0, \dots, n_1\} \times \{0, \dots, n_2\}.$$

Let:

- (i) $\omega_1^Y = \gamma_1^Y + \bar{\beta}_{11}^Y(n_1 - 1) + \bar{\beta}_{12}^Y n_2;$
- (ii) $\omega_1^N = \gamma_1^N + \bar{\beta}_{11}^N(n_1 - 1) + \bar{\beta}_{12}^N n_2;$
- (ii) $\omega_2^Y = \gamma_2^Y + \bar{\beta}_{22}^Y(n_2 - 1) + \bar{\beta}_{21}^Y n_1;$

$$(iv) \quad \omega_2^N = \gamma_2^N + \bar{\beta}_{22}^N(n_2 - 1) + \bar{\beta}_{21}^N n_1;$$

$$(v) \quad \alpha_{ij}^d = \beta_{ij}^d - \bar{\beta}_{ij}^d, \quad \text{for } i, j \in \{1, 2\} \quad \text{and} \quad d \in \{Y, N\}.$$

Let $U_1 : \mathbf{D} \times \mathbf{O} \rightarrow \mathbb{R}$ the *utility function* of an individual with type t_1 be given by

$$\begin{aligned} U_1(Y; l_1, l_2) &= \gamma_1^Y + \beta_{11}^Y(l_1 - 1) + \beta_{12}^Y l_2 + \bar{\beta}_{11}^Y(n_1 - l_1) + \bar{\beta}_{12}^Y(n_2 - l_2) \\ &= \omega_1^Y + \alpha_{11}^Y(l_1 - 1) + \alpha_{12}^Y l_2; \\ U_1(N; l_1, l_2) &= \gamma_1^N + \beta_{11}^N(n_1 - l_1 - 1) + \beta_{12}^N(n_2 - l_2) + \bar{\beta}_{11}^N l_1 + \bar{\beta}_{12}^N l_2 \\ &= \omega_1^N + \alpha_{11}^N(n_1 - l_1 - 1) + \alpha_{12}^N(n_2 - l_2). \end{aligned}$$

Let $U_2 : \mathbf{D} \times \mathbf{O} \rightarrow \mathbb{R}$ the *utility function* of an individual with type t_2 be given by

$$\begin{aligned} U_2(Y; l_1, l_2) &= \gamma_2^Y + \beta_{22}^Y(l_2 - 1) + \beta_{21}^Y l_1 + \bar{\beta}_{22}^Y(n_2 - l_2) + \bar{\beta}_{21}^Y(n_1 - l_1) \\ &= \omega_2^Y + \alpha_{22}^Y(l_2 - 1) + \alpha_{21}^Y l_1; \\ U_2(N; l_1, l_2) &= \gamma_2^N + \beta_{22}^N(n_2 - l_2 - 1) + \beta_{21}^N(n_1 - l_1) + \bar{\beta}_{22}^N l_2 + \bar{\beta}_{21}^N l_1 \\ &= \omega_2^N + \alpha_{22}^N(n_2 - l_2 - 1) + \alpha_{21}^N(n_1 - l_1). \end{aligned}$$

Given a strategy $S \in \mathbf{S}$, the *utility* $U_i(S)$ of an individual i with type $t_{p(i)}$ is given by $U_{p(i)}(S(i); l_1^Y(S), l_2^Y(S))$.

Definition 2.1.1 Let $x = \omega_1^Y - \omega_1^N$ be the horizontal relative decision preference of the individuals with type t_1 and let $y = \omega_2^Y - \omega_2^N$ be the vertical relative decision preference of the individuals with type t_2 .

If $x > 0$, the individuals with type t_1 prefer to decide Y , without taking into account the influence of the others. If $x = 0$, the individuals with type t_1 are indifferent to decide Y or N , without taking into account the influence of the others. If $x < 0$, the individuals with type t_1 prefer to decide N , without taking into account the influence of the others.

Definition 2.1.2 Let $A_{tt'} = \alpha_{tt'}^Y + \alpha_{tt'}^N$, for $t, t' \in \{1, 2\}$, be the coordinates of the influence matrix.

If $A_{tt'} > 0$, then individuals with type t' have a *positive influence* over the utility function of the individuals with type t . If $A_{tt'} = 0$, then individuals with type t' are *indifferent* for the utility function of the individuals with type t . If $A_{tt'} < 0$, then individuals with type t' have a *negative influence* over the utility function of the individuals with type t .

Definition 2.1.3 A strategy $S^* : \mathbf{I} \rightarrow \mathbf{D}$ is a (pure) Nash equilibrium if

$$U_i(S^*) \geq U_i(S)$$

for every individual $i \in \mathbf{I}$ and for every strategy $S \in \mathbf{S}$, with the property that $S^*(j) = S(j)$ for every individual $j \in \mathbf{I} \setminus \{i\}$. The (pure) Nash domain $\mathbf{N}(l_1, l_2)$ is the set of all pairs (x, y) for which the strategies with strategic decision vector (l_1, l_2) are Nash equilibria.

We will show that the relative decision preferences and the coordinates of the influence matrix together with the total number of individuals of each type encode all the relevant information for characterizing all the pure and mixed Nash equilibria domains.

2.2 Evolutionary dynamics and yes-no decision models

We implement the *evolutionary deterministic yes-no decision models* as follows: Fix an infinite sequence (i_t, d_t) , with $t \in \mathbb{N}$, of pairs $(i_t, d_t) \in \mathbf{I} \times \mathbf{D}$ with the property that every pair, contained in $\mathbf{I} \times \mathbf{D}$, occurs in the sequence infinitely often. Given a strategy $S_t : \mathbf{I} \rightarrow \mathbf{D}$, at moment t , the strategy $S_{t+1} : \mathbf{I} \rightarrow \mathbf{D}$ is defined as follows: (i) $S_{t+1} = S_t|_{\mathbf{I} \setminus \{i_{t+1}\}}$; (ii) $S_{t+1}(i_{t+1}) = d_{t+1}$, if i_{t+1} increases its utility by making decision d_{t+1} instead of $S_t(i_{t+1})$ (knowing that $S_{t+1} = S_t|_{\mathbf{I} \setminus \{i_{t+1}\}}$), and $S_{t+1}(i) = S_t(i)$, otherwise. Hence, the Nash equilibria are the fixed points, and vice-versa, of the evolutionary decision deterministic models.

We implement the *evolutionary stochastic yes-no decision models* as follows: Let P be a probability distribution that assigns a positive probability to each pair $(i, d) \in \mathbf{I} \times \mathbf{D}$. Given a strategy $S_t : \mathbf{I} \rightarrow \mathbf{D}$, at moment t , we choose randomly a pair (i, d) according to the probability distribution P . The strategy $S_{t+1} : \mathbf{I} \rightarrow \mathbf{D}$ is defined as follows: (i) $S_{t+1} = S_t|_{\mathbf{I} \setminus \{i\}}$; (ii) $S_{t+1}(i) = d$, if i increases its utility by deciding d instead of $S_t(i)$ (knowing that $S_{t+1} = S_t|_{\mathbf{I} \setminus \{i\}}$), and $S_{t+1}(i) = S_t(i)$, otherwise. Hence, the Nash equilibria are the absorbing states, and vice-versa, of the evolutionary decision stochastic model.

2.3 Pure Nash equilibria

In this section we study the pure Nash equilibria domain. The pure strategies are either cohesive or disparate.

2.3.1 Cohesive Nash equilibria

The goal of this subsection is to study the cohesive Nash equilibria domain.

Definition 2.3.1 *A cohesive strategy² is a strategy in which all individuals with the same type prefer to make the same decision.*

A cohesive strategy is described by a map $C : \mathbf{T} \rightarrow \mathbf{D}$ that indicates for every individual i with type $t_{p(i)}$, its decision $C(p(i))$. Hence, a cohesive strategy $C : \mathbf{T} \rightarrow \mathbf{D}$ determines a unique strategy $S : \mathbf{I} \rightarrow \mathbf{D}$ given by $S(i) = C(p(i))$.

We observe that there are four cohesive strategies:

- (i) (Y, Y) *strategy*: all individuals make the decision Y , i.e. the strategic decision vector is (n_1, n_2) ;
- (ii) (Y, N) *strategy*: all individuals, with type t_1 , make the decision Y , and all individuals, with type t_2 , make the decision N , i.e. the strategic decision vector is $(n_1, 0)$;
- (iii) (N, Y) *strategy*: all individuals, with type t_1 , make the decision N and all individuals, with type t_2 , make the decision Y , i.e. the strategic decision vector is $(0, n_2)$;

²or equivalently, *no-split strategy* or *herd strategy*

- (iv) (N, N) *strategy*: all individuals make the decision N , i.e. the strategic decision vector is $(0, 0)$.

The *horizontal* $H(Y, Y)$ and the *vertical* $V(Y, Y)$ *strategic thresholds* of the (Y, Y) strategy are given by

$$H(Y, Y) = -\alpha_{11}^Y(n_1 - 1) - \alpha_{12}^Y n_2 \quad \text{and} \quad V(Y, Y) = -\alpha_{22}^Y(n_2 - 1) - \alpha_{21}^Y n_1 .$$

The *Nash domain* of the cohesive strategy (Y, Y) is the right-upper quadrant given by

$$\mathbf{N}(Y, Y) = \{(x, y) \in \mathbb{R}^2 : x \geq H(Y, Y) \text{ and } y \geq V(Y, Y)\} .$$

Hence, the cohesive strategy (Y, Y) is a *Nash equilibrium* if, and only if, $(x, y) \in \mathbf{N}(Y, Y)$ (see Figure 2.1).

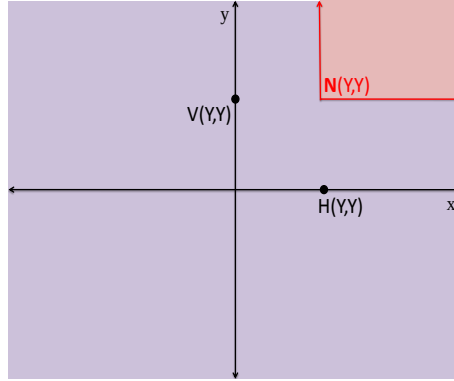


Figure 2.1: Cohesive Nash equilibria domain $\mathbf{N}(Y, Y)$.

The *horizontal* $H(Y, N)$ and the *vertical* $V(Y, N)$ *strategic thresholds* of the (Y, N) strategy are given by

$$H(Y, N) = -\alpha_{11}^Y(n_1 - 1) + \alpha_{12}^N n_2 \quad \text{and} \quad V(Y, N) = \alpha_{22}^N(n_2 - 1) - \alpha_{21}^Y n_1 .$$

The *Nash domain* of the cohesive strategy (Y, N) is the right-lower quadrant given by

$$\mathbf{N}(Y, N) = \{(x, y) \in \mathbb{R}^2 : x \geq H(Y, N) \text{ and } y \leq V(Y, N)\} .$$

Hence, the cohesive strategy (Y, N) is a *Nash equilibrium* if, and only if, $(x, y) \in \mathbf{N}(Y, N)$ (see Figure 2.2).

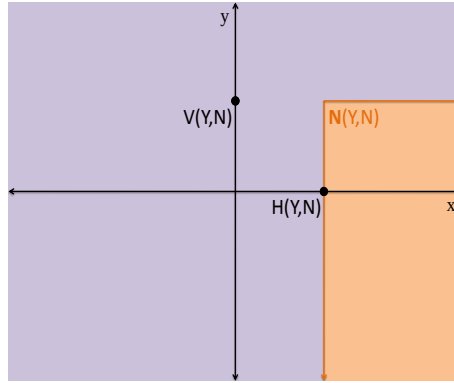


Figure 2.2: Cohesive Nash equilibria domain $\mathbf{N}(Y, N)$.

The *horizontal* $H(N, Y)$ and the *vertical* $V(N, Y)$ *strategic thresholds* of the (N, Y) strategy are given by

$$H(N, Y) = \alpha_{11}^N(n_1 - 1) - \alpha_{12}^Y n_2 \quad \text{and} \quad V(N, Y) = -\alpha_{22}^Y(n_2 - 1) + \alpha_{21}^N n_1 .$$

The *Nash domain* of the cohesive strategy (N, Y) is the left-upper quadrant given by

$$\mathbf{N}(N, Y) = \{(x, y) \in \mathbb{R}^2 : x \leq H(N, Y) \text{ and } y \geq V(N, Y)\} .$$

Hence, the cohesive strategy (N, Y) is a *Nash equilibrium* if, and only if, $(x, y) \in \mathbf{N}(N, Y)$ (see Figure 2.3).

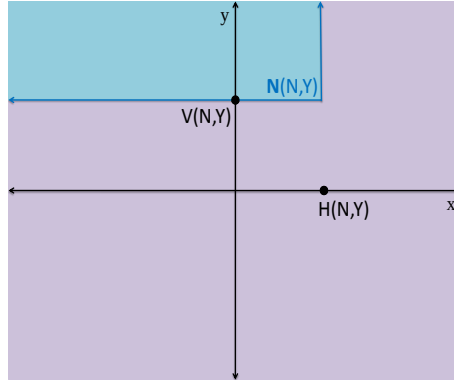


Figure 2.3: Cohesive Nash equilibria domain $\mathbf{N}(N, Y)$.

The *horizontal* $H(N, N)$ and the *vertical* $V(N, N)$ strategic thresholds of the (N, N) strategy are given by

$$H(N, N) = \alpha_{11}^N(n_1 - 1) + \alpha_{12}^N n_2 \text{ and } V(N, N) = \alpha_{22}^N(n_2 - 1) + \alpha_{21}^N n_1 .$$

The *Nash domain* of the cohesive strategy (N, N) is the left-lower quadrant given by

$$\mathbf{N}(N, N) = \{(x, y) \in \mathbb{R}^2 : x \leq H(N, N) \text{ and } y \leq V(N, N)\} .$$

Hence, the cohesive strategy (N, N) is a *Nash equilibrium* if, and only if, $(x, y) \in \mathbf{N}(N, N)$ (see Figure 2.4).

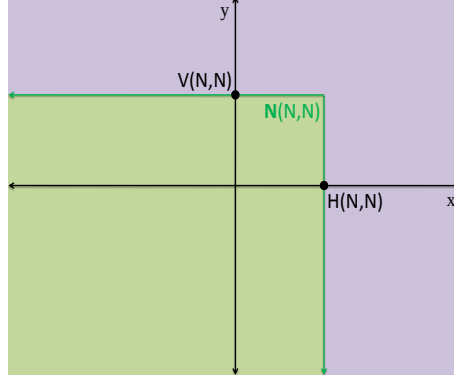


Figure 2.4: Cohesive Nash equilibria domain $\mathbf{N}(N, N)$.

The coordinates of the influence matrix can be written as

$$\begin{aligned}
 A_{11} &= \frac{H(N, Y) - H(Y, Y)}{n_1 - 1} = \frac{H(N, N) - H(Y, N)}{n_1 - 1}, \\
 A_{12} &= \frac{H(Y, N) - H(Y, Y)}{n_2} = \frac{H(N, N) - H(N, Y)}{n_2}, \\
 A_{21} &= \frac{V(N, Y) - V(Y, Y)}{n_1} = \frac{V(N, N) - V(Y, N)}{n_1}, \\
 A_{22} &= \frac{V(Y, N) - V(Y, Y)}{n_2 - 1} = \frac{V(N, N) - V(N, Y)}{n_2 - 1}.
 \end{aligned} \tag{2.1}$$

The representations of the Nash domains $\mathbf{N}(Y, Y)$, $\mathbf{N}(Y, N)$, $\mathbf{N}(N, Y)$, and $\mathbf{N}(N, N)$ in the plan (x, y) determine the *decision tilings*. Let $U(Y, Y) \subseteq \mathbf{N}(Y, Y)$, $U(Y, N) \subseteq \mathbf{N}(Y, N)$, $U(N, Y) \subseteq \mathbf{N}(N, Y)$ and $U(N, N) \subseteq \mathbf{N}(N, N)$ be the regions with unique Nash equilibrium.

In Figure 2.5, we present three decisions tilings where

- (i) regions with cohesive uniqueness Nash equilibria domains $U(Y, Y)$, $U(Y, N)$, $U(N, Y)$, and $U(N, N)$ colored red, orange, blue and green, respectively;
- (ii) regions without cohesive Nash equilibrium colored purple; and
- (iii) regions with two, three and four cohesive Nash equilibria colored yellow, brown and pink, respectively.

In Figure 2.5a, there is an unbounded region without cohesive Nash equilibrium. In Figure 2.5b, for every relative decision preferences, there is a unique cohesive Nash equilibrium, except along the axis, where there are two cohesive Nash equilibria, and at the origin, where there are four cohesive Nash equilibria. In Figure 2.5c, there are regions with one, two, three and four cohesive Nash equilibria.

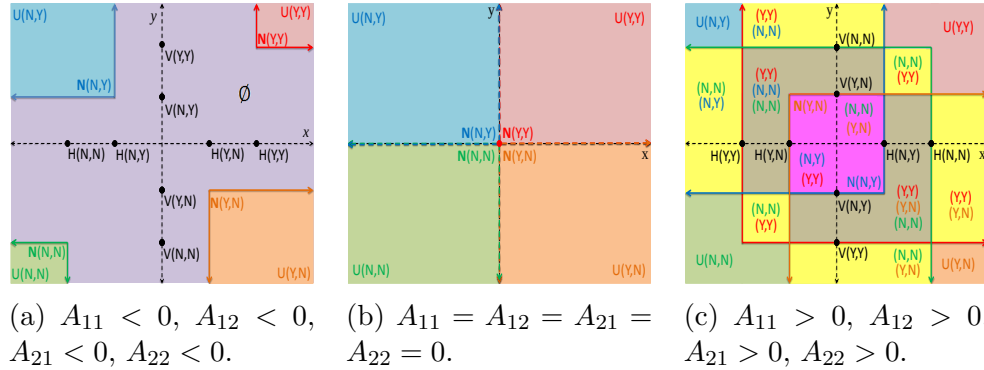


Figure 2.5: Three examples of decision tilings.

2.3.2 Disparate Nash equilibria

The goal of this subsection is to study the disparate Nash equilibria domain.

Definition 2.3.2 *A disparate strategy³ is a pure strategy that is not cohesive, i.e. a pure strategy in which there are some individuals with the same type who prefer to make different decisions.*

As in [38], we construct the Nash domains $\mathbf{N}(S)$ for the disparate strategies. We observe that the disparate strategy can have the following forms:

- (i) *(Disparate, cohesive) strategy*: All individuals with type t_2 make the same decision but individuals with type t_1 make different decisions, i.e. the strategic decision vector is (l_1, n_2) , where $l_1 \notin \{0, n_1\}$;
- (ii) *(Cohesive, disparate) strategy*: All individuals with type t_1 make the same decision but individuals with type t_2 make different decisions, i.e. the strategic decision vector is (n_1, l_2) , where $l_2 \notin \{0, n_2\}$;
- (iii) *(Disparate, disparate) strategy*: Individuals of both types make different decisions, i.e. the strategic decision vector is (l_1, l_2) , where $l_1 \notin \{0, n_1\}$ and $l_2 \notin \{0, n_2\}$.

(Disparate, cohesive) strategies

The (disparate, cohesive) strategies have two forms:

- (i) (l_1, Y) *strategy*: all the individuals with type t_2 and l_1 individuals with type t_1 make decision Y and $n_1 - l_1$ individuals with type t_1

³or equivalently, *split strategy*

make decision N , i.e. the strategic decision vector is (l_1, n_2) , where $l_1 \in \{1, 2, \dots, n_1 - 1\}$;

- (ii) (l_1, N) *strategy*: all the individuals with type t_2 and $n_1 - l_1$ individuals with type t_1 make decision N and l_1 individuals with type t_1 make decision Y , i.e. the strategic decision vector is $(l_1, 0)$, where $l_1 \in \{1, 2, \dots, n_1 - 1\}$.

We define the *left horizontal threshold* $H_L(l_1, Y)$ and the *right horizontal threshold* $H_R(l_1, Y)$ of the (disparate, cohesive) strategy (l_1, Y) by

$$\begin{aligned} H_L(l_1, Y) &= -\alpha_{11}^Y(l_1 - 1) - \alpha_{12}^Y n_2 + \alpha_{11}^N(n_1 - l_1) \\ H_R(l_1, Y) &= -\alpha_{11}^Y l_1 - \alpha_{12}^Y n_2 + \alpha_{11}^N(n_1 - l_1 - 1) . \end{aligned}$$

We define the *vertical threshold* $V(l_1, Y)$ of the (disparate, cohesive) strategy (l_1, Y) by

$$V(l_1, Y) = -\alpha_{21}^Y l_1 + \alpha_{21}^N(n_1 - l_1) - \alpha_{22}^Y(n_2 - 1) .$$

The (disparate, cohesive) *Nash domain* $\mathbf{N}(l_1, Y)$ is given by

$$\mathbf{N}(l_1, Y) = \{(x, y) \in \mathbb{R}^2 : H_L(l_1, Y) \leq x \leq H_R(l_1, Y) \text{ and } y \geq V(l_1, Y)\} .$$

Hence, the (disparate, cohesive) strategy (l_1, Y) is a *Nash equilibrium* if, and only if, $(x, y) \in \mathbf{N}(l_1, Y)$ (see Figure 2.6). Note that the yellow rectangles in Figures 2.6, 2.7, 2.8 and 2.9 represents the region where a mixed strategies occur (we study the mixed strategies in section 2.4).

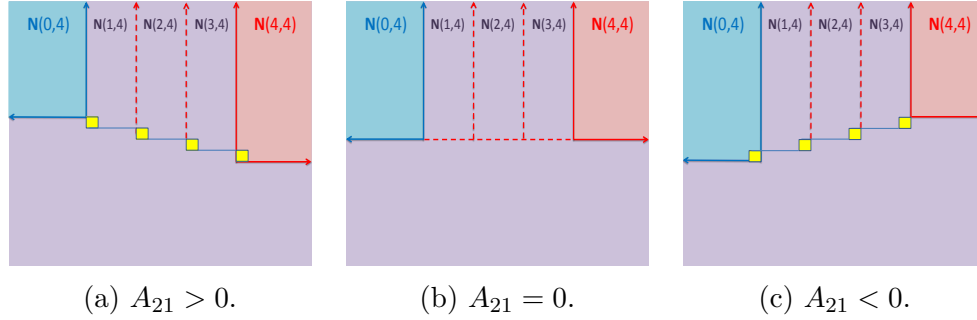


Figure 2.6: (Disparate, cohesive) Nash equilibria domain $\mathbf{N}(l_1, Y)$ when $A_{11} < 0$ and $n_1 = n_2 = 4$.

We define the *left horizontal threshold* $H_L(l_1, N)$ and the *right horizontal threshold* $H_R(l_1, N)$ of the (disparate, cohesive) strategy (l_1, N) by

$$\begin{aligned} H_L(l_1, N) &= -\alpha_{11}^Y(l_1 - 1) + \alpha_{12}^N n_2 + \alpha_{11}^N(n_1 - l_1) \\ H_R(l_1, N) &= -\alpha_{11}^Y l_1 + \alpha_{12}^N n_2 + \alpha_{11}^N(n_1 - l_1 - 1) . \end{aligned}$$

We define the *vertical threshold* $V(l_1, N)$ of the (disparate, cohesive) strategy (l_1, N) by

$$V(l_1, N) = -\alpha_{21}^Y l_1 + \alpha_{21}^N(n_1 - l_1) + \alpha_{22}^N(n_2 - 1) .$$

The (disparate, cohesive) *Nash domain* $\mathbf{N}(l_1, N)$ is given by

$$\mathbf{N}(l_1, N) = \{(x, y) \in \mathbb{R}^2 : H_L(l_1, N) \leq x \leq H_R(l_1, N) \text{ and } y \leq V(l_1, N)\} .$$

Hence, the (disparate, cohesive) strategy (l_1, N) is a *Nash equilibrium* if, and only if, $(x, y) \in \mathbf{N}(l_1, N)$ (see Figure 2.7).

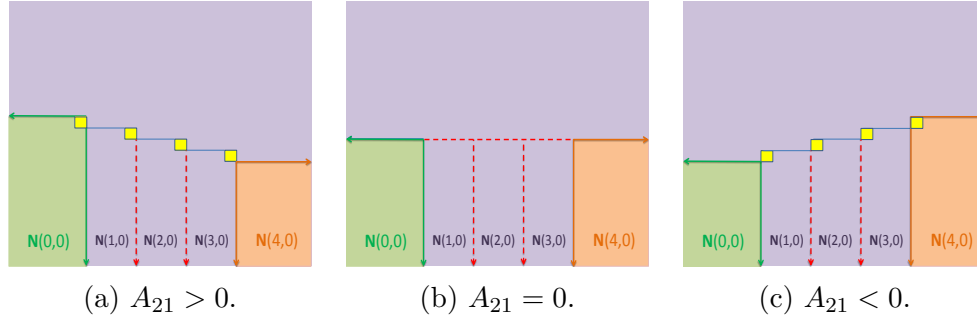


Figure 2.7: (Disparate, cohesive) Nash equilibria domain $\mathbf{N}(l_1, N)$ when $A_{11} < 0$ and $n_1 = n_2 = 4$.

(Cohesive, disparate) strategies

The (cohesive, disparate) strategies have two forms:

- (i) (Y, l_2) strategy: all the individuals with type t_1 and l_2 individuals with type t_2 make decision Y and $n_2 - l_2$ individuals with type t_2 make decision N , i.e. the strategic decision vector is (n_1, l_2) , where $l_2 \in \{1, 2, \dots, n_2 - 1\}$;
- (ii) (N, l_2) strategy: all the individuals with type t_1 and $n_2 - l_2$ individuals with type t_2 make decision N and l_2 individuals with type t_2 make decision Y , i.e. the strategic decision vector is $(0, l_2)$, where $l_2 \in \{1, 2, \dots, n_2 - 1\}$.

We define the *lower vertical threshold* $V_L(Y, l_2)$ and the *upper vertical threshold* $V_U(Y, l_2)$ of the (cohesive, disparate) (Y, l_2) strategy by

$$\begin{aligned}
 V_L(Y, l_2) &= -\alpha_{22}^Y(l_2 - 1) - \alpha_{21}^Y n_1 + \alpha_{22}^N(n_2 - l_2) \\
 V_U(Y, l_2) &= -\alpha_{22}^Y l_2 - \alpha_{21}^Y n_1 + \alpha_{22}^N(n_2 - l_2 - 1) .
 \end{aligned}$$

We define the *horizontal threshold* $H(Y, l_2)$ of the (cohesive, disparate) (Y, l_2) strategy by

$$H(Y, l_2) = -\alpha_{12}^Y l_2 + \alpha_{12}^N (n_2 - l_2) - \alpha_{11}^Y (n_1 - 1) .$$

The (cohesive, disparate) *Nash domain* $\mathbf{N}(Y, l_2)$ is given by

$$\mathbf{N}(Y, l_2) = \{(x, y) \in \mathbb{R}^2 : V_L(Y, l_2) \leq y \leq V_U(Y, l_2) \text{ and } x \geq H(Y, l_2)\} .$$

Hence, the (cohesive, disparate) strategy (Y, l_2) is a *Nash equilibrium* if, and only if, $(x, y) \in \mathbf{N}(Y, l_2)$ (see Figure 2.8).

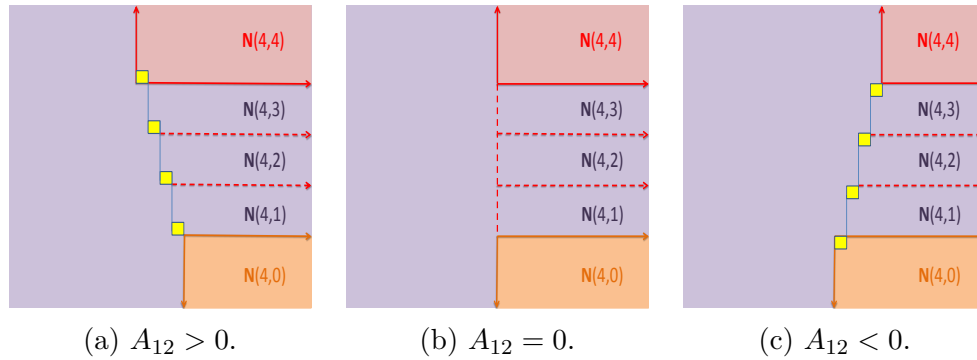


Figure 2.8: (Cohesive, disparate) Nash equilibria domain $\mathbf{N}(Y, l_2)$ when $A_{22} < 0$ and $n_1 = n_2 = 4$.

We define the *lower vertical threshold* $V_L(N, l_2)$ and the *upper vertical threshold* $V_U(N, l_2)$ of the (cohesive, disparate) strategy (N, l_2) by

$$\begin{aligned} V_L(N, l_2) &= -\alpha_{22}^Y (l_2 - 1) + \alpha_{21}^N n_1 + \alpha_{22}^N (n_2 - l_2) \\ V_U(N, l_2) &= -\alpha_{22}^Y l_2 + \alpha_{21}^N n_1 + \alpha_{22}^N (n_2 - l_2 - 1) . \end{aligned}$$

We define the *horizontal threshold* $H(N, l_2)$ of the (cohesive, disparate) strategy (N, l_2) by

$$H(N, l_2) = -\alpha_{12}^Y l_2 + \alpha_{12}^N (n_2 - l_2) + \alpha_{11}^N (n_1 - 1) .$$

The (cohesive, disparate) *Nash domain* $\mathbf{N}(N, l_2)$ is given by

$$\mathbf{N}(N, l_2) = \{(x, y) \in \mathbb{R}^2 : V_L(N, l_2) \leq y \leq V_U(N, l_2) \text{ and } x \leq H(N, l_2)\} .$$

Hence, the (cohesive, disparate) strategy (N, l_2) is a *Nash equilibrium* if, and only if, $(x, y) \in \mathbf{N}(N, l_2)$ (see Figure 2.9).

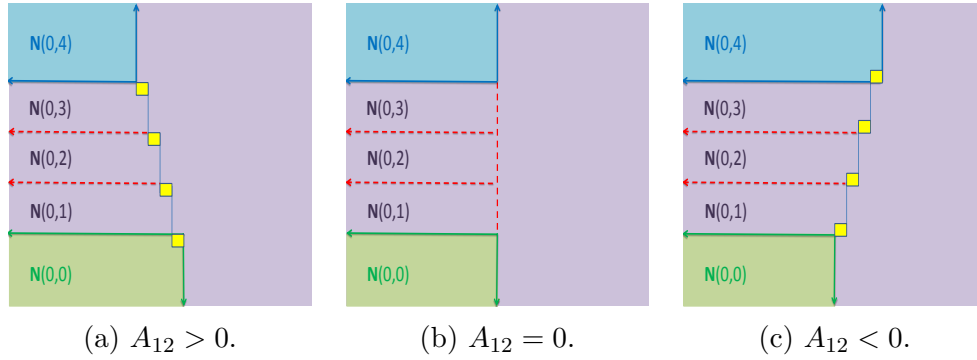


Figure 2.9: (Cohesive, disparate) Nash equilibria domain $\mathbf{N}(N, l_2)$ when $A_{22} < 0$ and $n_1 = n_2 = 4$.

(Disparate, disparate) strategies

The (disparate, disparate) strategies⁴ (see Figure 2.10) are of the form (l_1, l_2) *strategy*: l_1 individuals with type t_1 and l_2 individuals with type t_2 make

⁴The number of (disparate, disparate) strategies is $(n_1 - 1)(n_2 - 1)$.

decision Y , and $n_1 - l_1$ individuals with type t_1 and $n_2 - l_2$ individuals with type t_2 make decision N , where $l_1 \in \{1, \dots, n_1 - 1\}$ and $l_2 \in \{1, \dots, n_2 - 1\}$.

We define the *left horizontal threshold* $H_L(l_1, l_2)$ and the *right horizontal threshold* $H_R(l_1, l_2)$ of the (disparate, disparate) (l_1, l_2) by

$$\begin{aligned} H_L(l_1, l_2) &= \alpha_{11}^N n_1 + \alpha_{12}^N n_2 + \alpha_{11}^Y - (\alpha_{12}^Y + \alpha_{12}^N) l_2 - (\alpha_{11}^Y + \alpha_{11}^N) l_1 \\ H_R(l_1, l_2) &= \alpha_{11}^N n_1 + \alpha_{12}^N n_2 - \alpha_{11}^N - (\alpha_{12}^Y + \alpha_{12}^N) l_2 - (\alpha_{11}^Y + \alpha_{11}^N) l_1 . \end{aligned}$$

We define the *down vertical threshold* $V_D(l_1, l_2)$ and the *up vertical threshold* $V_U(l_1, l_2)$ of the (disparate, disparate) (l_1, l_2) by

$$\begin{aligned} V_D(l_1, l_2) &= \alpha_{22}^N n_2 + \alpha_{21}^N n_1 + \alpha_{22}^Y - (\alpha_{21}^Y + \alpha_{21}^N) l_1 - (\alpha_{22}^Y + \alpha_{22}^N) l_2 \\ V_U(l_1, l_2) &= \alpha_{22}^N n_2 + \alpha_{21}^N n_1 - \alpha_{22}^N - (\alpha_{21}^Y + \alpha_{21}^N) l_1 - (\alpha_{22}^Y + \alpha_{22}^N) l_2 . \end{aligned}$$

The (disparate, disparate) *Nash domain* $\mathbf{N}(l_1, l_2)$ is given by

$$\mathbf{N}(l_1, l_2) = \{(x, y) \in \mathbb{R}^2 : x \in I(l_1, l_2) \text{ and } y \in J(l_1, l_2)\} ,$$

where $I(l_1, l_2) = [H_L(l_1, l_2), H_R(l_1, l_2)]$ and $J(l_1, l_2) = [V_D(l_1, l_2), V_U(l_1, l_2)]$. Hence, the (disparate, disparate) strategy (l_1, l_2) is a *Nash equilibrium* if, and only if, $(x, y) \in \mathbf{N}(l_1, l_2)$ (see Figure 2.10).

In Figures 2.10a and 2.10b, we observe that for every pair of preference (x, y) belongs to the yellow rectangles, there are two pure strategies that are Nash equilibria. Furthermore, there are coordinates x and y for which a mixed strategy is a Nash equilibrium (we study the mixed strategies in

section 2.4). However, in Figures 2.10c and 2.10d, we observe that for every pair of preference (x, y) belongs to the yellow rectangles, there are no pure strategies that are Nash equilibria, but there are coordinates x and y for which a strictly mixed strategy is a Nash equilibrium.

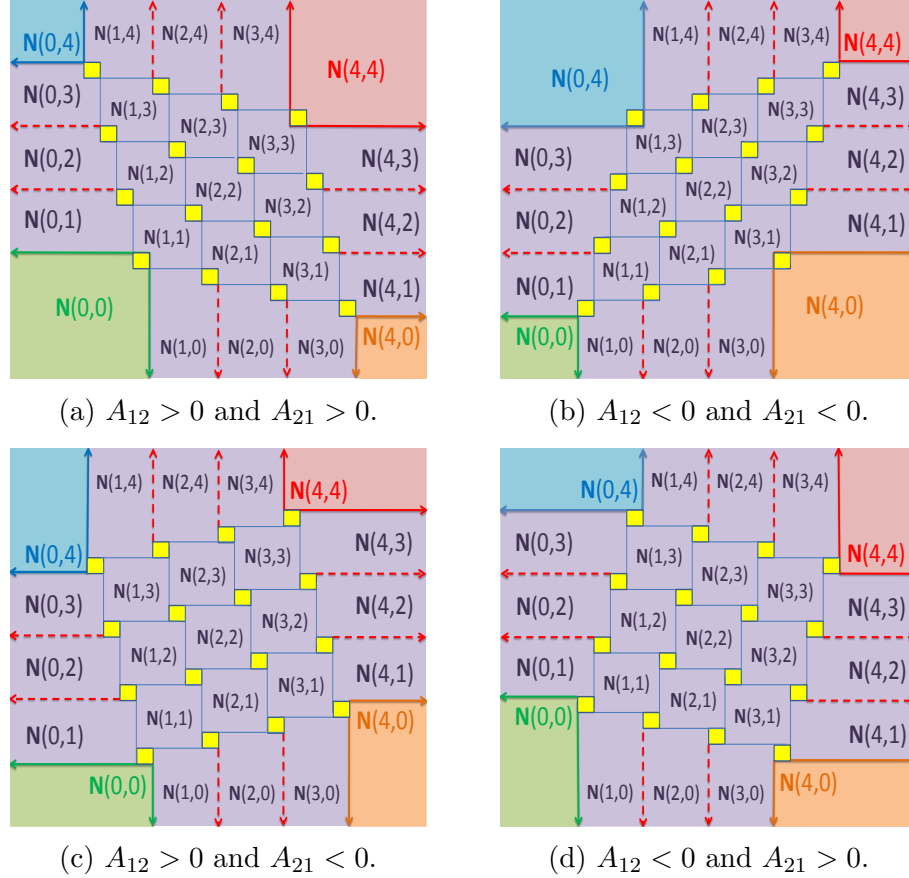


Figure 2.10: (Disparate, disparate) Nash equilibria domain $\mathbf{N}(l_1, l_2)$ when $A_{11} < 0$, $A_{22} < 0$ and $n_1 = n_2 = 4$.

Note that the thresholds that determine the pure Nash domain $\mathbf{N}(l_1, l_2)$ have the following properties: $H_R(l_1, l_2) = H_L(l_1, l_2) - A_{11}$, $V_U(l_1, l_2) = V_D(l_1, l_2) - A_{22}$, $H_L(l_1, l_2) = H_L(l_1, l_2 + 1) + A_{12}$, $H_R(l_1, l_2) = H_R(l_1, l_2 +$

$1) + A_{12}$, $V_U(l_1, l_2) = V_U(l_1 + 1, l_2) + A_{21}$, $V_D(l_1, l_2) = V_D(l_1 + 1, l_2) + A_{21}$,
 $H_R(l_1, l_2) = H_L(l_1 + 1, l_2)$ and $V_U(l_1, l_2) = V_D(l_1, l_2 + 1)$.

2.3.3 The geometry of the pure Nash equilibria

An (l_1, l_2) *strategic set* is the set of all pure strategies $S \in \mathbf{S}$ with $l_1(S) = l_1$ and $l_2(S) = l_2$; an (l_1, l_2) *cohesive strategic set* is an (l_1, l_2) strategic set with $l_1 \in \{0, n_1\}$ and $l_2 \in \{0, n_2\}$; an (l_1, l_2) *disparate strategic set* is an (l_1, l_2) strategic set that is not cohesive. We observe that a cohesive strategic set has a single strategy and a disparate strategic set has more than one strategy.

Since individuals with the same type are identical, a strategy to be a Nash equilibrium depends only upon the number of individuals of each type that decide either Y or N , and not upon the individual who is making the decision.

The (l_1, l_2) pure Nash equilibrium set is *cohesive* if $l_1 \in \{0, n_1\}$ and $l_2 \in \{0, n_2\}$. The (l_1, l_2) pure Nash equilibrium set is *disparate* if $l_1 \notin \{0, n_1\}$ or $l_2 \notin \{0, n_2\}$.

Remark 2.3.1 *Let (l_1, l_2) be a Nash equilibrium.*

(i) *If $A_{11} > 0$, then $l_1 \in \{0, n_1\}$.*

(ii) *If $A_{22} > 0$, then $l_2 \in \{0, n_2\}$.*

Furthermore, if $A_{11} > 0$ and $A_{22} > 0$, then (l_1, l_2) is cohesive.

Hence, if the individuals with a given type have a positive influence over the utility of the individuals with the same type, i.e. $A_{11} > 0$ and $A_{22} > 0$, then there are no disparate Nash equilibria.

Before proceeding to define the cohesive and disparate vectors, let $C \in \mathbb{R}^2$ be the *corner point* defined by

$$C = (H(N, N), V(N, N)) . \quad (2.2)$$

The *cohesive horizontal vector* $\vec{H}$ is defined by

$$\vec{H} = (H(Y, N), V(Y, N)) - C = -(A_{11}(n_1 - 1), A_{21}n_1) . \quad (2.3)$$

The *cohesive vertical vector* $\vec{V}$ is defined by

$$\vec{V} = (H(N, Y), V(N, Y)) - C = -(A_{12}n_2, A_{22}(n_2 - 1)) . \quad (2.4)$$

The *disparate vector* $\vec{Z}(l_1, l_2)$ is defined by

$$\begin{aligned} \vec{Z}(l_1, l_2) &= -l_1(A_{11}, A_{21}) - l_2(A_{12}, A_{22}) \\ &= \frac{l_1}{n_1}(\vec{H} - (A_{11}, 0)) + \frac{l_2}{n_2}(\vec{V} - (0, A_{22})) \\ &= \frac{l_1}{n_1 - 1}(\vec{H} + (0, A_{21})) + \frac{l_2}{n_2 - 1}(\vec{V} + (A_{12}, 0)) . \end{aligned} \quad (2.5)$$

We will now state the following remark obtaining the geometry of the pure Nash equilibria.

Remark 2.3.2 Let $l_1 \in \{1, 2, \dots, n_1 - 1\}$ and $l_2 \in \{1, 2, \dots, n_2 - 1\}$.

(i) If $A_{11} \leq 0$, then the disparate Nash domain $\mathbf{N}(l_1, 0)$ is given by

$$\mathbf{N}(l_1, 0) = \{C + \vec{Z}(l_1, 0) + (pA_{11}, q) : p \in [0, 1], q \in (-\infty, 0]\}$$

and the disparate Nash domain $\mathbf{N}(l_1, n_2)$ is given by

$$\mathbf{N}(l_1, n_2) = \{C + \vec{Z}(l_1, n_2 - 1) + (pA_{11}, q) : p \in [0, 1], q \in [0, +\infty)\} .$$

(ii) If $A_{22} \leq 0$, then the disparate Nash domain $\mathbf{N}(0, l_2)$ is given by

$$\mathbf{N}(0, l_2) = \{C + \vec{Z}(0, l_2) + (p, qA_{22}) : q \in [0, 1], p \in (-\infty, 0]\}$$

and the disparate Nash domain $\mathbf{N}(n_1, l_2)$ is given by

$$\mathbf{N}(n_1, l_2) = \{C + \vec{Z}(n_1 - 1, l_2) + (p, qA_{22}) : q \in [0, 1], p \in [0, +\infty)\} .$$

(iii) If $A_{11} \leq 0$ and $A_{22} \leq 0$, then the disparate Nash domain $\mathbf{N}(l_1, l_2)$ is

$$\mathbf{N}(l_1, l_2) = \{C + \vec{Z}(l_1, l_2) + (pA_{11}, qA_{22}) : p, q \in [0, 1]\} .$$

Hence, if the individuals with a given type have a non-positive influence over the utility of the individuals with the same type, i.e. $A_{11} \leq 0$ and $A_{22} \leq 0$, then for every (l_1, l_2) disparate strategic set there are relative preferences for which (l_1, l_2) is a Nash equilibrium set.

2.4 Mixed Nash equilibria

We will show that the relative decision preferences and the influence matrix together with the total number of individuals of each type encode all the relevant information for characterizing all the mixed Nash equilibria.

Recall that $I = I_1 \sqcup I_2$. We describe the (mixed) decision of the individuals by a (*mixed*) *strategy map* $S : \mathbf{I} \rightarrow [0, 1]$ that associates to each individual $i \in \mathbf{I}_1$ the probability $p_i = S(i)$ to decide $Y \in \mathbf{D}$ and to each individual $j \in \mathbf{I}_2$ the probability $q_j = S(j)$ to decide $Y \in \mathbf{D}$. Hence, each individual $i \in \mathbf{I}_1$ decides $N \in \mathbf{D}$ with probability $1 - p_i = 1 - S(i)$ and each individual $j \in \mathbf{I}_2$ decides $N \in \mathbf{D}$ with probability $1 - q_j = 1 - S(j)$. We assume that the decisions of the individuals are independent.

Define $P = \sum_{i=1}^{n_1} p_i$, $Q = \sum_{j=1}^{n_2} q_j$, $P_i = P - p_i$ and $Q_j = Q - q_j$. For every individual $i \in \mathbf{I}_1$, the *Y-fitness function* $f_{Y,1} : [0, 1] \times [0, n_1] \times [0, n_2] \rightarrow \mathbb{R}^+$ is given by

$$f_{Y,1}(p_i; P, Q) = \omega_1^Y + \alpha_{11}^Y P_i + \alpha_{12}^Y Q ;$$

and the *N-fitness function* $f_{N,1} : [0, 1] \times [0, n_1] \times [0, n_2] \rightarrow \mathbb{R}^+$ is given by

$$f_{N,1}(p_i; P, Q) = \omega_1^N + \alpha_{11}^N (n_1 - 1 - P_i) + \alpha_{12}^N (n_2 - Q) .$$

For every individual $j \in \mathbf{I}_2$, the *Y-fitness function* $f_{Y,2} : [0, 1] \times [0, n_1] \times [0, n_2] \rightarrow \mathbb{R}^+$ is given by

$$f_{Y,2}(q_j; P, Q) = \omega_2^Y + \alpha_{22}^Y Q_j + \alpha_{21}^Y P ;$$

and the N -fitness function $f_{N,2} : [0, 1] \times [0, n_1] \times [0, n_2] \rightarrow \mathbb{R}^+$ is given by

$$f_{N,2}(q_j; P, Q) = \omega_2^N + \alpha_{22}^N(n_2 - 1 - Q_j) + \alpha_{21}^N(n_1 - P) .$$

Lemma 2.4.1 *Let $S : \mathbf{I} \rightarrow [0, 1]$ be a mixed strategy. For every individual $i \in \mathbf{I}_1$, the utility function $U_1 : [0, 1] \times [0, n_1] \times [0, n_2] \rightarrow \mathbb{R}^+$ is given by*

$$U_1(p_i; P, Q) = p_i f_{Y,1}(p_i; P, Q) + (1 - p_i) f_{N,1}(p_i; P, Q) .$$

For every individual $j \in \mathbf{I}_2$, the utility function $U_2 : [0, 1] \times [0, n_1] \times [0, n_2] \rightarrow \mathbb{R}^+$ is given by

$$U_2(q_j, P, Q) = q_j f_{Y,2}(q_j; P, Q) + (1 - q_j) f_{N,2}(q_j; P, Q) .$$

Proof. The proof follows by induction on the number of individuals. Let $(n_1, n_2) = (1, 1)$. The utility function of individual $i = 1$, with type t_1 , is given by

$$\begin{aligned} U_1(p_i; p_i, q_j) &= p_i \left(q_j f_{Y,1}(1; 1, 1) + (1 - q_j) f_{Y,1}(1; 1, 0) \right) \\ &\quad + (1 - p_i) \left(q_j f_{N,1}(0; 0, 1) + (1 - q_j) f_{N,1}(0; 0, 0) \right). \end{aligned}$$

By substituting the fitness functions in the previous identity, we obtain

$$\begin{aligned} U_1(p_i; p_i, q_j) &= p_i \left(q_j \left(\omega_1^Y + \alpha_{11}^Y 0 + \alpha_{12}^Y 1 \right) + (1 - q_j) \left(\omega_1^Y + \alpha_{11}^Y 0 + \alpha_{12}^Y 0 \right) \right) \\ &\quad + (1 - p_i) \left(q_j \left(\omega_1^N + \alpha_{11}^N 0 + \alpha_{12}^N 0 \right) + (1 - q_j) \left(\omega_1^N + \alpha_{11}^N 0 + \alpha_{12}^N 1 \right) \right). \end{aligned}$$

After rearranging the terms, the last identity becomes

$$\begin{aligned} U_1(p_i; p_i, q_j) &= p_i \left(\omega_1^Y + \alpha_{12}^Y q_j \right) + (1 - p_i) \left(\omega_1^N + \alpha_{12}^N (1 - q_j) \right) \\ &= p_i f_{Y,1}(p_i; p_i, q_j) + (1 - p_i) f_{N,1}(p_i; p_i, q_j) . \end{aligned}$$

Similarly, the utility function of individual $j = 1$, with type t_2 , is given by

$$U_2(q_j; p_i, q_j) = q_j f_{Y,2}(q_j; p_i, q_j) + (1 - q_j) f_{N,2}(q_j; p_i, q_j) .$$

Let us add one more individual $i = n_1 + 1$, with type t_1 , and compute its utility function. Let us suppose, by induction, that the utility functions are known for n_1 individuals with type t_1 and for n_2 individuals with type t_2 . Let $\underline{P} = \sum_{k=1}^{n_1} p_k$, $Q = \sum_{k=1}^{n_2} q_k$ and $P = \underline{P} + p_{n_1+1}$. The utility function of the individual $n_1 + 1$ is given by

$$\begin{aligned} U_1(p_{n_1+1}; P, Q) &= p_{n_1+1} \left(p_{n_1} (f_{Y,1}(p_{n_1}; \underline{P}, Q) + \alpha_{11}^Y) + (1 - p_{n_1}) f_{Y,1}(p_{n_1}; \underline{P}, Q) \right) \\ &\quad + (1 - p_{n_1+1}) \left(p_{n_1} f_{N,1}(p_{n_1}; \underline{P}, Q) + (1 - p_{n_1}) (f_{N,1}(p_{n_1}; \underline{P}, Q) + \alpha_{11}^N) \right) . \end{aligned}$$

Thus,

$$\begin{aligned} U_1(p_{n_1+1}; P, Q) &= p_{n_1+1} \left(f_{Y,1}(p_{n_1}; \underline{P}, Q) + p_{n_1} \alpha_{11}^Y \right) \\ &\quad + (1 - p_{n_1+1}) \left(f_{N,1}(p_{n_1}; \underline{P}, Q) + (1 - p_{n_1}) \alpha_{11}^N \right) . \end{aligned}$$

By substituting the fitness functions in the previous identity, we obtain

$$\begin{aligned} U_1(p_{n_1+1}; P, Q) &= p_{n_1+1} \left(\omega_1^Y + \alpha_{11}^Y \mathbb{P} + \alpha_{12}^Y Q \right) \\ &\quad + (1 - p_{n_1+1}) \left(\omega_1^N + \alpha_{11}^N \left((n_1 + 1) - 1 - \mathbb{P} \right) + \alpha_{12}^N (n_2 - Q) \right). \end{aligned}$$

Therefore,

$$\begin{aligned} U_1(p_{n_1+1}; P, Q) &= p_{n_1+1} \left(\omega_1^Y + \alpha_{11}^Y (P - p_{n_1+1}) + \alpha_{12}^Y Q \right) \\ &\quad + (1 - p_{n_1+1}) \left(\omega_1^N + \alpha_{11}^N \left((n_1 + 1) - 1 - (P - p_{n_1+1}) \right) + \alpha_{12}^N (n_2 - Q) \right). \end{aligned}$$

Hence,

$$U_1(p_{n_1+1}; P, Q) = p_{n_1+1} f_{Y,1}(p_{n_1+1}; P, Q) + (1 - p_{n_1+1}) f_{N,1}(p_{n_1+1}; P, Q) .$$

The proof follows similarly if we add one more individual $j = n_2 + 1$ with type t_2 and compute its utility function. ■

Definition 2.4.1 A strategy $S^* : \mathbf{I} \rightarrow [0, 1]$ is a (mixed) Nash equilibrium, if

$$U_i(S^*) \geq U_i(S)$$

for every individual $i \in \mathbf{I}$ and for every strategy $S \in \mathbf{S}$ with the property that $S^*(j) = S(j)$, for every individual $j \in \mathbf{I} \setminus \{i\}$.

Lemma 2.4.2 Let $S : \mathbf{I} \rightarrow [0, 1]$ be a mixed Nash equilibrium.

(i) If $0 < p_i < 1$, then $x = -A_{11}(P - p_i) - A_{12}Q + H(N, N)$.

(ii) If $0 < q_j < 1$, then $y = -A_{21}P - A_{22}(Q - q_j) + V(N, N)$.

Hence, if $A_{11} \neq 0$, then there is not a mixed Nash equilibrium with the property that $0 < p_{i_1} \neq p_{i_2} < 1$. Furthermore, if $A_{22} \neq 0$, then there is not a mixed Nash equilibrium with the property that $0 < q_{j_1} \neq q_{j_2} < 1$.

Proof. Let $S : \mathbf{I} \rightarrow [0, 1]$ be a mixed Nash equilibrium. For every $p \in [0, 1]$, we have $U_1(p_i; P, Q) \geq U_1(p; P - p_i + p, Q)$. If $0 < p_i < 1$, we get

$$f_{Y,1}(p_i; P, Q) = f_{N,1}(p_i; P, Q)$$

which implies Lemma 2.4.2 (i). The proof of Lemma 2.4.2 (ii) follows similarly. ■

The $(l_1, k_1, p; l_2, k_2, q)$ *mixed strategic set* is the set of all strategies $S : \mathbf{I} \rightarrow [0, 1]$ with the following properties:

- (i) $l_1 = \#\{i \in I_1 : p_i = 1\}$ and $k_1 = \#\{i \in I_1 : p_i = p\}$;
- (ii) $l_2 = \#\{j \in I_2 : q_j = 1\}$ and $k_2 = \#\{j \in I_2 : q_j = q\}$;
- (iii) $n_1 - (l_1 + k_1) = \#\{i \in I_1 : p_i = 0\}$ and $n_2 - (l_2 + k_2) = \#\{j \in I_2 : q_j = 0\}$.

For $p, q \in \{0, 1\}$, we observe that the $(l_1, k_1, p; l_2, k_2, q)$ mixed strategic set is equal to the $(l_1 + pk_1, l_2 + qk_2)$ pure strategic set.

Remark 2.4.1 By Lemma 2.4.2, supposing that $A_{11} \neq 0$ and $A_{22} \neq 0$, a mixed strategy S is a Nash equilibrium, if S is contained in some mixed strategic set $(l_1, k_1, p; l_2, k_2, q)$.

Since individuals with the same type are identical, if a mixed strategy contained in the $(l_1, k_1, p; l_2, k_2, q)$ mixed strategic set is a Nash equilibrium,

then all the strategies in the $(l_1, k_1, p; l_2, k_2, q)$ mixed strategic set are Nash equilibria.

Definition 2.4.2 *An $(l_1, k_1, p; l_2, k_2, q)$ mixed Nash equilibrium (set) is an $(l_1, k_1, p; l_2, k_2, q)$ strategic set whose strategies are Nash equilibria. The (mixed) Nash domain $\mathbf{N}(l_1, k_1, p; l_2, k_2, q)$ is the set of all pairs (x, y) for which the $(l_1, k_1, p; l_2, k_2, q)$ strategic set is a mixed Nash equilibrium set.*

An $(l_1, k_1, p; l_2, k_2, q)$ *strict mixed Nash equilibrium set* is a mixed Nash equilibrium set that does not contain pure strategies, i.e. $(p, q) \in [0, 1]^2 \setminus \{0, 1\}^2$. A *strict mixed Nash domain* $\mathbf{N}(l_1, k_1, p; l_2, k_2, q)$ is the mixed Nash domain of a strict mixed Nash equilibrium set.

The *mixed vector* $\vec{W}(l_1, k_1, p; l_2, k_2, q)$ is defined by

$$\vec{W}(l_1, k_1, p; l_2, k_2, q) = \vec{Z}(l_1, l_2) + p(\vec{Z}(k_1, 0) - (0, A_{21})) + q(\vec{Z}(0, k_2) - (A_{12}, 0)) ,$$

where $\vec{Z}(l_1, l_2)$ is as given in (2.5). Rearranging terms in the equality above, we obtain

$$\begin{aligned} \vec{W}(l_1, k_1, p; l_2, k_2, q) &= -l_1(A_{11}, A_{21}) - l_2(A_{12}, A_{22}) - p(k_1 A_{11}, (k_1 + 1)A_{21}) \\ &\quad - q((k_2 + 1)A_{12}, k_2 A_{22}) . \end{aligned} \tag{2.6}$$

We observe that

$$\vec{W}(0, n_1 - 1, p; 0, n_2 - 1, q) = p\vec{H} + q\vec{V} ,$$

where $\vec{H}$ and $\vec{V}$ are as given in (2.3) and (2.4), respectively.

Let C be as given in (2.2), $\vec{W}$ be as given in (2.6), $I_0 = (-\infty, 0]$ and $I_{n_i} = [A_{ii}, +\infty)$ for $i \in \{1, 2\}$. The following result provides the mixed Nash domain when $A_{11} > 0$ and $A_{22} > 0$.

Theorem 2.4.1 *Let $A_{11} > 0$ and $A_{22} > 0$. Then, $\mathbf{N}(l_1, k_1, p; l_2, k_2, q) \neq \emptyset$ if, and only if, $l_1 = n_1 - k_1$, $l_2 = n_2 - k_2$, $k_1 \in \{0, n_1\}$ and $k_2 \in \{0, n_2\}$. Furthermore,*

(i) *for every $l_1 \in \{0, n_1\}$ and $q \in (0, 1)$, the mixed Nash domain*

$\mathbf{N}(l_1, 0, 0; 0, n_2, q)$ is the semi-line given by

$$\mathbf{N}(l_1, 0, 0; 0, n_2, q) = \{C + (x, 0) + \vec{W}(l_1, 0, 0; 0, n_2 - 1, q) : x \in I_{l_1}\};$$

(ii) *for every $p \in (0, 1)$ and $l_2 \in \{0, n_2\}$, the mixed Nash domain*

$\mathbf{N}(0, n_1, p; l_2, 0, 0)$ is the semi-line given by

$$\mathbf{N}(0, n_1, p; l_2, 0, 0) = \{C + (0, y) + \vec{W}(0, n_1 - 1, p; l_2, 0, 0) : y \in I_{l_2}\};$$

(iii) *for every $p, q \in (0, 1)$, the mixed Nash domain $\mathbf{N}(0, n_1, p; 0, n_2, q)$ is the singleton given by*

$$\mathbf{N}(0, n_1, p; 0, n_2, q) = C + \vec{W}(0, n_1 - 1, p; 0, n_2 - 1, q) .$$

By Theorem 2.4.1, if $l_1 = n_1$, then the only strict mixed Nash equilibria are the ones presented in (i). Furthermore, if $l_1 = 0$, then the only strict mixed Nash equilibria are the ones presented in (ii) and (iii). Hence, if the individuals with a given type have a positive influence over the utility of the

individuals with the same type, i.e. $A_{11} > 0$ and $A_{22} > 0$, then there are no mixed Nash equilibrium, unless all the individuals with the same type opt for a mixed strategy.

In Figure 2.11, we show the geometric interpretation of Theorem 2.4.1, noting that

$$\begin{aligned}\vec{H} &= \vec{W}(0, n_1 - 1, 1; 0, 0, 0), \\ \vec{V} &= \vec{W}(0, 0, 0; 0, n_2 - 1, 1), \\ Z &= \vec{W}(0, n_1 - 1, p; 0, n_2 - 1, q) + C = p\vec{H} + q\vec{V} + C.\end{aligned}$$

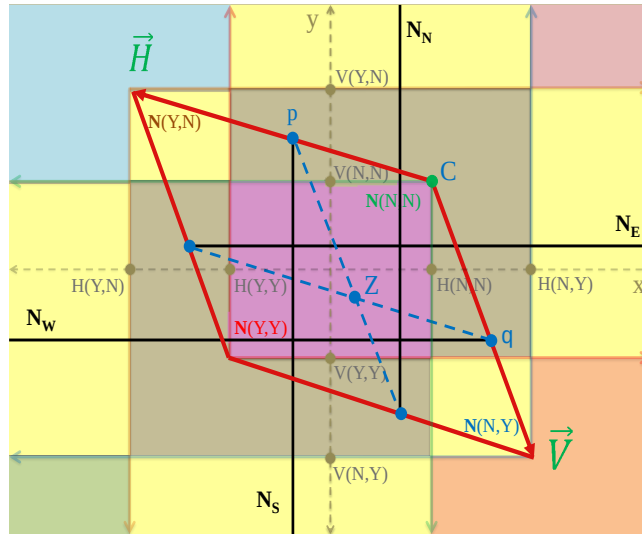


Figure 2.11: A decision tiling with mixed Nash equilibria domain when $A_{11} > 0$ and $A_{22} > 0$, where $N_N = N(0, n_1, p; n_2, 0, 0)$, $N_S = N(0, n_1, p; 0, 0, 0)$, $N_W = N(0, 0, 0; 0, n_2, q)$ and $N_E = N(n_1, 0, 0; 0, n_2, q)$.

Proof. The mixed strategy $(0, 0, 0; 0, n_2, q)$ is a Nash equilibrium if, and

only if, the following two inequalities hold

$$U_1(0; P, Q) \geq U_1(1; P+1, Q) \quad \text{and} \quad U_2(q; P, Q) \geq U_2(q'; P, Q-q+q') .$$

We note that $U_2(q; P, Q) \geq U_2(q'; P, Q-q+q')$ if, and only if, $f_{Y,2}(q; P, Q) = f_{N,2}(q; P, Q)$. Hence, by Lemma 2.4.2,

$$y = -A_{21}P - A_{22}(Q - q) + V(N, N).$$

The mixed strategy $(n_1, 0, 0; 0, n_2, q)$ is a Nash equilibrium if, and only if, the following two inequalities hold

$$U_1(1; P, Q) \geq U_1(0; P-1, Q) \quad \text{and} \quad U_2(q; P, Q) \geq U_2(q'; P, Q-q+q') .$$

We note that $U_2(q; P, Q) \geq U_2(q'; P, Q-q+q')$ if, and only if, $f_{Y,2}(q; P, Q) = f_{N,2}(q; P, Q)$. Hence, by Lemma 2.4.2,

$$y = -A_{21}P - A_{22}(Q - q) + V(N, N).$$

Thus, the proof of Theorem 2.4.1 (i) follows from rearranging the terms in the previous inequalities. The proof of Theorem 2.4.1 (ii) follows similarly to the proof of Theorem 2.4.1 (i).

We will now prove Theorem 2.4.1 (iii). The mixed strategy $(0, n_1, p; 0, n_2, q)$ is a Nash equilibrium if, and only if, the following two inequalities hold

$$U_1(p; P, Q) \geq U_1(p'; P-p+p', Q) \quad \text{and} \quad U_2(q; P, Q) \geq U_2(q'; P, Q-q+q') .$$

Therefore,

$$f_{Y,1}(p; P, Q) = f_{N,1}(p; P, Q) \quad \text{and} \quad f_{Y,2}(q; P, Q) = f_{N,2}(q; P, Q) .$$

Thus, by Lemma 2.4.2 we obtain

$$x = -A_{11}(P-p_i) - A_{12}Q + H(N, N) \quad \text{and} \quad y = -A_{21}P - A_{22}(Q-q) + V(N, N).$$

Hence, Theorem 2.4.1 (iii) follows from rearranging the terms in the previous inequalities. ■

Let C be as given in (2.2), $\vec{W}$ be as given in (2.6), $I_0 = (-\infty, 0]$, $I_{n_i} = [A_{ii}, +\infty)$ for $i \in \{1, 2\}$ and $I_{l_2} = [A_{22}, 0]$ for $l_2 \in \{1, \dots, n_2 - 1\}$. The following result provides the mixed Nash domain when $A_{11} > 0$ and $A_{22} \leq 0$.

Theorem 2.4.2 *Let $A_{11} > 0$ and $A_{22} \leq 0$. Then, $\mathbf{N}(l_1, k_1, p; l_2, k_2, q) \neq \emptyset$ if, and only if, $l_1 = n_1 - k_1$ and $k_1 \in \{0, n_1\}$. Furthermore,*

(i) *for every $l_1 \in \{0, n_1\}$, $k_2 \geq 1$ and $q \in (0, 1)$, the mixed Nash domain $\mathbf{N}(l_1, 0, 0; l_2, k_2, q)$ is the semi-line*

$$\mathbf{N}(l_1, 0, 0; l_2, k_2, q) = \{C + (x, 0) + \vec{W}(l_1, 0, 0; l_2, k_2 - 1, q) : x \in I_{l_1}\};$$

(ii) *for every $p \in (0, 1)$ and $l_2 \in \{0, \dots, n_2\}$, the mixed Nash domain $\mathbf{N}(0, n_1, p; l_2, 0, 0)$ is the semi-line*

$$\mathbf{N}(0, n_1, p; l_2, 0, 0) = \{C + (0, y) + \vec{W}(0, n_1 - 1, p; l_2, 0, 0) : y \in I_{l_2}\};$$

(iii) for every $p, q \in (0, 1)$, the mixed Nash domain $\mathbf{N}(0, n_1, p; l_2, k_2, q)$ is the singleton

$$\mathbf{N}(0, n_1, p; l_2, k_2, q) = C + \vec{W}(0, n_1 - 1, p; l_2, k_2 - 1, q) .$$

By Theorem 2.4.2, if $l_1 = n_1$, then the only strict mixed Nash equilibria are the ones presented in (i). Furthermore, if $l_1 = 0$, then the only strict mixed Nash equilibria are the ones presented in (ii) and (iii). Hence, if the individuals with type t_1 have a positive influence over the utility of the individuals with the same type, i.e. $A_{11} > 0$, then, for every Nash equilibrium all the individuals with the type t_1 opt for the same strategy either pure or mixed.

Proof. The mixed strategy $(0, 0, 0; l_2, k_2, q)$ is a Nash equilibrium if the following two inequalities hold

$$U_1(0; P, Q) \geq U_1(1; P+1, Q) \quad \text{and} \quad U_2(q; P, Q) \geq U_2(q'; P, Q - q + q') .$$

We note that $U_2(q; P, Q) \geq U_2(q'; P, Q - q + q')$ if, and only if, $f_{Y,2}(q; P, Q) = f_{N,2}(q; P, Q)$. Hence, by Lemma 2.4.2,

$$y = -A_{21}P - A_{22}(Q - q) + V(N, N).$$

The mixed strategy $(n_1, 0, 0; l_2, k_2, q)$ is a Nash equilibrium if the following two inequalities hold

$$U_1(1; P, Q) \geq U_1(0; P-1, Q) \quad \text{and} \quad U_2(q; P, Q) \geq U_2(q'; P, Q - q + q') .$$

Furthermore, (a) if $l_2 \geq 1$ then $U_2(1; P, Q) \geq U_2(0; P, Q - 1)$ and (b) if $n_2 > l_2 + k_2$ then $U_2(0; P, Q) \geq U_2(1; P, Q + 1)$. Hence, the proof of Theorem 2.4.2 (i) follows from rearranging the terms in the previous inequalities and noting that $A_{22} \leq 0$.

The mixed strategy $(0, n_1, p; 0, 0, 0)$ is a Nash equilibrium if the following two inequalities hold

$$U_2(0; P, Q) \geq U_2(1; P, Q + 1) \quad \text{and} \quad U_1(p; P, Q) \geq U_1(p'; P - p + p', Q) .$$

We note that $U_1(p; P, Q) \geq U_1(p'; P - p + p', Q)$ if, and only if, $f_{Y,1}(p; P, Q) = f_{N,1}(q; P, Q)$. Hence, by Lemma 2.4.2,

$$x = -A_{11}(P - p) - A_{12}Q + H(N, N) .$$

The mixed strategy $(0, n_1, p; n_2, 0, 0)$ is a Nash equilibrium if the following two inequalities hold

$$U_2(1; P, Q) \geq U_2(0; P, Q - 1) \quad \text{and} \quad U_1(p; P, Q) \geq U_1(p'; P - p + p', Q) .$$

Furthermore, (a) if $l_2 \geq 1$ then $U_2(1; P, Q) \geq U_2(0; P, Q - 1)$ and (b) if $n_2 > l_2$ then $U_2(0; P, Q) \geq U_2(1; P, Q + 1)$. Hence, the proof of Theorem 2.4.2 (ii) follows from rearranging the terms in the previous inequalities and noting that $A_{22} \leq 0$. Let us prove Theorem 2.4.2 (iii).

The mixed strategy $(0, n_1, p; l_2, k_2, q)$ is a Nash equilibrium if the follow-

ing inequalities hold

$$U_1(p; P, Q) \geq U_1(p'; P - p + p', Q) \quad \text{and} \quad U_2(q; P, Q) \geq U_2(q'; P, Q - q + q') .$$

Therefore,

$$f_{Y,1}(p; P, Q) = f_{N,1}(p; P, Q) \quad \text{and} \quad f_{Y,2}(q; P, Q) = f_{N,2}(q; P, Q) .$$

Thus, by Lemma 2.4.2 we obtain

$$x = -A_{11}(P - p) - A_{12}Q + H(N, N) \quad \text{and} \quad y = -A_{21}P - A_{22}(Q - q) + V(N, N) .$$

Furthermore, (a) if $l_2 \geq 1$ then $U_2(1; P, Q) \geq U_2(0; P, Q - 1)$ and (b) if $n_2 > l_2$ then $U_2(0; P, Q) \geq U_2(1; P, Q + 1)$. Hence, the proof of Theorem 2.4.2 (iii) follows from rearranging the terms in the previous inequalities and noting that $A_{22} \leq 0$. ■

Let $I_{l_1} = [A_{11}, 0]$ for $l_1 \in \{1, \dots, n_1 - 1\}$. The proof of the following remark follows similarly to the proof of Theorem 2.4.2.

Remark 2.4.2 *Let $A_{11} \leq 0$ and $A_{22} > 0$. $\mathbf{N}(l_1, k_1, p; l_2, k_2, q) \neq \emptyset$ if, and only if, $l_2 = n_2 - k_2$ and $k_2 \in \{0, n_2\}$. Furthermore,*

(i) *for every $l_2 \in \{0, n_2\}$, $k_1 \geq 1$ and $p \in (0, 1)$, the mixed Nash domain $\mathbf{N}(l_1, k_1, p; l_2, 0, 0)$ is the semi-line*

$$\mathbf{N}(l_1, k_1, p; l_2, 0, 0) = \{C + (0, y) + \vec{W}(l_1, k_1 - 1, p; l_2, 0, 0) : y \in I_{l_2}\};$$

(ii) *for every $q \in (0, 1)$ and $l_1 \in \{0, \dots, n_1\}$, the mixed Nash domain*

$N(l_1, 0, 0; 0, n_2, q)$ is the semi-line

$$N(l_1, 0, 0; 0, n_2, q) = \{C + (x, 0) + \vec{W}(l_1, 0, 0; 0, n_2 - 1, q) : x \in I_{l_1}\};$$

(iii) for every $p, q \in (0, 1)$, the mixed Nash domain $N(l_1, k_1, p; 0, n_2, q)$ is the singleton

$$N(l_1, k_1, p; 0, n_2, q) = C + \vec{W}(l_1, k_1 - 1, p; 0, n_2 - 1, q) .$$

Let C be as given in (2.2), $\vec{W}$ be as given in (2.6), $I_0 = (-\infty, 0]$, $I_{n_i} = [A_{ii}, +\infty)$ for $i \in \{1, 2\}$, $I_{l_1} = [A_{11}, 0]$ for $l_1 \in \{1, \dots, n_1 - 1\}$ and $I_{l_2} = [A_{22}, 0]$ for $l_2 \in \{1, \dots, n_2 - 1\}$. The following result provides the mixed Nash domain when $A_{11} \leq 0$ and $A_{22} \leq 0$.

Theorem 2.4.3 *Let $A_{11} \leq 0$ and $A_{22} \leq 0$.*

(i) For $l_1 \in \{0, \dots, n_1\}$, $k_2 \geq 1$ and $q \in (0, 1)$, the mixed Nash domain $N(l_1, 0, 0; l_2, k_2, q)$ is the semi-line

$$N(l_1, 0, 0; l_2, k_2, q) = \{C + (x, 0) + \vec{W}(l_1, 0, 0; l_2, k_2 - 1, q) : x \in I_{l_1}\} ;$$

(ii) For $p \in (0, 1)$, $k_1 \geq 1$ and $l_2 \in \{0, \dots, n_2\}$, the mixed Nash domain $N(l_1, k_1, p; l_2, 0, 0)$ is the semi-line

$$N(l_1, k_1, p; l_2, 0, 0) = \{C + (0, y) + \vec{W}(l_1, k_1 - 1, p; l_2, 0, 0) : y \in I_{l_2}\} ;$$

(iii) For $k_1 \geq 1$, $k_2 \geq 1$ and $p, q \in (0, 1)$, the mixed Nash domain

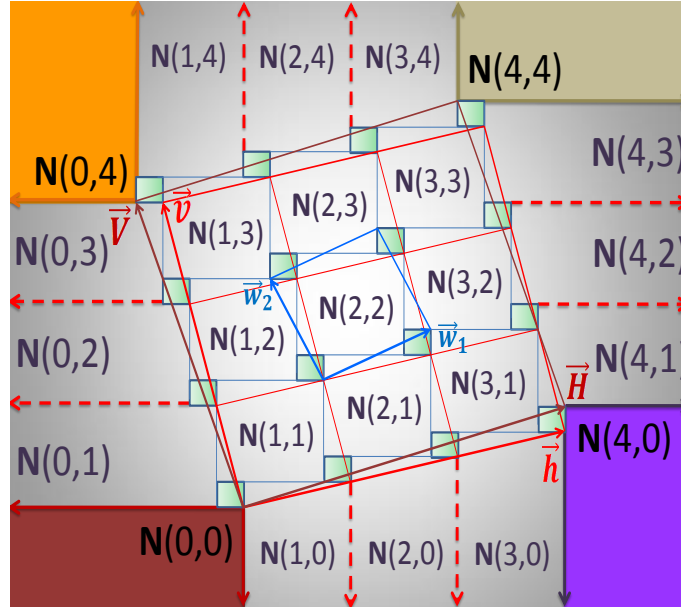
$N(l_1, k_1, p; l_2, k_2, q)$ is the singleton

$$N(l_1, k_1, p; l_2, k_2, q) = C + \vec{W}(l_1, k_1 - 1, p; l_2, k_2 - 1, q) .$$

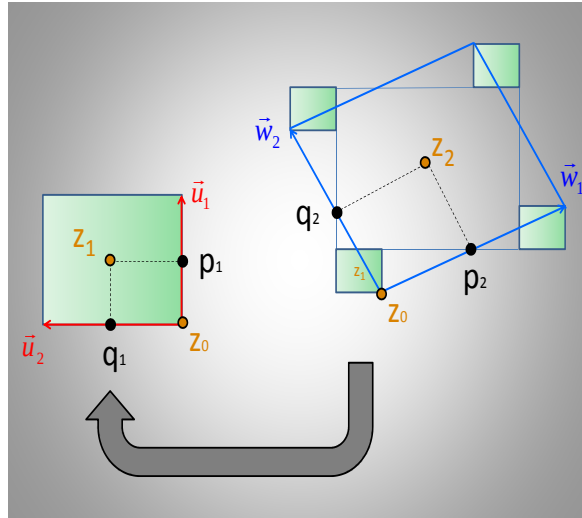
By Theorem 2.4.3, if the individuals with a given type have a non-positive influence over the utility of the individuals with the same type, i.e. $A_{11} \leq 0$ and $A_{22} \leq 0$, then for every $(l_1, k_1, p; l_2, k_2, q)$ mixed strategic set there are relative preferences for which $(l_1, k_1, p; l_2, k_2, q)$ is a Nash equilibrium set.

Recall that C , $\vec{H}$, $\vec{V}$ and $\vec{Z}(l_1, l_2)$ are as given in (2.2), (2.3), (2.4) and (2.5), respectively. In Figures 2.12a and 2.12b, we show the geometric interpretation of Theorem 2.4.3, with

$$\begin{aligned} \vec{u}_1 &= \vec{Z}(0, 0) - (0, A_{21}) = -(0, A_{21}) , \\ \vec{u}_2 &= \vec{Z}(0, 0) - (A_{12}, 0) = -(A_{12}, 0) , \\ \vec{u} &= p_1 \vec{u}_1 + q_1 \vec{u}_2 , \\ \vec{h} &= \vec{H} - \vec{u}_1 = \vec{Z}(n_1 - 1, 0) , \\ \vec{v} &= \vec{V} - \vec{u}_2 = \vec{Z}(0, n_2 - 1) , \\ \vec{w}_1 &= \vec{Z}(1, 0) - \vec{u}_1 = -(A_{11}, 2A_{21}) , \\ \vec{w}_2 &= \vec{Z}(0, 1) - \vec{u}_2 = -(2A_{12}, A_{22}) , \\ \vec{w} &= p_2 \vec{w}_1 + q_2 \vec{w}_2 , \\ Z_0 &= C + \vec{Z}(1, 1), \\ Z_1 &= \vec{u} + Z_0 , \\ Z_2 &= \vec{w} + Z_0 . \end{aligned}$$



(a) The coordinates of the influence matrix are given by $A_{11} = -2$, $A_{12} = 1/2$, $A_{21} = -1/2$ and $A_{22} = -2$.



(b) Zooming Figure 2.12a. The mixed Nash equilibria $\{Z_1\} = \mathbf{N}(1, 1, p_1; 1, 1, q_1)$ and $\{Z_2\} = \mathbf{N}(1, 2, p_2; 1, 2, q_2)$.

Figure 2.12: A decision tiling with mixed Nash equilibria domain when $A_{11} < 0$, $A_{22} < 0$ and $n_1 = n_2 = 4$.

Proof. The mixed strategy $(l_1, 0, 0; l_2, k_2, q)$ is a Nash equilibrium if the following inequality holds

$$U_2(q; P, Q) \geq U_2(q'; P, Q - q + q').$$

Hence,

$$f_{Y,2}(q; P, Q) = f_{N,2}(q; P, Q).$$

Therefore, by Lemma 2.4.2,

$$y = -A_{21}P - A_{22}(Q - q) + V(N, N).$$

Furthermore, (a) if $l_1 \geq 1$ then $U_1(1; P, Q) \geq U_1(0; P - 1, Q)$; (b) if $n_1 > l_1$ then $U_1(0; P, Q) \geq U_1(1; P + 1, Q)$; (c) if $l_2 \geq 1$ then $U_2(1; P, Q) \geq U_2(0; P, Q - 1)$; and (d) if $n_2 > l_2 + k_2$ then $U_2(0; P, Q) \geq U_2(1; P, Q + 1)$. Hence, the proof of Theorem 2.4.3 (i) follows from rearranging the terms in the previous inequalities and noting that $A_{11} \leq 0$ and $A_{22} \leq 0$. The proof of Theorem 2.4.3 (ii) follows similarly. Let us prove Theorem 2.4.3 (iii).

The mixed strategy $(l_1, k_1, p; l_2, k_2, q)$ is a Nash equilibrium if the following inequalities hold

$$U_1(p; P, Q) \geq U_1(p'; P - p + p', Q) \quad , \quad U_2(q; P, Q) \geq U_2(q'; P, Q - q + q') \quad .$$

Hence,

$$f_{Y,1}(p; P, Q) = f_{N,1}(p; P, Q) \quad \text{and} \quad f_{Y,2}(q; P, Q) = f_{N,2}(q; P, Q).$$

Therefore, by Lemma 2.4.2,

$$x = -A_{11}(P-p) - A_{12}Q + H(N, N) \quad \text{and} \quad y = -A_{21}P - A_{22}(Q-q) + V(N, N).$$

Furthermore, (a) if $l_1 \geq 1$ then $U_1(1; P, Q) \geq U_1(0; P-1, Q)$; (b) if $n_1 > l_1 + k_1$ then $U_1(0; P, Q) \geq U_1(1; P+1, Q)$; (c) if $l_2 \geq 1$ then $U_2(1; P, Q) \geq U_2(0; P, Q-1)$; and (d) if $n_2 > l_2 + k_2$ then $U_2(0; P, Q) \geq U_2(1; P, Q+1)$. Hence, the proof of Theorem 2.4.3 (iii) follows by rearranging the terms in the previous inequalities and noting that $A_{11} \leq 0$ and $A_{22} \leq 0$. ■

Putting together Lemma 2.4.2, Theorems 2.4.2, 2.4.3 and Remark 2.4.2, we obtain the following Remark.

Remark 2.4.3 *Let S be a strategy given by $S(i) = p_i$, for $i \in I_1$, and $S(j) = q_j$, for $j \in I_2$.*

(i) *If $A_{22} \neq 0$ and $A_{11} = 0$, the strategy S at (x, y) is a Nash equilibrium if, and only if, there is $0 < q < 1$, such that*

$$\#\{j \in I_2 : q_j = 1\} = l_2;$$

$$\#\{j \in I_2 : q_j = q\} = k_2 \quad (k_2 \text{ might be } 0);$$

$$\#\{j \in I_2 : q_j = 0\} = n_2 - (l_2 + k_2)$$

and

$$(x, y) \in \mathbf{N}(0, n_1, P/n_1; l_2, k_2, q).$$

(ii) *If $A_{11} \neq 0$ and $A_{22} = 0$, the strategy S at (x, y) is a Nash equilibrium*

if, and only if, there is $0 < p < 1$, such that

$$\#\{i \in I_1 : p_i = 1\} = l_1;$$

$$\#\{i \in I_1 : p_i = p\} = k_1 \quad (k_1 \text{ might be } 0);$$

$$\#\{i \in I_1 : p_i = 0\} = n_1 - (l_1 + k_1)$$

and

$$(x, y) \in \mathbf{N}(l_1, k_1, p; 0, n_2, Q/n_2).$$

(iii) If $A_{11} = 0$ and $A_{22} = 0$, the strategy S at (x, y) is a Nash equilibrium if, and only if

$$(x, y) \in \mathbf{N}(0, n_1, P/n_1; 0, n_2, Q/n_2).$$

In the following result, we observe, for a mixed Nash equilibria, who receives higher utility between (i) the individuals who decide Y , (ii) the individuals who decide N , and (iii) the individuals who decide based on probability.

Lemma 2.4.3 *Let $A_{11} \leq 0$. Let $(l_1, k_1, p; l_2, k_2, q)$ be a mixed Nash equilibrium.*

(i) *If $\alpha_{11}^Y \geq 0$, then $U_1(1; P, Q) \leq U_1(p; P, Q) \leq U_1(0; P, Q)$.*

(ii) *If $\alpha_{11}^N \geq 0$, then $U_1(0; P, Q) \leq U_1(p; P, Q) \leq U_1(1; P, Q)$.*

(iii) If $\alpha_{11}^Y \leq 0$, $\alpha_{11}^N \leq 0$ and $(1-p)\alpha_{11}^Y - p\alpha_{11}^N \leq 0$, then

$$U_1(p; P, Q) \leq U_1(0; P, Q) \leq U_1(1; P, Q) ;$$

(iv) If $\alpha_{11}^Y \leq 0$, $\alpha_{11}^N \leq 0$ and $(1-p)\alpha_{11}^Y - p\alpha_{11}^N \geq 0$, then

$$U_1(p; P, Q) \leq U_1(1; P, Q) \leq U_1(0; P, Q) .$$

Proof. The proof follows by putting together Lemma 2.4.1 and Lemma 2.4.2. ■

The results of Remark 2.4.3 are counter intuitive: (i) the individuals, with the same type, who make a decision that they like to share with the other individuals, with the same type, have less utility than the individuals who make the opposite decision; (ii) if the individuals, with the same type, do not like to share any of the two decisions, then the individuals who decide based on probability p have the least utility. Furthermore, if the probability p is small, then the individuals who decide N have less utility than the individuals who decide Y and, if the probability p is high, then the individuals who decide Y have less utility than the individuals who decide N (see Figure 2.13).

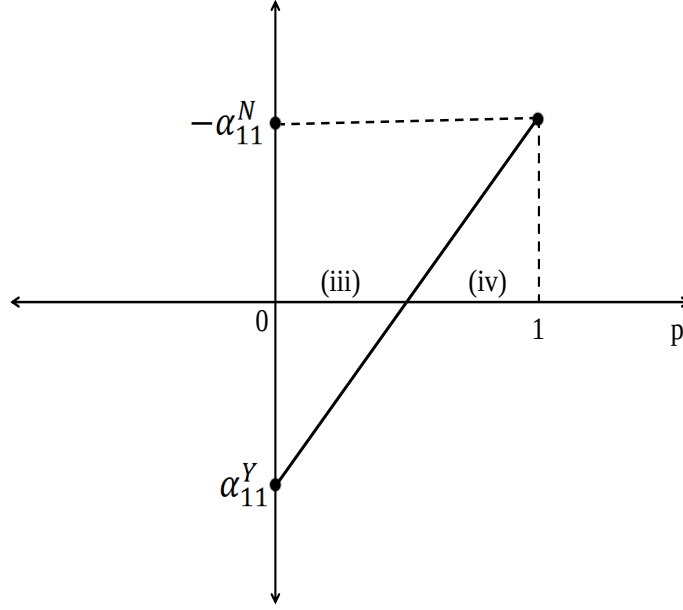


Figure 2.13: Cases (iii) and (iv) of Remark 2.4.3.

2.5 Replicator Dynamics

Recall that $I = I_1 \sqcup I_2$ and that a strategy $S : \mathbf{I} \rightarrow [0, 1]$ associates to each individual $i \in \mathbf{I}_1$ the probability $p_i = S(i)$ to decide $Y \in \mathbf{D}$ and to each individual $j \in \mathbf{I}_2$ the probability $q_j = S(j)$ to decide $Y \in \mathbf{D}$. Recall that $P = \sum_{i=1}^{n_1} p_i$, $Q = \sum_{j=1}^{n_2} q_j$, $P_i = P - p_i$ and $Q_j = Q - q_j$. The *replicator dynamics* $\dot{S} = G(S; x, y)$ is given by

$$\begin{cases} \frac{\dot{p}_i}{p_i} = f_{Y,1}(p_i; P, Q) - U_1(p_i; P, Q) = (1 - p_i) \left(f_{Y,1}(p_i; P, Q) - f_{N,1}(p_i; P, Q) \right), \\ \frac{\dot{q}_j}{q_j} = f_{Y,2}(q_j; P, Q) - U_2(q_j; P, Q) = (1 - q_j) \left(f_{Y,2}(q_j; P, Q) - f_{N,2}(q_j; P, Q) \right). \end{cases}$$

Hence, the *replicator dynamics* $\dot{S} = G(S; x, y)$ can be rewritten as

$$\begin{cases} \dot{p}_i = p_i(1 - p_i)(P_i A_{11} + Q A_{12} + x - H(N, N)) , & i \in \{1, \dots, n_1\} , \\ \dot{q}_j = q_j(1 - q_j)(Q_j A_{22} + P A_{21} + y - V(N, N)) , & j \in \{1, \dots, n_2\} . \end{cases}$$

For every $t \in \mathbb{R}$, we observe that

- (i) if $p_i(0) < p_j(0)$, then $p_i(t) < p_j(t)$;
- (ii) if $p_i(0) = p_j(0)$, then $p_i(t) = p_j(t)$;
- (iii) if $q_i(0) < q_j(0)$, then $q_i(t) < q_j(t)$;
- (iv) if $q_i(0) = q_j(0)$, then $q_i(t) = q_j(t)$;

A strategy $S : \mathbf{I} \rightarrow [0, 1]$ is a *dynamical equilibrium* if $G(S; x, y) = 0$,
i.e.

$$\begin{cases} p_i(1 - p_i)(P_i A_{11} + Q A_{12} + x - H(N, N)) = 0 , & i \in \{1, \dots, n_1\} , \\ q_j(1 - q_j)(Q_j A_{22} + P A_{21} + y - V(N, N)) = 0 , & j \in \{1, \dots, n_2\} . \end{cases}$$

The coefficients of the *linearized replicator dynamics* $DG(S; x, y)$ are

$$\left\{ \begin{array}{l} \partial \dot{p}_i / \partial p_i = (1 - 2p_i)(P_i A_{11} + Q A_{12} + x - H(N, N)) , \\ \partial \dot{p}_i / \partial p_j = p_i(1 - p_i) A_{11} , \quad i \neq j \\ \partial \dot{p}_i / \partial q_j = p_i(1 - p_i) A_{12} , \\ \partial \dot{q}_j / \partial q_j = (1 - 2q_j)(Q_j A_{22} + P A_{21} + y - V(N, N)) , \\ \partial \dot{q}_j / \partial q_i = q_j(1 - q_j) A_{22} , \quad j \neq i \\ \partial \dot{q}_j / \partial p_i = q_j(1 - q_j) A_{21} . \end{array} \right.$$

An equilibrium strategy S is (*strongly*) *stable* if all the eigenvalues of the linearized replicator dynamics $DG(S; x, y)$ have real parts that are negative. An equilibrium strategy S is (*strongly*) *unstable* if there is at least one eigenvalue of the linearized replicator dynamics $DG(S; x, y)$ with positive real part.

Lemma 2.5.1 *Let $S : \mathbf{I} \rightarrow [0, 1]$ be a dynamical equilibrium of the replicator dynamics.*

(i) *If $0 < p_i < 1$, then $x = -A_{11}(P - p_i) - A_{12}Q + H(N, N)$.*

(ii) *If $0 < q_j < 1$, then $y = -A_{22}(Q - q_j) - A_{21}P + V(N, N)$.*

Hence, if $A_{11} \neq 0$, then there is not a dynamical equilibrium with the property that $0 < p_{i_1} \neq p_{i_2} < 1$. Furthermore, if $A_{22} \neq 0$, then there is not a dynamical equilibrium with the property that $0 < q_{j_1} \neq q_{j_2} < 1$.

Proof. The proof is analogous to the proof of Lemma 2.4.2. ■

Hence, assuming that $A_{11} \neq 0$ and $A_{22} \neq 0$, the dynamical equilibria of the replicator dynamics are contained in the union of all $(l_1, k_1, p; l_2, k_2, q)$

strategic sets, where $p, q \in [0, 1]$, $l_1 + k_1 \in \{0, 1, \dots, n_1\}$ and $l_2 + k_2 \in \{0, 1, \dots, n_2\}$. Thus, to find and study the dynamical equilibria we introduce the following notation:

$$\begin{aligned} v[1l] &= (v_1^{1l}, \dots, v_{l_1}^{1l}), \quad v[1m] = (v_1^{1m}, \dots, v_{n_1-(l_1+k_1)}^{1m}), \quad v[1r] = (v_1^{1r}, \dots, v_{k_1}^{1r}) ; \\ v[2l] &= (v_1^{2l}, \dots, v_{l_2}^{2l}), \quad v[2m] = (v_1^{2m}, \dots, v_{n_2-(l_2+k_2)}^{2m}), \quad v[2r] = (v_1^{2r}, \dots, v_{k_2}^{2r}) . \end{aligned}$$

Let us define the vectors $v[1] \in \mathbb{R}^{n_1}$, $v[2] \in \mathbb{R}^{n_2}$ and $v \in \mathbb{R}^{n_1+n_2}$ as follows:

$$v[1] = (v[1l], v[1m], v[1r]), \quad v[2] = (v[2l], v[2m], v[2r]) \quad \text{and} \quad v = (v[1], v[2]) .$$

Let us define $V[1]$ and $V[2]$ as follows:

$$\begin{aligned} V[1] &= \sum_{i=1}^{l_1} v_i^{1l} + \sum_{j=1}^{n_1-(l_1+k_1)} v_j^{1m} + \sum_{k=1}^{k_1} v_k^{1r} , \\ V[2] &= \sum_{i=1}^{l_2} v_i^{2l} + \sum_{j=1}^{n_2-(l_2+k_2)} v_j^{2m} + \sum_{k=1}^{k_2} v_k^{2r} . \end{aligned}$$

Using the previous notations, the *replicator dynamics* $G(v; x, y)$ are given by the following $n_1 + n_2$ ODE:

$$\begin{cases} \dot{v}_{i_1}^{1s} = v_{i_1}^{1s} (1 - v_{i_1}^{1s}) ((V[1] - v_{i_1}^{1s}) A_{11} + V[2] A_{12} + x - H(N, N)) , & i_1 \in I_1 \\ \dot{v}_{i_2}^{2s} = v_{i_2}^{2s} (1 - v_{i_2}^{2s}) ((V[2] - v_{i_2}^{2s}) A_{22} + V[1] A_{21} + y - V(N, N)) , & i_2 \in I_2 \end{cases}$$

where $s \in \{l, m, r\}$ and $(x, y) \in \mathbb{R}^2$.

Definition 2.5.1 *The $(l_1, k_1, p; l_2, k_2, q)$ canonical strategy is defined as fol-*

lows:

(i) for all $i \in \{1, \dots, l_1\}$, $j \in \{1, \dots, n_1 - (l_1 + k_1)\}$ and $k \in \{1, \dots, k_1\}$

$$v_i^{1l} = 1, \quad v_j^{1m} = 0, \quad \text{and} \quad v_k^{1r} = p;$$

(ii) for all $i \in \{1, \dots, l_2\}$, $j \in \{1, \dots, n_2 - (l_2 + k_2)\}$ and $j \in \{1, \dots, k_2\}$

$$v_i^{2l} = 1, \quad v_j^{2m} = 0, \quad \text{and} \quad v_k^{2r} = q.$$

For $k_1 \neq 0$ and $k_2 \neq 0$, the linearized replicator dynamics $DG = DG(l_1, k_1, p; l_2, k_2, q; x, y)$ at the $(l_1, k_1, p; l_2, k_2, q)$ canonical strategy is given as follows

$$DG = \begin{pmatrix} f[1l, 1l] & 0 & 0 & 0 & 0 & 0 \\ 0 & f[1m, 1m] & 0 & 0 & 0 & 0 \\ f[1r, 1l] & f[1r, 1m] & f[1r, 1r] & f[1r, 2l] & f[1r, 2m] & f[1r, 2r] \\ 0 & 0 & 0 & f[2l, 2l] & 0 & 0 \\ 0 & 0 & 0 & 0 & f[2m, 2m] & 0 \\ f[2r, 1l] & f[2r, 1m] & f[2r, 1r] & f[2r, 2l] & f[2r, 2m] & f[2r, 2r] \end{pmatrix},$$

where the coefficients are matrices with the following coordinates:

$$f_{ii}[1l, 1l] = -(l_1 - 1 + k_1 p)A_{11} - (l_2 + k_2 q)A_{12} - x + H(N, N)$$

for all $i \in \{1, \dots, l_1\}$;

$$f_{ul}[1m, 1m] = (l_1 + k_1p)A_{11} + (l_2 + k_2q)A_{12} + x - H(N, N)$$

for all $l \in \{1, \dots, n_1 - (l_1 + k_1)\}$;

$$f_{ij}[1l, 1l] = f_{l,k}[1m, 1m] = 0$$

for all $i, j \in \{1, \dots, l_1\}$, with $i \neq j$, and for all $l, k \in \{1, \dots, n_1 - (l_1 + k_1)\}$, with $l \neq k$;

$$f_{ii}[1r, 1r] = (1 - 2p)((l_1 + (k_1 - 1)p)A_{11} + (l_2 + k_2q)A_{12} + x - H(N, N))$$

for all $i \in \{1, \dots, k_1\}$;

$$f_{ij}[1r, 1r] = f_{il}[1r, 1l] = f_{ik}[1r, 1m] = p(1 - p)A_{11}$$

for all $i, j \in \{1, \dots, k_1\}$, with $i \neq j$, $l \in \{1, \dots, l_1\}$ and for all $k \in \{1, \dots, n_1 - (l_1 + k_1)\}$;

$$f_{ij}[1r, 2l] = f_{ik}[1r, 2m] = f_{il}[1r, 2r] = p(1 - p)A_{12}$$

for all $i \in \{1, \dots, k_1\}$, $j \in \{1, \dots, l_2\}$, $k \in \{1, \dots, n_2 - (l_2 + k_2)\}$ and $l \in \{1, \dots, k_2\}$;

$$f_{ii}[2l, 2l] = -(l_2 - 1 + k_2q)A_{22} - (l_1 + k_1p)A_{21} - y + V(N, N)$$

for all $i \in \{1, \dots, l_2\}$;

$$f_{il}[2m, 2m] = (l_2 + k_2q)A_{22} + (l_1 + k_1p)A_{21} + y - V(N, N)$$

for all $l \in \{1, \dots, n_2 - (l_2 + k_2)\}$;

$$f_{ij}[2l, 2l] = f_{l,k}[2m, 2m] = 0$$

for all $i, j \in \{1, \dots, l_2\}$, with $i \neq j$, and for all $l, k \in \{1, \dots, n_2 - (l_2 + k_2)\}$, with $l \neq k$;

$$f_{ii}[2r, 2r] = (1 - 2q)((l_2 + (k_2 - 1)q)A_{22} + (l_1 + k_1p)A_{21} + y - V(N, N))$$

for all $i \in \{1, \dots, k_2\}$;

$$f_{ij}[2r, 2r] = f_{il}[2r, 2l] = f_{ik}[2r, 2m] = q(1 - q)A_{22}$$

for all $i, j \in \{1, \dots, k_2\}$, with $i \neq j$, $l \in \{1, \dots, l_2\}$ and for all $k \in \{1, \dots, n_2 - (l_2 + k_2)\}$;

$$f_{ij}[2r, 1l] = f_{ik}[2r, 1m] = f_{il}[2r, 1r] = q(1 - q)A_{21}$$

for all $i \in \{1, \dots, k_2\}$, $j \in \{1, \dots, l_1\}$, $k \in \{1, \dots, n_1 - (l_1 + k_1)\}$ and $l \in \{1, \dots, k_1\}$.

We observe that (a) if $l_1 = 0$, then $f[1l, 1l] = \emptyset$; (b) if $l_1 = n_1$, then $f[1m, 1m] = \emptyset$; (c) if $l_2 = 0$, then $f[2l, 2l] = \emptyset$; and (d) if $l_2 = n_2$, then $f[2m, 2m] = \emptyset$. Hence, we note that in all cases (a)-(d) the corresponding

rows and columns vanish.

For $k_1 = 0$ and $k_2 \neq 0$, the linearized replicator dynamics $DG = DG(l_1; l_2, k_2, q; x, y)$ at the $(l_1; l_2, k_2, q)$ canonical strategy is given as follows

$$DG = \begin{pmatrix} f[1l, 1l] & 0 & 0 & 0 & 0 \\ 0 & f[1m, 1m] & 0 & 0 & 0 \\ 0 & 0 & f[2l, 2l] & 0 & 0 \\ 0 & 0 & 0 & f[2m, 2m] & 0 \\ f[2r, 1l] & f[2r, 1m] & f[2r, 2l] & f[2r, 2m] & f[2r, 2r] \end{pmatrix},$$

where the coefficients are matrices with the following coordinates:

$$f_{ii}[1l, 1l] = -(l_1 - 1)A_{11} - (l_2 + k_2q)A_{12} - x + H(N, N)$$

for all $i \in \{1, \dots, l_1\}$;

$$f_{il}[1m, 1m] = l_1A_{11} + (l_2 + k_2q)A_{12} + x - H(N, N)$$

for all $l \in \{1, \dots, n_1 - l_1\}$;

$$f_{ij}[1l, 1l] = f_{l,k}[1m, 1m] = 0$$

for all $i, j \in \{1, \dots, l_1\}$, with $i \neq j$, and for all $l, k \in \{1, \dots, n_1 - l_1\}$, with $l \neq k$;

$$f_{ii}[2l, 2l] = -(l_2 - 1 + k_2q)A_{22} - l_1A_{21} - y + V(N, N)$$

for all $i \in \{1, \dots, l_2\}$;

$$f_{il}[2m, 2m] = (l_2 + k_2q)A_{22} + l_1A_{21} + y - V(N, N)$$

for all $l \in \{1, \dots, n_2 - (l_2 + k_2)\}$;

$$f_{ij}[2l, 2l] = f_{l,k}[2m, 2m] = 0$$

for all $i, j \in \{1, \dots, l_2\}$, with $i \neq j$, and for all $l, k \in \{1, \dots, n_2 - (l_2 + k_2)\}$, with $l \neq k$;

$$f_{ii}[2r, 2r] = (1 - 2q)((l_2 + (k_2 - 1)q)A_{22} + l_1A_{21} + y - V(N, N))$$

for all $i \in \{1, \dots, k_2\}$;

$$f_{ij}[2r, 2r] = f_{il}[2r, 2l] = f_{ik}[2r, 2m] = q(1 - q)A_{22}$$

for all $i, j \in \{1, \dots, k_2\}$, with $i \neq j$, $l \in \{1, \dots, l_2\}$ and for all $k \in \{1, \dots, n_2 - (l_2 + k_2)\}$;

$$f_{ij}[2r, 1l] = f_{ik}[2r, 1m] = q(1 - q)A_{21}$$

for all $i \in \{1, \dots, k_2\}$, $j \in \{1, \dots, l_1\}$ and $k \in \{1, \dots, n_1 - l_1\}$.

We observe that (a) if $l_1 = 0$, then $f[1l, 1l] = \emptyset$; (b) if $l_1 = n_1$, then $f[1m, 1m] = \emptyset$; (c) if $l_2 = 0$, then $f[2l, 2l] = \emptyset$; and (d) if $l_2 = n_2$, then $f[2m, 2m] = \emptyset$. Hence, we note that in all cases (a)-(d) the corresponding rows and columns vanish.

For $k_1 \neq 0$ and $k_2 = 0$, the linearized replicator dynamics $DG = DG(l_1, k_1, p; l_2; x, y)$ at the $(l_1, k_1, p; l_2)$ canonical strategy is given as follows

$$DG = \begin{pmatrix} f[1l, 1l] & 0 & 0 & 0 & 0 \\ 0 & f[1m, 1m] & 0 & 0 & 0 \\ f[1r, 1l] & f[1r, 1m] & f[1r, 1r] & f[1r, 2l] & f[1r, 2m] \\ 0 & 0 & 0 & f[2l, 2l] & 0 \\ 0 & 0 & 0 & 0 & f[2m, 2m] \end{pmatrix},$$

where the coefficients are matrices with the following coordinates:

$$f_{ii}[1l, 1l] = -(l_1 - 1 + k_1 p)A_{11} - l_2 A_{12} - x + H(N, N)$$

for all $i \in \{1, \dots, l_1\}$;

$$f_u[1m, 1m] = (l_1 + k_1 p)A_{11} + l_2 A_{12} + x - H(N, N)$$

for all $l \in \{1, \dots, n_1 - (l_1 + k_1)\}$;

$$f_{ij}[1l, 1l] = f_{l,k}[1m, 1m] = 0$$

for all $i, j \in \{1, \dots, l_1\}$, with $i \neq j$, and for all $l, k \in \{1, \dots, n_1 - (l_1 + k_1)\}$, with $l \neq k$;

$$f_{ii}[1r, 1r] = (1 - 2p)((l_1 + (k_1 - 1)p)A_{11} + l_2 A_{12} + x - H(N, N))$$

for all $i \in \{1, \dots, k_1\}$;

$$f_{ij}[1r, 1r] = f_{il}[1r, 1l] = f_{ik}[1r, 1m] = p(1-p)A_{11}$$

for all $i, j \in \{1, \dots, k_1\}$, with $i \neq j$, $l \in \{1, \dots, l_1\}$ and for all $k \in \{1, \dots, n_1 - (l_1 + k_1)\}$;

$$f_{ij}[1r, 2l] = f_{ik}[1r, 2m] = p(1-p)A_{12}$$

for all $i \in \{1, \dots, k_1\}$, $j \in \{1, \dots, l_2\}$ and $k \in \{1, \dots, n_2 - l_2\}$;

$$f_{ii}[2l, 2l] = -(l_2 - 1)A_{22} - (l_1 + k_1p)A_{21} - y + V(N, N)$$

for all $i \in \{1, \dots, l_2\}$;

$$f_{il}[2m, 2m] = l_2A_{22} + (l_1 + k_1p)A_{21} + y - V(N, N)$$

for all $l \in \{1, \dots, n_2 - l_2\}$;

$$f_{ij}[2l, 2l] = f_{l,k}[2m, 2m] = 0$$

for all $i, j \in \{1, \dots, l_2\}$, with $i \neq j$, and for all $l, k \in \{1, \dots, n_2 - l_2\}$, with $l \neq k$.

We observe that (a) if $l_1 = 0$, then $f[1l, 1l] = \emptyset$; (b) if $l_1 = n_1$, then $f[1m, 1m] = \emptyset$; (c) if $l_2 = 0$, then $f[2l, 2l] = \emptyset$; and (d) if $l_2 = n_2$, then $f[2m, 2m] = \emptyset$. Hence, we note that in all cases (a)-(d) the corresponding rows and columns vanish.

For $k_1 = k_2 = 0$, the linearized replicator dynamics $DG = DG(l_1; l_2; x, y)$ at the (l_1, l_2) canonical strategy is given as follows

$$DG = \begin{pmatrix} f[1l, 1l] & 0 & 0 & 0 \\ 0 & f[1m, 1m] & 0 & 0 \\ 0 & 0 & f[2l, 2l] & 0 \\ 0 & 0 & 0 & f[2m, 2m] \end{pmatrix},$$

where the coefficients are matrices with the following coordinates:

$$f_{ii}[1l, 1l] = -(l_1 - 1)A_{11} - l_2A_{12} - x + H(N, N)$$

for all $i \in \{1, \dots, l_1\}$;

$$f_{ll}[1m, 1m] = l_1A_{11} + l_2A_{12} + x - H(N, N)$$

for all $l \in \{1, \dots, n_1 - l_1\}$;

$$f_{ij}[1l, 1l] = f_{l,k}[1m, 1m] = 0$$

for all $i, j \in \{1, \dots, l_1\}$, with $i \neq j$, and for all $l, k \in \{1, \dots, n_1 - l_1\}$, with $l \neq k$;

$$f_{ii}[2l, 2l] = -(l_2 - 1)A_{22} - l_1A_{21} - y + V(N, N)$$

for all $i \in \{1, \dots, l_2\}$;

$$f_{ll}[2m, 2m] = l_2A_{22} + l_1A_{21} + y - V(N, N)$$

for all $l \in \{1, \dots, n_2 - l_2\}$;

$$f_{ij}[2l, 2l] = f_{l,k}[2m, 2m] = 0$$

for all $i, j \in \{1, \dots, l_2\}$, with $i \neq j$, and for all $l, k \in \{1, \dots, n_2 - l_2\}$, with $l \neq k$.

We observe that (a) if $l_1 = 0$, then $f[1l, 1l] = \emptyset$; (b) if $l_1 = n_1$, then $f[1m, 1m] = \emptyset$; (c) if $l_2 = 0$, then $f[2l, 2l] = \emptyset$ and (d) if $l_2 = n_2$, then $f[2m, 2m] = \emptyset$.

For some pair $(x, y) \in \mathbb{R}^2$, if the $(l_1, k_1, p; l_2, k_2, q)$ canonical strategy is an equilibrium of the replicator dynamics, then every strategy contained in the $(l_1, k_1, p; l_2, k_2, q)$ strategic set is an equilibrium of the replicator dynamics. Furthermore, the eigenvectors of the linearized replicator dynamics at the $(l_1, k_1, p; l_2, k_2, q)$ canonical strategy coincide with the eigenvectors of the linearized replicator dynamics at any strategy contained in the $(l_1, k_1, p; l_2, k_2, q)$ strategic set, up to permutation of coordinates. Hence, the eigenvalues of the linearized replicator dynamics at the $(l_1, k_1, p; l_2, k_2, q)$ canonical strategy coincide with the eigenvalues of the linearized replicator dynamics at any strategy contained in the $(l_1, k_1, p; l_2, k_2, q)$ strategic set.

Definition 2.5.2 *The equilibria domain $\mathbf{E}(l_1, k_1, p; l_2, k_2, q)$ is the set of all pairs $(x, y) \in \mathbb{R}^2$ for which the strategies contained in the $(l_1, k_1, p; l_2, k_2, q)$ strategic set are equilibria of the replicator dynamics. The (strongly) stable domain $\mathbf{S}(l_1, k_1, p; l_2, k_2, q)$ is the set of all pairs $(x, y) \in \mathbb{R}^2$ for which the strategies contained in the $(l_1, k_1, p; l_2, k_2, q)$ strategic set are (strongly) stable equilibria of the replicator dynamics. The (strongly) unstable domain*

$\mathbf{U}(l_1, k_1, p; l_2, k_2, q)$ is the set of all pairs $(x, y) \in \mathbb{R}^2$ for which the strategies contained in the $(l_1, k_1, p; l_2, k_2, q)$ strategic set are (strongly) unstable equilibria of the replicator dynamics.

We observe that the stable domains are contained in the Nash domains and the Nash domains are contained in the equilibria domains, i.e.

$$\mathbf{S}(l_1, k_1, p; l_2, k_2, q) \subseteq \mathbf{N}(l_1, k_1, p; l_2, k_2, q) \subseteq \mathbf{E}(l_1, k_1, p; l_2, k_2, q) .$$

For the (l_1, l_2) (pure) strategic set, the dynamic equilibria set coincides with $\mathbb{R}^2$, i.e.

$$\mathbf{E}(l_1, l_2) = \mathbb{R}^2 .$$

The following geometric relation associates to every pair $(x, y) \in \mathbb{R}^2$ a unique pair $(x_1, y_1) \in \mathbb{R}^2$ and vice-versa:

$$(x, y) = C + \vec{Z}(l_1, l_2) + (x_1 A_{11}, y_1 A_{22}) = \left(H(N, N) - l_1 A_{11} - l_2 A_{12} + x_1 A_{11}, V(N, N) - l_2 A_{22} - l_1 A_{21} + y_1 A_{22} \right). \quad (2.7)$$

For $k_1 = k_2 = 0$, we have $v[1r] = \emptyset$ and $v[2r] = \emptyset$, and so $v[1] = (v[1l], v[1m])$ and $v[2] = (v[2l], v[2m])$.

Theorem 2.5.1 *Let $k_1 = k_2 = 0$. The eigenvalues of $DG(l_1, l_2; x, y)$ are*

$$\begin{aligned} \lambda[1l] &= A_{11}(1 - x_1) , & \lambda[1m] &= A_{11}x_1 , \\ \lambda[2l] &= A_{22}(1 - y_1) , & \lambda[2m] &= A_{22}y_1 . \end{aligned} \quad (2.8)$$

The eigenspaces of $DG(l_1, l_2; x, y)$ are

$$\begin{aligned} E(\lambda[1l]) &= \{v : (v[1m], v[2]) = 0\} , & E(\lambda[1m]) &= \{v : (v[1l], v[2]) = 0\} , \\ E(\lambda[2l]) &= \{v : (v[1], v[2m]) = 0\} , & E(\lambda[2m]) &= \{v : (v[1], v[2l]) = 0\} \end{aligned} \quad (2.9)$$

Furthermore,

$$\mathbf{S}(l_1, l_2) = \text{int}(\mathbf{N}(l_1, l_2)) \quad \text{and} \quad \mathbf{U}(l_1, l_2) \subset \mathbb{R}^2 \setminus \mathbf{N}(l_1, l_2).$$

Putting together Remark 2.3.2 and Theorem 2.5.1, we obtain the following:

- (i) if (a) $A_{11} \geq 0$ and $l_1 \in \{1, \dots, n_1 - 1\}$, or if (b) $A_{22} \geq 0$ and $l_2 \in \{1, \dots, n_2 - 1\}$, or if (c) $A_{11} \geq 0$ and $A_{22} \geq 0$ and $(l_1, l_2) \in \{1, \dots, n_1 - 1\} \times \{1, \dots, n_2 - 1\}$, then

$$\mathbf{S}(l_1, l_2) = \text{int}(\mathbf{N}(l_1, l_2)) = \emptyset.$$

- (ii) if (a) $A_{11} \geq 0$ and $A_{22} \geq 0$ and $(l_1, l_2) \in \{(0, 0), (n_1, 0), (0, n_2), (n_1, n_2)\}$, or if (b) $A_{11} < 0$, $l_1 \in \{0, 1, \dots, n_1\}$ and $l_2 \in \{0, n_2\}$, or if (c) $A_{22} < 0$, $l_2 \in \{0, 1, \dots, n_2\}$ and $l_1 \in \{0, n_1\}$, or if (d) $A_{11} < 0$ and $A_{22} < 0$ and $l_1 \in \{0, 1, \dots, n_1\}$ and $l_2 \in \{0, 1, \dots, n_2\}$, then

$$\mathbf{S}(l_1, l_2) = \text{int}(\mathbf{N}(l_1, l_2)) \neq \emptyset.$$

Proof. Recall the geometric relation given in (2.7). The coefficients of the linearized replicator dynamics $DG = DG(l_1, 0, 0; l_2, 0, 0; x, y)$ are the following: If $i \in \{1, \dots, l_1\}$, then

$$f_{ii}[1l, 1l] = -(l_1 - 1)A_{11} - l_2A_{12} - x + H(N, N) = A_{11}(1 - x_1) .$$

If $i \in \{1, \dots, n_1 - l_1\}$, then

$$f_{ii}[1m, 1m] = l_1A_{11} + l_2A_{12} + x - H(N, N) = A_{11}x_1 .$$

If $j \in \{1, \dots, l_2\}$, then

$$f_{jj}[2l, 2l] = -(l_2 - 1)A_{22} - l_1A_{21} - y + V(N, N) = A_{22}(1 - y_1) .$$

If $j \in \{1, \dots, n_2 - l_2\}$, then

$$f_{jj}[2m, 2m] = l_2A_{22} + l_1A_{21} + y - V(N, N) = A_{22}y_1 .$$

Furthermore, all the other coefficients of DG are equal to 0. Hence, DG is a diagonal matrix. The eigenvalues and the eigenspaces of DG are the ones presented in (2.8) and (2.9). Hence, the proof of the second part of Theorem 2.5.1 follows from applying Remark 2.3.2. ■

From Theorems 2.4.1, 2.4.2 and 2.4.3, the equilibrium domain $\mathbf{E}(l_1, k_1, p; l_2, k_2, q)$ is the following:

- (i) For $l_1 \in \{0, \dots, n_1\}$ and $q \in (0, 1)$, the equilibrium domain

$\mathbf{E}(l_1, 0, 0; l_2, k_2, q)$ is the line

$$\mathbf{E}(l_1, 0, 0; l_2, k_2, q) = \{C + (x, 0) + \vec{W}(l_1, 0, 0; l_2, k_2 - 1, q) : x \in \mathbb{R}\} ;$$

(ii) For $p \in (0, 1)$ and $l_2 \in \{0, \dots, n_2\}$, the equilibrium domain

$\mathbf{E}(l_1, k_1, p; l_2, 0, 0)$ is the line

$$\mathbf{E}(l_1, k_1, p; l_2, 0, 0) = \{C + (0, y) + \vec{W}(l_1, k_1 - 1, p; l_2, 0, 0) : y \in \mathbb{R}\} ;$$

(iii) For $p, q \in (0, 1)$, the equilibrium domain $\mathbf{E}(l_1, k_1, p; l_2, k_2, q)$ is the singleton

$$\mathbf{E}(l_1, k_1, p; l_2, k_2, q) = C + \vec{W}(l_1, k_1 - 1, p; l_2, k_2 - 1, q) .$$

Define the matrix $M(k_1, k_2) = M(l_1, k_1, p; l_2, k_2, q; x, y)$ as follows:

(i) if $k_1 \geq 1$ and $k_2 \geq 1$, then

$$M(k_1, k_2) = \begin{pmatrix} (k_1 - 1)p(1 - p)A_{11} & k_1q(1 - q)A_{21} \\ k_2p(1 - p)A_{12} & (k_2 - 1)q(1 - q)A_{22} \end{pmatrix} .$$

Note that the eigenvalues of the matrix $M(k_1, k_2)$ are given by

$$\lambda_{\pm} = \frac{1}{2} \left(a + c \pm \sqrt{4bd + (a - c)^2} \right) ,$$

where

$$\begin{aligned} a &= (k_1 - 1)p(1 - p)A_{11} , & b &= k_1q(1 - q)A_{21} , \\ c &= (k_2 - 1)q(1 - q)A_{22} , & d &= k_2p(1 - p)A_{12} ; \end{aligned}$$

(ii) if $k_1 = 0$ and $k_2 \geq 1$, then

$$M(0, k_2) = ((k_2 - 1)q(1 - q)A_{22}) ;$$

(iii) if $k_1 \geq 1$ and $k_2 = 0$, then

$$M(k_1, 0) = ((k_1 - 1)p(1 - p)A_{11}) .$$

The following relation associates to each pair $(x, y) \in \mathbb{R}^2$ a unique pair $(p, q) \in \mathbb{R}^2$ and vice-versa:

$$\begin{aligned} (x, y) - C = \\ - \left((l_1 + p(k_1 - 1)) A_{11} + (l_2 + qk_2) A_{12}, (l_2 + q(k_2 - 1)) A_{22} + (l_1 + pk_1) A_{21} \right). \end{aligned}$$

Theorem 2.5.2 *Let $k_1 \neq 0$ and $k_2 \neq 0$. Let $DG = DG(l_1, k_1, p; l_2, k_2, q; x, y)$ and*

$$(p, q) \in [0, 1]^2 \setminus \{(0, 0), (0, 1), (1, 0), (1, 1)\}.$$

The eigenvalues of DG are the following:

(i) $\lambda[1l] = (1 - p)A_{11}$, with algebraic dimension l_1 ;

(ii) $\lambda[1m] = pA_{11}$, with algebraic dimension $n_1 - (l_1 + k_1)$;

(iii) $\lambda[2l] = (1 - q)A_{22}$, with algebraic dimension l_2 ;

(iv) $\lambda[2m] = qA_{22}$, with algebraic dimension $n_2 - (l_2 + k_2)$;

(v) $\lambda[1r] = -p(1 - p)A_{11}$, with algebraic dimension $k_1 - 1$;

(vi) $\lambda[2r] = -q(1 - q)A_{22}$, with algebraic dimension $k_2 - 1$;

(vii) The eigenvalues of the matrix $M(l_1, k_1, p; l_2, k_2, q; x, y)$.

Furthermore, $S(l_1, k_1, p; l_2, k_2, q) = \emptyset$.

We observe that (a) if $l_1 = 0$, then $E(\lambda[1l]) = \emptyset$, and so $\lambda[1l]$ is not an eigenvalue; (b) if $l_1 = n_1$, then $E(\lambda[1m]) = \emptyset$, and so $\lambda[1m]$ is not an eigenvalue; (c) if $l_2 = 0$, then $E(\lambda[2l]) = \emptyset$, and so $\lambda[2l]$ is not an eigenvalue and (d) if $l_2 = n_2$, then $E(\lambda[2m]) = \emptyset$, and so $\lambda[2m]$ is not an eigenvalue.

Proof. Let us denote the vectors of the canonical basis of $\mathbb{R}^{n_1+n_2}$ by

$$e_1[1l], \dots, e_{l_1}[1l] , e_1[1m], \dots, e_{n_1-(l_1+k_1)}[1m] , e_1[1r], \dots, e_{k_1}[1r]$$

and

$$e_1[2l], \dots, e_{l_2}[2l] , e_1[2m], \dots, e_{n_2-(l_2+k_2)}[2m] , e_1[2r], \dots, e_{k_2}[2r] .$$

Let $w_1 = \sum_{i=1}^{k_1} e_i[1r]$ and $w_2 = \sum_{i=1}^{k_2} e_i[2r]$. We obtain that

$$\begin{aligned}
 DG(e_i[1l]) &= \lambda[1l]e_i[1l] + p(1-p)A_{11}w_1 + q(1-q)A_{21}w_2, \\
 DG(e_i[1m]) &= \lambda[1m]e_i[1m] + p(1-p)A_{11}w_1 + q(1-q)A_{21}w_2, \\
 DG(e_i[1r]) &= -p(1-p)A_{11}e_i[1r] + p(1-p)A_{11}w_1 + q(1-q)A_{21}w_2, \\
 DG(e_i[2l]) &= \lambda[2l]e_i[2l] + p(1-p)A_{12}w_1 + q(1-q)A_{22}w_2, \\
 DG(e_i[2m]) &= \lambda[2m]e_i[2m] + p(1-p)A_{12}w_1 + q(1-q)A_{22}w_2, \\
 DG(e_i[2r]) &= -q(1-q)A_{22}e_i[2r] + p(1-p)A_{12}w_1 + q(1-q)A_{22}w_2.
 \end{aligned}$$

Furthermore,

$$\begin{aligned}
 DG(w_1) &= \sum_{i=1}^{k_1} DG(e_i[1r]) \\
 &= \sum_{i=1}^{k_1} \left(\sum_{j=1}^{k_1} p(1-p)A_{11}e_j[1r] + \sum_{j=1}^{k_2} q(1-q)A_{21}e_j[2r] \right) \\
 &= (k_1 - 1)p(1-p)A_{11}w_1 + k_1q(1-q)A_{21}w_2.
 \end{aligned}$$

$$\begin{aligned}
 DG(w_2) &= \sum_{i=1}^{k_2} DG(e_i[2r]) \\
 &= \sum_{i=1}^{k_2} \left(\sum_{j=1}^{k_2} q(1-q)A_{22}e_j[2r] + \sum_{j=1}^{k_1} p(1-p)A_{12}e_j[1r] \right) \\
 &= k_2p(1-p)A_{12}w_1 + (k_2 - 1)q(1-q)A_{22}w_2.
 \end{aligned}$$

Hence, the space spanned by w_1 and w_2 is DG invariant and $DG|_{\langle w_1, w_2 \rangle}$ is given by the matrix $M(k_1, k_2)$. Therefore, the eigenvalues of $DG|_{\langle w_1, w_2 \rangle}$

$w_1, w_2 >$ are the eigenvalues of the matrix $M(k_1, k_2)$. For $t \in \{1, 2\}$ and $s \in \{l, m, r\}$, the vectors $e_i[ts] - e_j[ts]$, with $i \neq j$, belong to the eigenspace of the eigenvalue $\lambda[ts]$. Since the space spanned by w_1 and w_2 is DG invariant, the vectors $e_i[ts]$ belong to the extended eigenspace of the eigenvalue $\lambda[ts]$.

■

The proof of the following two remarks follows similarly to the proof of Theorem 2.5.2. The following relation associates to each pair $(x, y) \in \mathbb{R}^2$ a unique pair $(x_1, q) \in \mathbb{R}^2$ and vice-versa:

$$(x, y) - C = - \left((l_1 - x_1) A_{11} + (l_2 + qk_2) A_{12}, (l_2 + q(k_2 - 1)) A_{22} + l_1 A_{21} \right).$$

Remark 2.5.1 Let $k_1 = 0$ and $k_2 \neq 0$. Let $DG = DG(l_1; l_2, k_2, q; x, y)$ and

$$(x_1, q) \in [0, 1]^2 \setminus \{(0, 0), (0, 1), (1, 0), (1, 1)\}.$$

The eigenvalues of DG are the following:

- (i) $\lambda[1l] = (1 - x_1)A_{11}$, with algebraic dimension l_1 ;
- (ii) $\lambda[1m] = x_1A_{11}$, with algebraic dimension $n_1 - l_1$;
- (iii) $\lambda[2l] = (1 - x_1)A_{22}$, with algebraic dimension l_2 ;
- (iv) $\lambda[2m] = qA_{22}$, with algebraic dimension $n_2 - (l_2 + k_2)$;
- (v) $\lambda[2r] = -q(1 - q)A_{22}$, with algebraic dimension $k_2 - 1$;
- (vi) The eigenvalues of the matrix $M(l_1; l_2, k_2, q; x, y)$.

Furthermore, $S(l_1; l_2, k_2, q) = \emptyset$.

We observe that (a) if $l_1 = 0$, then $\lambda[1l]$ is not an eigenvalue and (b) if $l_1 = n_1$, then $\lambda[1m]$ is not an eigenvalue.

The following relation associates to each pair $(x, y) \in \mathbb{R}^2$ a unique pair $(p, y_1) \in \mathbb{R}^2$ and vice-versa:

$$(x, y) - C = - \left((l_1 + p(k_1 - 1)) A_{11} + l_2 A_{12}, (l_2 - y_1) A_{22} + (l_1 + p k_1) A_{21} \right).$$

Remark 2.5.2 *Let $k_1 \neq 0$ and $k_2 = 0$. Let $DG = DG(l_1, k_1, p; l_2; x, y)$ and*

$$(p, y_1) \in [0, 1]^2 \setminus \{(0, 0), (0, 1), (1, 0), (1, 1)\}.$$

The eigenvalues of DG are the following:

- (i) $\lambda[1l] = (1 - p)A_{11}$, with algebraic dimension l_1 ;
- (ii) $\lambda[1m] = pA_{11}$, with algebraic dimension $n_1 - (l_1 + k_1)$;
- (iii) $\lambda[2l] = (1 - y_1)A_{22}$, with algebraic dimension l_2 ;
- (iv) $\lambda[2m] = y_1A_{22}$, with algebraic dimension $n_2 - l_2$;
- (v) $\lambda[1r] = -p(1 - p)A_{11}$, with algebraic dimension $k_1 - 1$;
- (vi) *The eigenvalues of the matrix $M(l_1, k_1, p; l_2; x, y)$.*

Furthermore, $S(l_1, k_1, p; l_2) = \emptyset$.

We observe that (a) if $l_2 = 0$, then $\lambda[2l]$ is not an eigenvalue and (b) if $l_2 = n_2$, then $\lambda[2m]$ is not an eigenvalue.

Theorem 2.5.2 and Remarks 2.5.1 and 2.5.2 also implies the following unstability properties for the equilibria $E(l_1, k_1, p; l_2, k_2, q; x, y)$.

- (i) Let $A_{11} > 0$ and $A_{22} > 0$. The sum of the eigenvalues of $M(l_1, k_1, p; l_2, k_2, q; x, y)$ is greater than zero. Hence, $U(l_1, k_1, p; l_2, k_2, q) = E(l_1, k_1, p; l_2, k_2, q)$.
- (ii) Let $A_{11} > 0$ and $A_{22} < 0$. (a) If $k_2 \geq 2$ then $\lambda[2r] > 0$. (b) If $k_1 < n_1$ then $\lambda[1l]$ or $\lambda[1m]$ has real part greater than zero. (c) If $k_1 = n_1$ and $k_2 = 0$ then the eigenvalue of $M(l_1, k_1, p; l_2, k_2, q; x, y)$ has real part greater than zero. (d) If $k_1 = n_1$ and $k_2 = 1$ then the sum of the eigenvalues of $M(l_1, k_1, p; l_2, k_2, q; x, y)$ is greater than zero. Hence, $U(l_1, k_1, p; l_2, k_2, q) = E(l_1, k_1, p; l_2, k_2, q)$.
- (iii) Let $A_{11} < 0$ and $A_{22} < 0$.

- (1) If $k_1 \geq 2$ or $k_2 \geq 2$, then $\lambda[1r]$ or $\lambda[2r]$ is positive. Hence,

$$U(l_1, k_1, p; l_2, k_2, q) = E(l_1, k_1, p; l_2, k_2, q) .$$

- (2) If $k_1 = 1$ and $k_2 = 1$, then the eigenvalues $\lambda_{\pm}$ of $M(l_1, k_1, p; l_2, k_2, q; x, y)$ are

$$\lambda_{\pm}^2 = p(1-p)q(1-q)A_{12}A_{21}.$$

Hence,

- (a) if $A_{12}A_{21} > 0$ then $U(l_1, k_1, p; l_2, k_2, q) = E(l_1, k_1, p; l_2, k_2, q)$;
 (b) if $A_{12}A_{21} \leq 0$ then

$$U(l_1, k_1, p; l_2, k_2, q) = S(l_1, k_1, p; l_2, k_2, q) = \emptyset,$$

i.e. the mixed Nash equilibria is neither strongly stable nor

strongly unstable.

(3) if $k_1 = 1$ and $k_2 = 0$, then $M(l_1, k_1, p; l_2, k_2, q; x, y) = (0)$. Hence,

$$U(l_1, k_1, p; l_2, k_2, q) = S(l_1, k_1, p; l_2, k_2, q) = \emptyset,$$

i.e. the mixed Nash equilibria is neither strongly stable nor strongly unstable.

In Figure 2.14, we observe the appearance of periodic cycles for the replicator dynamics. The individuals keep modifying their decisions along time exhibiting a periodic pattern in their decisions.

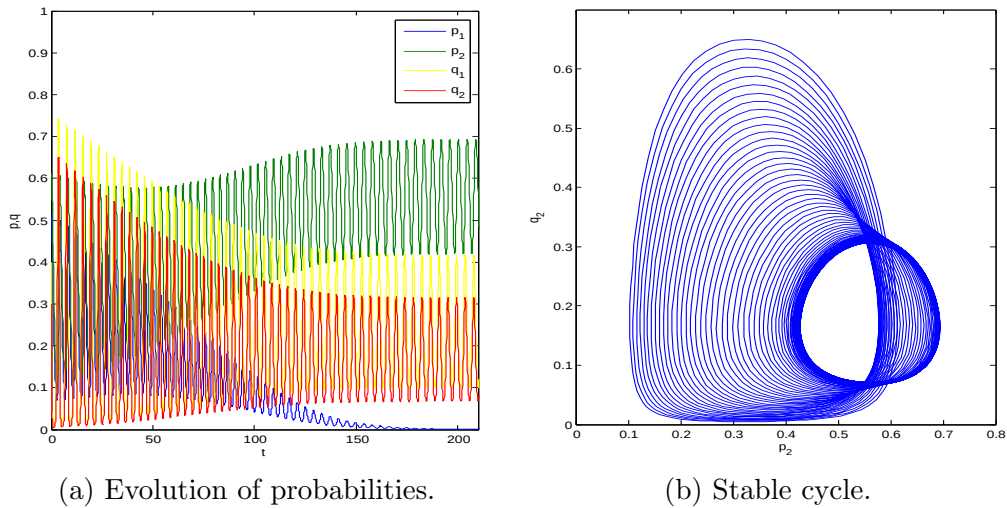


Figure 2.14: The time evolution of the probabilities of individuals under the replicator dynamics form a stable cycle. The parameters of the model were taken as $n_1 = n_2 = 2$, $A_{11} = -0.1$, $A_{12} = 3$, $A_{21} = -10$, $A_{22} = 0$, $x = 0.4$, $y = 0.6$, $p_1(0) = 0.5$, $p_2(0) = 0.6$, $q_1(0) = 0.4$ and $q_2(0) = 0.3$.

2.6 Bifurcations and combinatorial equivalent tilings

In this section, we study the bifurcations in tilings. Recall that the coordinates of the *influence matrix* are given by $A_{tt'} = \alpha_{tt'}^Y + \alpha_{tt'}^N$ for $t, t' \in \{1, 2\}$. The signs of the coordinates of the *influence matrix* determine a certain order for the horizontal and vertical strategic thresholds. Using (2.1), we obtain that

$$\begin{aligned}
 H(N, Y) \leq H(Y, Y) &\Leftrightarrow A_{11} \leq 0 \Leftrightarrow H(N, N) \leq H(Y, N) ; \\
 H(Y, N) \leq H(Y, Y) &\Leftrightarrow A_{12} \leq 0 \Leftrightarrow H(N, N) \leq H(N, Y) ; \\
 V(N, Y) \leq V(Y, Y) &\Leftrightarrow A_{21} \leq 0 \Leftrightarrow V(N, N) \leq V(Y, N) ; \\
 V(Y, N) \leq V(Y, Y) &\Leftrightarrow A_{22} \leq 0 \Leftrightarrow V(N, N) \leq V(N, Y) .
 \end{aligned}$$

Definition 2.6.1 *Let $\mathcal{B}(n_1, n_2)$ be the balanced threshold weight matrix whose coordinates are given by*

$$\begin{aligned}
 B_{11}(n_1, n_2) &= A_{11}(n_1 - 1) - A_{12}n_2, & B_{12}(n_1, n_2) &= A_{11}(n_1 - 1) + A_{12}n_2, \\
 B_{21}(n_1, n_2) &= A_{22}(n_2 - 1) + A_{21}n_1, & B_{22}(n_1, n_2) &= A_{22}(n_2 - 1) - A_{21}n_1.
 \end{aligned}$$

The signs of the *balanced threshold weight matrix* determine a certain order for the horizontal and vertical strategic thresholds. We observe that

$$H(N, Y) \leq H(Y, N) \Leftrightarrow B_{11}(n_1, n_2) \leq 0 ;$$

$$H(N, N) \leq H(Y, Y) \Leftrightarrow B_{12}(n_1, n_2) \leq 0 ;$$

$$V(N, N) \leq V(Y, Y) \Leftrightarrow B_{21}(n_1, n_2) \leq 0 ;$$

$$V(Y, N) \leq V(N, Y) \Leftrightarrow B_{22}(n_1, n_2) \leq 0 .$$

From now on, we will use the simplified notation B_{ij} to refer to $B_{ij}(n_1, n_2)$ for all $i, j \in \{1, 2\}$.

We say that a decision tiling is *structurally stable* if all horizontal and vertical thresholds are pairwise distinct. We say that a decision tiling is a *bifurcation* if there are at least two horizontal thresholds coincide or there are at least two vertical thresholds coincide.

Two decision tilings are *combinatorial equivalent*, if the lexicographic orders of the horizontal and vertical thresholds along the axis are the same in both tilings.

The *parameter space* PS is the set

$$PS = \{\underline{\alpha} = (\alpha_{11}^Y, \alpha_{12}^Y, \alpha_{21}^Y, \alpha_{22}^Y, \alpha_{11}^N, \alpha_{12}^N, \alpha_{21}^N, \alpha_{22}^N; n_1, n_2) \in \mathbb{R}^{10}\} .$$

The *bifurcation parameter space* BPS

$$BPS = \{\underline{\alpha} \in PS : A_{ij} = 0 \vee B_{ij} = 0, \quad \text{with } i, j \in \{1, 2\}\}$$

is the set of all parameters corresponding to bifurcation decision tilings. All parameters, in a same connected component of $PS \setminus BPS$, determine decision tilings that are combinatorial equivalent.

We say that a decision tiling is *structurally horizontal (resp. vertical) stable*, if all the horizontal (resp. vertical) thresholds are pairwise distinct.

A bifurcation is *horizontally (resp. vertically) single* if only two horizontal (resp. vertical) thresholds coincide (see Figure 2.15).

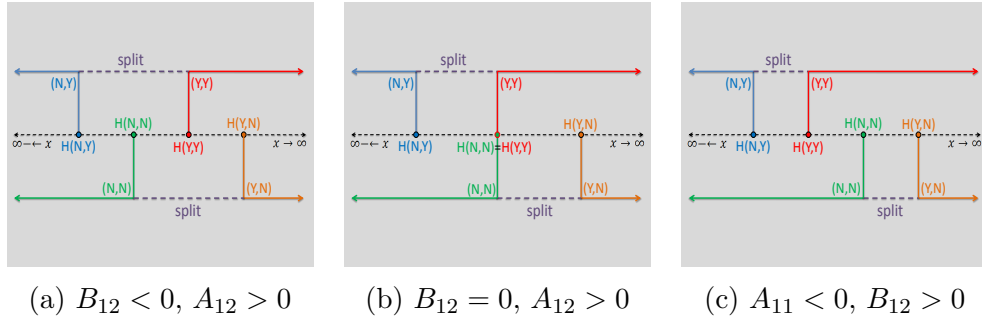


Figure 2.15: Single horizontal bifurcation.

A bifurcation is *horizontally (resp. vertically) double* if only two pairs of horizontal (resp. vertical) thresholds coincide (see Figure 2.16).

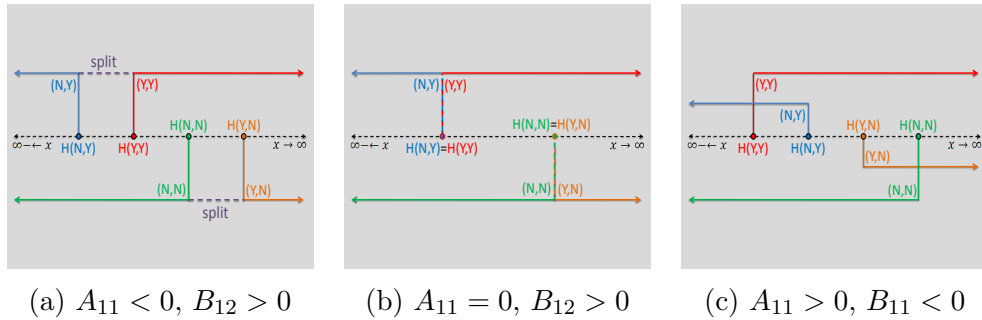


Figure 2.16: Double horizontal bifurcation.

A bifurcation is *horizontally* (resp. *vertically*) *degenerated* if all horizontal (resp. vertical) thresholds coincide (see Figure 2.17).

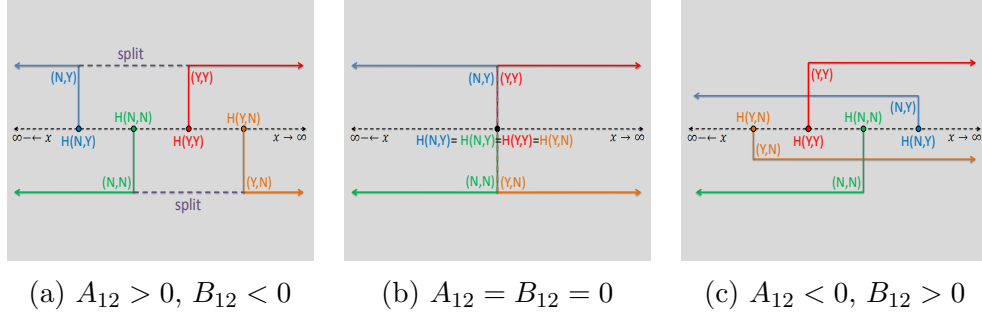


Figure 2.17: Degenerate horizontal bifurcation.

In Figures 2.18 and 2.19, we show a single bifurcations in tilings and in Figure 2.20, we show a double and a degenerate bifurcations in tilings. The regions with only one cohesive Nash equilibria are determined by $U(Y, Y) \subseteq \mathbf{N}(Y, Y)$, $U(Y, N) \subseteq \mathbf{N}(Y, N)$, $U(N, Y) \subseteq \mathbf{N}(N, Y)$ and $U(N, N) \subseteq \mathbf{N}(N, N)$. Furthermore, $U(Y, Y)$, $U(Y, N)$, $U(N, Y)$, and $U(N, N)$ are colored red, orange, blue and green, respectively; regions with two, three and four cohesive Nash equilibria are colored yellow, brown and pink, respectively, and regions without cohesive Nash equilibria are colored purple.

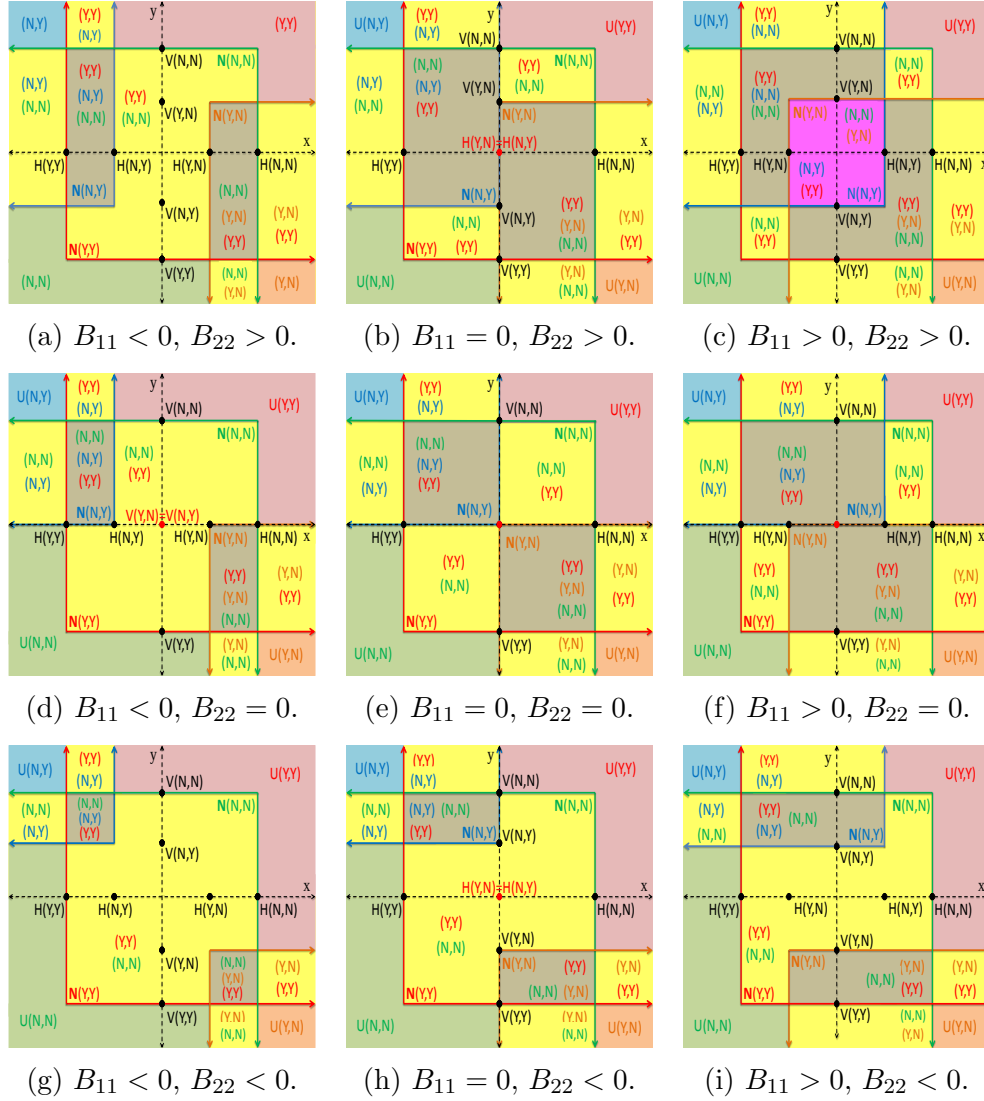


Figure 2.18: Single bifurcations when $A_{11} > 0$ and $A_{22} > 0$. A single horizontal bifurcation occurs in Figures 2.18b and 2.18h. A single vertical bifurcation occurs in Figures 2.18d and 2.18f. A single horizontal and a single vertical bifurcations occur in Figure 2.18e.

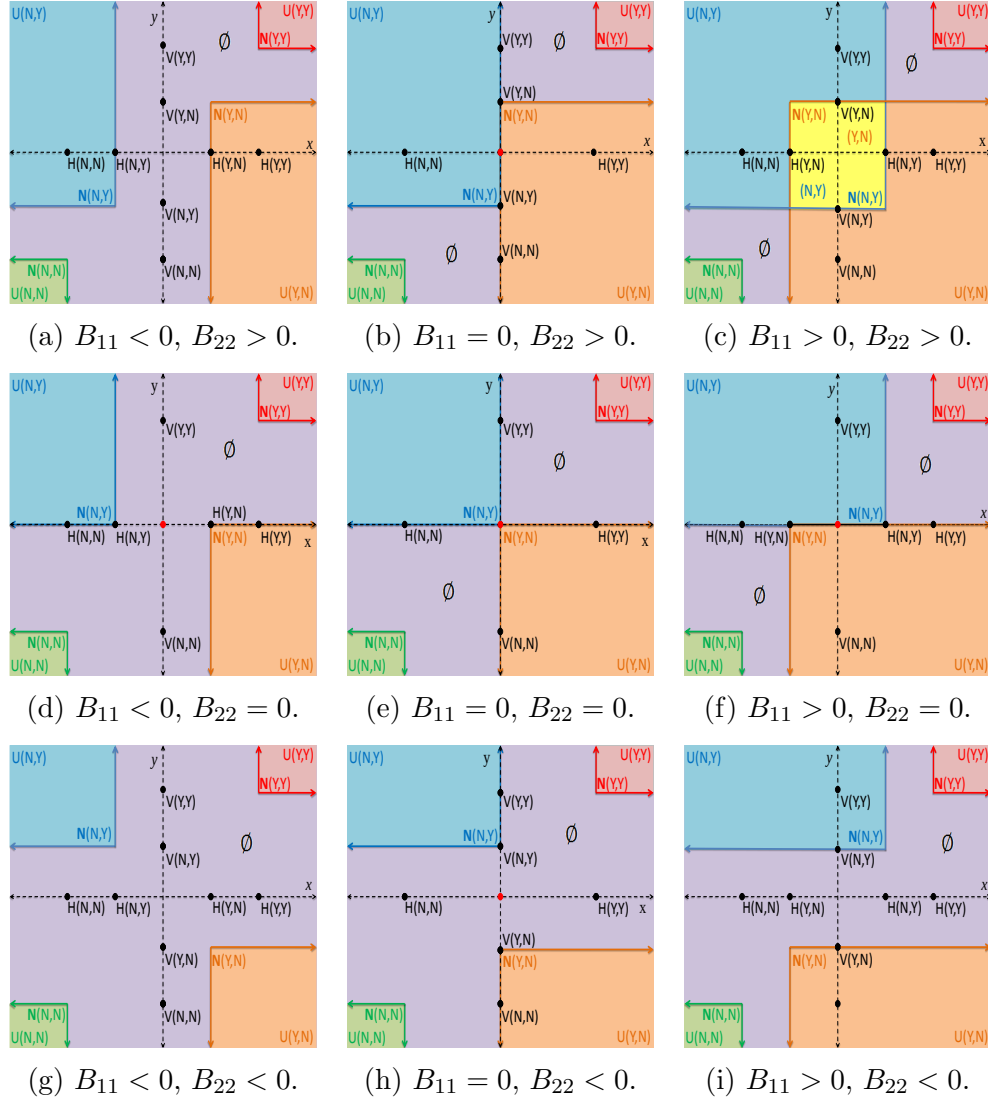


Figure 2.19: Single bifurcations when $A_{11} < 0$ and $A_{22} < 0$. A single horizontal bifurcation occurs in Figures 2.19b and 2.19h. A single vertical bifurcation occurs in Figures 2.19d and 2.19f. A single horizontal and a single vertical bifurcations occur in Figure 2.19e.

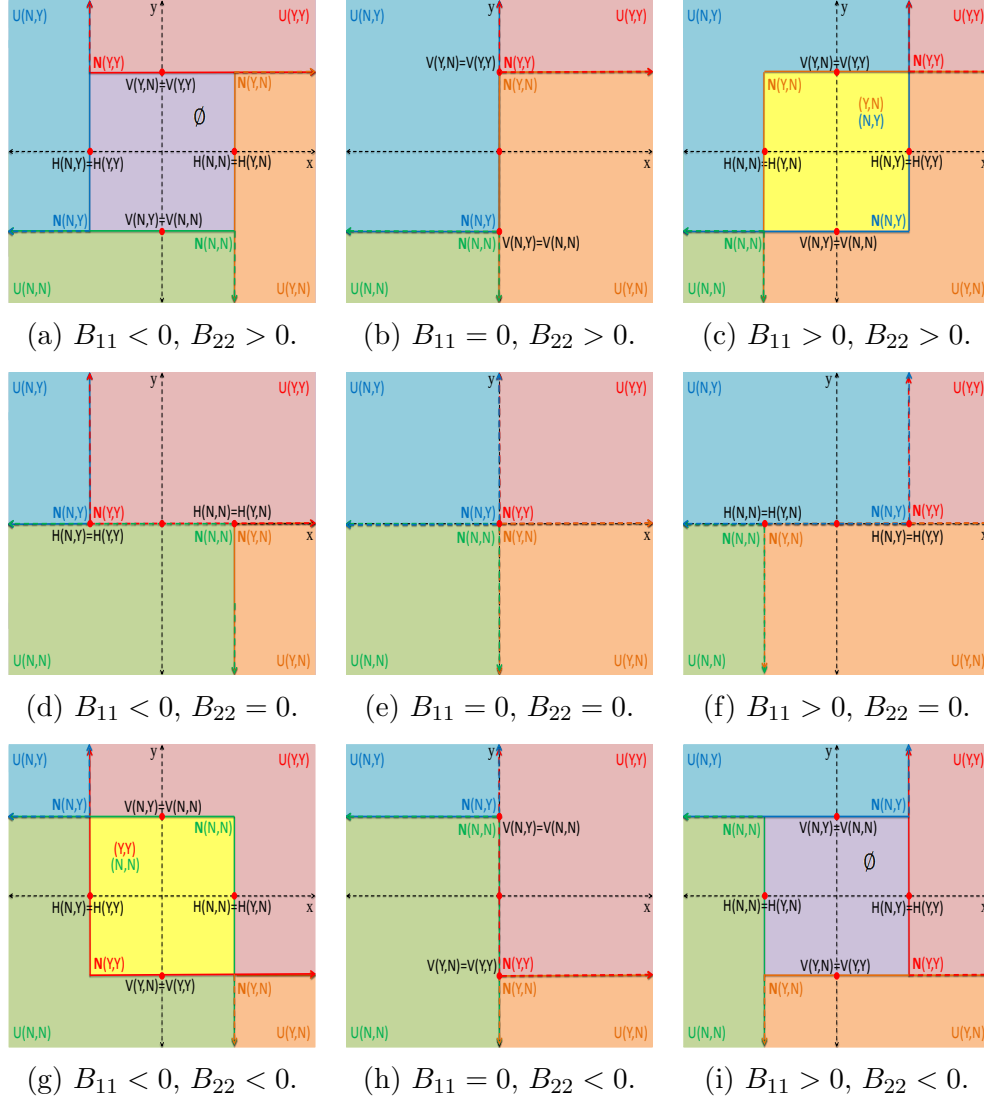


Figure 2.20: Double and a degenerate bifurcations when $A_{11} = 0$ and $A_{22} = 0$. A degenerate horizontal and a double vertical bifurcations occur in Figures 2.20b and 2.20h. A degenerate vertical and a double horizontal bifurcations occur in Figures 2.20d and 2.20f. A degenerate horizontal and a degenerate vertical bifurcations occur in Figure 2.20e. A double horizontal and a double vertical bifurcations occur in Figures 2.20a, 2.20c, 2.20g and 2.20i.

In Figure 2.21, the thresholds $H(Y, Y)$ (resp. $V(Y, Y)$) are marked by the red dots, the thresholds $H(Y, N)$ (resp. $V(N, Y)$) are marked by the orange dots, the thresholds $H(N, Y)$ (resp. $V(Y, N)$) are marked by the blue dots, and the thresholds $H(N, N)$ (resp. $V(N, N)$) are marked by the green dots. There are four horizontal (resp. vertical) thresholds whose order is determined in each direction of the *bussola*.

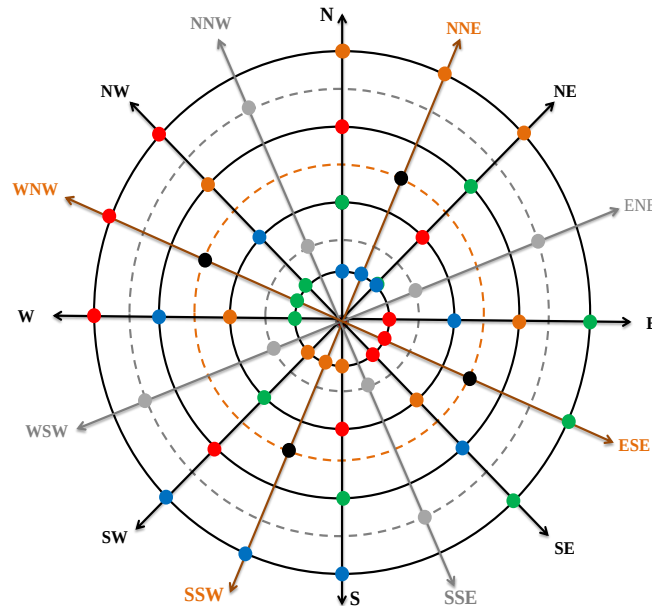


Figure 2.21: Horizontal (or vertical) decision Bussola.

The way the colored thresholds spiral in the Bussola correspond to the way they change with the coordinates of the influence matrix and with the coordinates of the threshold balanced weight matrix. Hence, a pair (d_1, d_2) of directions in the Bussola determine a unique decision tiling, up to combinatorial equivalence, and vice-versa.

The Bussola has the following properties:

- (i) d_1 and d_2 are both in the north side of the Bussola implies the existence of (disparate, disparate) Nash equilibria in the corresponding tiling;
- (ii) d_1 is in the north side and d_2 is in the south side of the Bussola implies the existence of (disparate, cohesive) Nash equilibria in the corresponding tiling;
- (iii) d_1 is in the south side and d_2 is in the north side of the Bussola implies the existence of (cohesive, disparate) Nash equilibria in the corresponding tiling;
- (iv) d_1 and d_2 are both in the south side of the Bussola implies the existence of (cohesive, cohesive) Nash equilibria in the corresponding tiling;
- (v) the orange directions represent the occurrence of single bifurcation;
- (vi) the gray directions represent the occurrence of double bifurcation;
- (vii) the center of the Bussola is $-\infty$.

Note that the four horizontal (resp. vertical) thresholds whose order is determined in each line transversal to the braid horizontal (resp. vertical) braid or, equivalently, by each direction in the horizontal (resp. vertical) Bussola. Each pair of lines transversal to the horizontal and vertical braids, respectively, or, equivalently, each pair of directions in the horizontal and vertical Bussolas, respectively, determines a unique decision tilings, up to combinatorial equivalence, and vice-versa.

We call the pair of horizontal and vertical braids the *human decision chromosomes* because they determine the human decision behavior (see Figure 2.22).

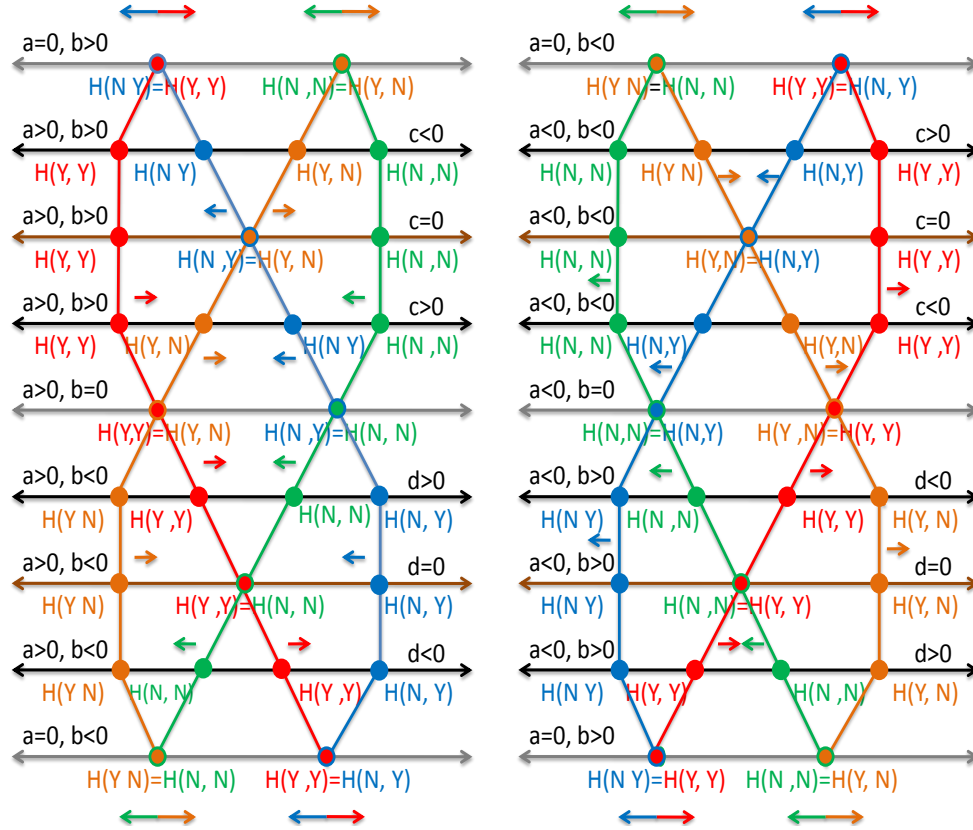


Figure 2.22: Human horizontal decision braid (or chromosomes) with $a = A_{11}$, $b = A_{12}$, $c = B_{11}$, $d = B_{12}$ (resp. vertical decision braid (or chromosomes) with $H(N, N)$, $H(N, Y)$, $H(Y, N)$ and $H(Y, Y)$ replaced by $V(N, N)$, $V(Y, N)$, $V(N, Y)$ and $V(Y, Y)$, respectively, and $a = A_{22}$, $b = A_{21}$, $c = B_{22}$, $d = B_{21}$).

We observe that there are $289 = 17 \times 17$ combinatorial classes of decision tilings described by the decision Bussola (or the decision chromosomes), and $64 = 8 \times 8$ of them are being structurally stable (see Figure 2.23) and 225 combinatorial classes of bifurcation decision tilings. In Figure 2.24, we show a collection of eight tilings that are structurally stable.

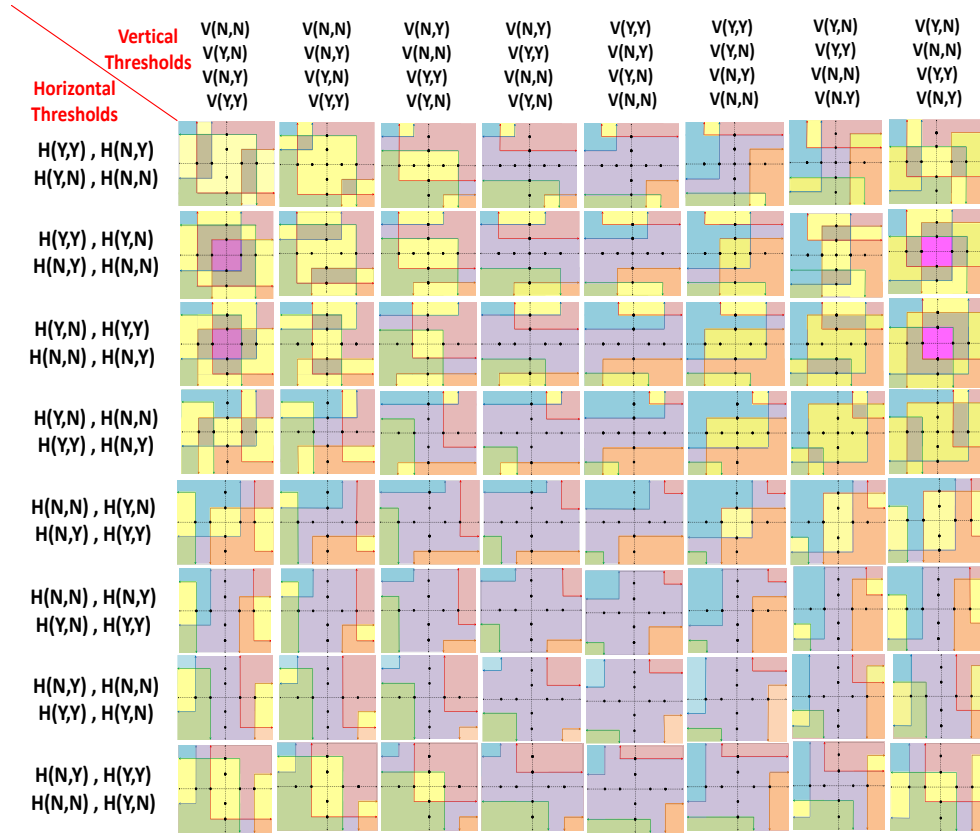


Figure 2.23: 64 Combinatorial classes of tilings that are structurally stable.

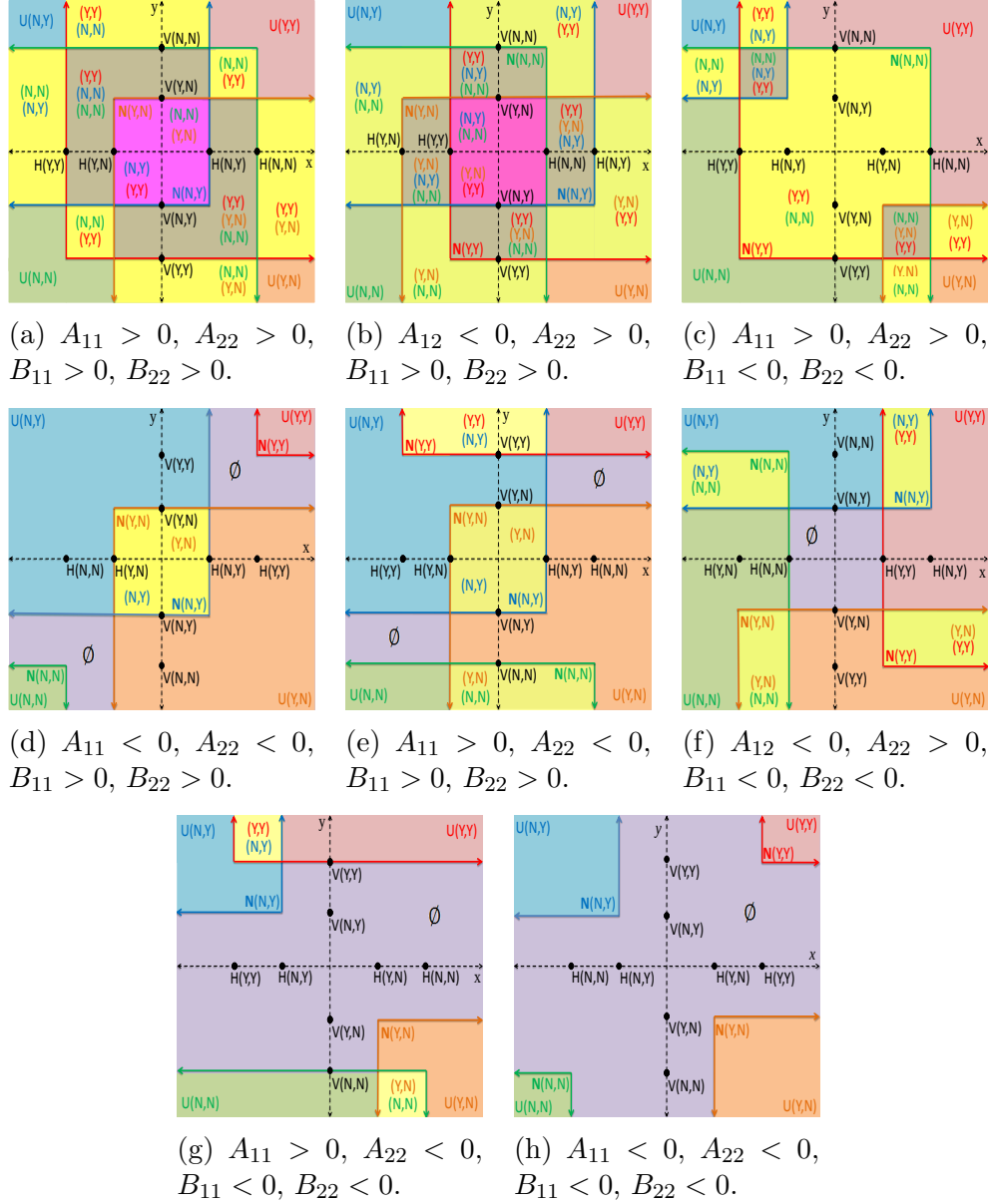


Figure 2.24: A collection of eight structurally stable tilings.

2.7 Leadership Model

A leader is an individual who can influence others to make a certain decision. For simplicity, we assume that the leader will influence other individuals to make decision Y. We study how the choice of the leader can influence the potential followers (individuals of type t_1 and t_2) to make the decision he pretends, see [3].

As in [3], the parameters (P_i, θ_i, L_i) , with $i \in \{1, 2\}$, characterize the leaders and the potential followers as follows:

- (i) *Altruistic, individualist and biased leaders.* The leader donates P_1 to the individuals of type t_1 and P_2 to the individuals of type t_2 . The *altruistic leader*, for the individuals with type t_i , is the one who distributes a valuation to potential followers making the decision Y, i.e. $P_i > 0$; while the *individualist leader*, for the individuals with type t_i , is the one who distributes a devaluation or debt to potential followers making the decision Y, i.e. $P_i < 0$. The *biased leader* is the one who distributes a valuation to one type of potential followers and a debt to the other type of potential followers, i.e. $P_1 P_2 < 0$.
- (ii) *Consumption or wealth creation by the followers.* Define θ_1, θ_2 as the parameters of the consumption or wealth creation on the valuation distributed by the leader to other individuals. Therefore, the new valuation of the individuals, with type t_1 and t_2 , to make decision Y

is given by

$$\omega_i^Y + \frac{P_i \theta_i}{n_i},$$

where ω_i^Y corresponds to the valuation before the influence of the leader of the individuals to make decision Y. There is *wealth creation by the followers* of type t_i when $P_i > 0$ and $\theta_i > 1$ or when $P_i < 0$ and $\theta_i < 1$. There is *wealth consumption by the followers* of type t_i when $P_i > 0$ and $\theta_i < 1$ or when $P_i < 0$ and $\theta_i > 1$.

- (iii) *Influential and persuasive leader.* The influence or persuasiveness of the leader on other individuals is measured by the parameters L_1 and L_2 . The individuals have a new valuation, when they make the decision N, under the influence of the leader, given by

$$\omega_i^N - L_i.$$

If $L_i < 0$, the individuals with type t_i will like more to make the decision that the leader pretends; however, if $L_i > 0$, the individuals with type t_i will like more to make the opposite decision from the one that the leader pretends.

Let the horizontal point $\mathcal{H}$ be given by

$$\bar{\mathcal{H}} = \max\{H(Y, Y), H(Y, N), H(N, Y), H(N, N)\}$$

and the vertical point $\mathcal{V}$ be given by

$$\bar{\mathcal{V}} = \max\{V(Y, Y), V(Y, N), V(N, Y), V(N, N)\}.$$

We now state the following remark.

Remark 2.7.1 *Let S be a pure or mixed Nash equilibrium.*

(i) *If*

$$\frac{\theta_1 P_1}{n_1} + L_1 > \bar{\mathcal{H}} - x ,$$

then individuals with type t_1 make the decision Y .

(ii) *If*

$$\frac{\theta_2 P_2}{n_2} + L_2 > \bar{\mathcal{V}} - y ,$$

then individuals with type t_2 make the decision Y .

As in [3], the inequalities above provide a sufficient condition in the values of the donated parts P_1 and P_2 , in the values of the influence and persuasiveness L_1 and L_2 of the leader and in the values of the consumption or creation of wealth θ_1 and θ_2 by the followers, implying that the potential followers make the same decision as the leader.

2.8 Resort pricing and bankruptcy

In this section, we study a resort pricing model, where different types of tourists choose between different resorts. We study the influence of the resort prices on the choices of the different types of tourists. In particular,

the dependence of the resort prices on the characteristics of the tourists. We characterize the cohesive strategies of the tourists that are Nash equilibria. We find the prices that lead to the bankruptcy of the resorts and, in particular, their dependence on the characteristics of the tourists.

2.8.1 Resort pricing model

The *resort pricing model* has two types $\mathbf{T} = \{t_1, t_2\}$ of tourists $i \in \mathbf{I}$ that have to choose between two goods or services. For instance, the tourists have to choose between spending their holidays in a beach resort B or in a mountain resort M , i.e. $r \in \mathbf{R} = \{B, M\}$. Let $n_q \geq 1$ be the number of tourists with type t_q , $q \in \{1, 2\}$. Let $\mathcal{P}$ be the *price vector* whose *coordinates* $p^r > 0$ indicates the *standard price* of the resort r for each tourist, independently of his type

$$\mathcal{P} = (p^B, p^M).$$

Let $\mathcal{L}$ be the *preference location matrix* whose *coordinates* ω_q^r indicate how much the tourist with type t_q likes ($\omega_q^r > 0$), or dislikes ($\omega_q^r < 0$), to choose resort r

$$\mathcal{L} = \begin{pmatrix} \omega_1^B & \omega_1^M \\ \omega_2^B & \omega_2^M \end{pmatrix}.$$

The preference location matrix indicates for each type, the resort that the tourists prefer, i.e. the tourists taste type (see [15–17]).

Let $\mathcal{N}_r$ be the *preference neighbors matrix* whose *coordinates* $\alpha_{qq'}^r$ indicate how much the tourist with type t_q likes ($\alpha_{qq'}^r > 0$), or dislikes ($\alpha_{qq'}^r < 0$), that tourist with type $t_{q'}$ chooses resort r

$$\mathcal{N}_r = \begin{pmatrix} \alpha_{11}^r & \alpha_{12}^r \\ \alpha_{21}^r & \alpha_{22}^r \end{pmatrix}.$$

The preference neighbors matrix indicates for each type of tourists, whom they prefer, or not, to be with in each resort, i.e. the tourists crowding type (see [15–17]).

We describe the tourists' location by a *strategy map* $S : \mathbf{I} \rightarrow \mathbf{R}$ that associates to each tourist $i \in \mathbf{I}$ its location $S(i) \in \mathbf{R}$. Let $\mathbf{S}$ be the space of all strategies S . Given a strategy S , let $\mathcal{O}_S$ be the *strategic occupation matrix* whose coordinates $l_q^r = l_q^r(S)$ indicate the number of tourists with type t_q who choose resort r

$$\mathcal{O}_S = \begin{pmatrix} l_1^B & l_1^M \\ l_2^B & l_2^M \end{pmatrix}.$$

The *strategic occupation vector* associated to a strategy S is the vector $(l_1, l_2) = (l_1^B(S), l_2^B(S))$. Hence, l_1 (resp. $n_1 - l_1$) is the number of tourists with type t_1 who choose the resort B (resp. M). Similarly, l_2 (resp. $n_2 - l_2$) is the number of tourists with type t_2 who choose the resort B (resp. M). The set $\mathbf{O}$ of all possible *occupation vectors* is

$$\mathbf{O} = \{0, \dots, n_1\} \times \{0, \dots, n_2\}.$$

Let $U_1 : \mathbf{R} \times \mathbf{O} \rightarrow \mathbb{R}$ the *utility function* of the tourist with type t_1 be given by

$$\begin{aligned} U_1(B; l_1, l_2) &= -p^B + \omega_1^B + \alpha_{11}^B(l_1 - 1) + \alpha_{12}^B l_2 ; \\ U_1(M; l_1, l_2) &= -p^M + \omega_1^M + \alpha_{11}^M(n_1 - l_1 - 1) + \alpha_{12}^M(n_2 - l_2) . \end{aligned}$$

Let $U_2 : \mathbf{R} \times \mathbf{O} \rightarrow \mathbb{R}$ the *utility function* of the tourists with type t_2 be given by

$$\begin{aligned} U_2(B; l_1, l_2) &= -p^B + \omega_2^B + \alpha_{22}^B(l_2 - 1) + \alpha_{21}^B l_1 ; \\ U_2(M; l_1, l_2) &= -p^M + \omega_2^M + \alpha_{22}^M(n_2 - l_2 - 1) + \alpha_{21}^M(n_1 - l_1) . \end{aligned}$$

Given a strategy $S \in \mathbf{S}$, the *utility* $U_i(S)$, of the tourist i with type $t_{p(i)}$, is given by $U_{p(i)}(S(i); l_1^{S(i)}(S), l_2^{S(i)}(S))$.

We note that if the price becomes depend on the tourist type, then the prices of the resorts can be encoded in the preference location matrix and therefore, the model can be studied as the yes-no decision model presented in [38].

By Definition 2.1.3, a strategy $S^* : \mathbf{I} \rightarrow \mathbf{R}$ is a *Nash equilibrium* if, for every tourist $i \in \mathbf{I}$ and for every strategy S , with the property that $S^*(j) = S(j)$ for every tourist $j \in I \setminus \{i\}$, we have

$$U_i(S^*) \geq U_i(S) .$$

A *cohesive strategy*⁵ is a strategy in which all tourists, with the same type, prefer to choose the same resort. A *cohesive strategy* is described by a map $C : \mathbf{T} \rightarrow \mathbf{R}$, where for every tourist i with type $t_{q(i)}$, indicates its location $C(q(i))$. Hence, a cohesive strategy $C : \mathbf{T} \rightarrow \mathbf{R}$ determines a unique strategy $S : \mathbf{I} \rightarrow \mathbf{R}$ given by $S(i) = C(q(i))$.

Let $x = \omega_1^B - \omega_1^M$ be the *horizontal relative location preference* of the

⁵or equivalently, *coherent strategy* or *no-split strategy*

tourists with type t_1 and $y = \omega_2^B - \omega_2^M$ be the *vertical relative location preference* of the tourists with type t_2 . Let $p \in \mathbb{R}$ be the *relative price* defined by

$$p = p^B - p^M .$$

Given a pair (x, y) of relative location preferences, the *Nash equilibria prices domain (interval)* $P(R_1, R_2) = P(x, y; R_1, R_2)$ of a cohesive strategy (R_1, R_2) is the set of all relative prices p for which the strategy (R_1, R_2) is a Nash equilibrium.

Our aim is to characterize all Nash equilibria prices interval.

2.8.2 Nash equilibria prices

As in [37], we construct the Nash equilibrium prices interval for the cohesive strategies. We observe that there are four *cohesive* strategies:

- (i) (B, B) *strategy* - all tourists choose the resort B ;
- (ii) (B, M) *strategy* - all tourists with type t_1 choose the resort B and all tourists with type t_2 choose the resort M ;
- (iii) (M, B) *strategy* - all tourists with type t_1 choose the resort M and all tourists with type t_2 choose the resort B ;
- (iv) (M, M) *strategy* - all tourists choose the resort M .

The *horizontal* $H(B, B)$ and *vertical* $V(B, B)$ *strategic thresholds* of the (B, B) strategy are given by

$$H(B, B) = -\alpha_{11}^B(n_1 - 1) - \alpha_{12}^B n_2 \quad \text{and} \quad V(B, B) = -\alpha_{22}^B(n_2 - 1) - \alpha_{21}^B n_1 .$$

The *Nash equilibrium prices interval* $P(B, B)$ is the semi-line given by

$$P(B, B) = \{p \in \mathbb{R} : p \leq x - H(B, B) \text{ and } p \leq y - V(B, B)\} .$$

In Figure 2.25, for a given relative preferences pair (x, y) in the red half-plane, the first coordinate of the blue vector, i.e. the yellow horizontal projection, represents the maximum price in $P(B, B)$; and for a given relative preferences pair (x, y) in the green half-plane, the second coordinate of the blue vector, i.e. the orange vertical projection, represents the maximum

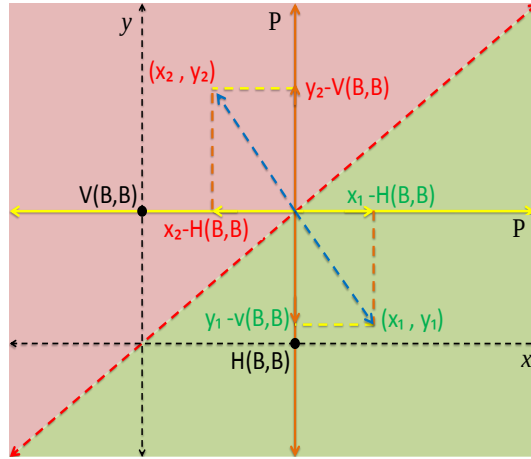


Figure 2.25: Nash equilibria prices domain $P(B, B)$.

price in $P(B, B)$.

The horizontal $H(B, M)$ and vertical $V(B, M)$ *strategic thresholds* of the (B, M) strategy are given by

$$H(B, M) = -\alpha_{11}^B(n_1 - 1) + \alpha_{12}^M n_2 \text{ and } V(B, M) = \alpha_{22}^M(n_2 - 1) - \alpha_{21}^B n_1 .$$

The *Nash equilibrium prices interval* $P(B, M)$ is the segment line (that can be empty) given by

$$P(B, M) = \{p \in \mathbb{R} : p \leq x - H(B, M) \text{ and } p \geq y - V(B, M)\} .$$

In Figure 2.26, for a given relative preferences pair (x, y) in the blue half-plane, the second coordinate of the blue vector, i.e. the orange vertical projection, represents the minimum price in $P(B, M)$ and the first coordinate of the blue vector, i.e. the yellow horizontal projection, represents the

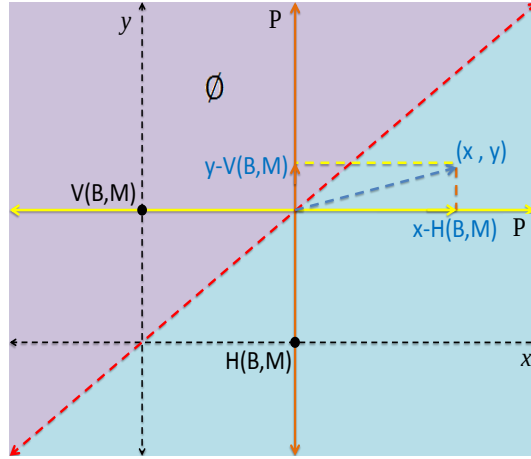


Figure 2.26: Nash equilibria prices domain $P(B, M)$.

maximum price in $P(B, M)$; in the purple half-plane, there are no Nash equilibria prices.

The horizontal $H(M, B)$ and vertical $V(M, B)$ *strategic thresholds* of the (M, B) strategy are given by

$$H(M, B) = \alpha_{11}^M(n_1 - 1) - \alpha_{12}^B n_2 \text{ and } V(M, B) = -\alpha_{22}^B(n_2 - 1) + \alpha_{21}^M n_1 .$$

The *Nash equilibrium prices interval* $P(M, B)$ is the segment line (that can be empty) given by

$$P(M, B) = \{p \in \mathbb{R} : p \geq x - H(M, B) \text{ and } p \leq y - V(M, B)\} .$$

In Figure 2.27, for a given relative preferences pair (x, y) in the blue half-plane, the first coordinate of the blue vector, i.e. the yellow horizontal projection, represents the minimum price in $P(M, B)$ and the second coordinate

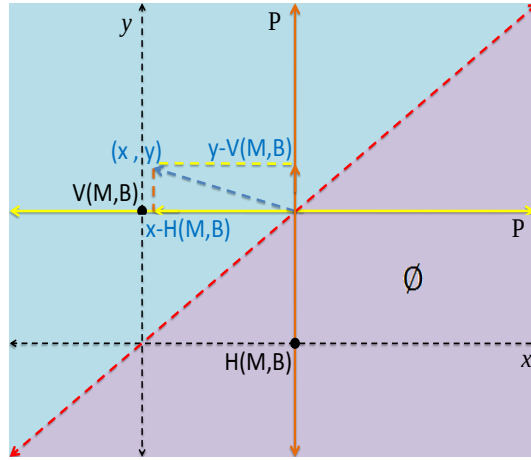


Figure 2.27: Nash equilibria prices domain $P(M, B)$.

of the blue vector, i.e. the orange vertical projection, represents the maximum price in $P(M, B)$; in the purple half-plane, there are no Nash equilibria prices.

The horizontal $H(M, M)$ and vertical $V(M, M)$ strategic thresholds of the (M, M) strategy are given by

$$H(M, M) = \alpha_{11}^M(n_1 - 1) + \alpha_{12}^M n_2 \text{ and } V(M, M) = \alpha_{22}^M(n_2 - 1) + \alpha_{21}^M n_1 .$$

The *Nash equilibrium prices interval* $P(M, M)$ is the semi-line given by

$$P(M, M) = \{p \in \mathbb{R} : p \geq x - H(M, M) \text{ and } p \geq y - V(M, M)\}.$$

In Figure 2.28, for a given relative preferences pair (x, y) in the red half-plane, the first coordinate of the blue vector, i.e. the yellow horizontal projection, represents the minimum price in $P(M, M)$; and for a given rela-

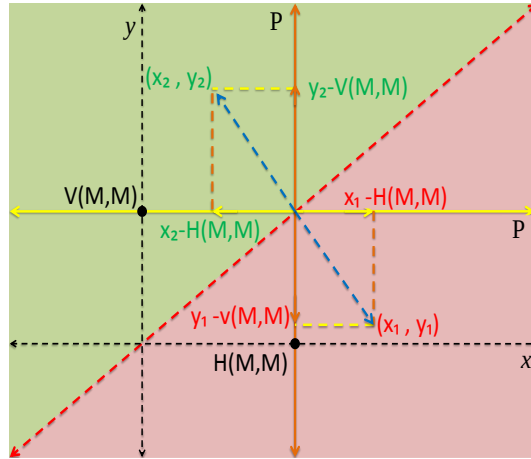


Figure 2.28: Nash equilibria prices domain $P(M, M)$.

tive preferences pair (x, y) in the green half-plane, the second coordinate of the blue vector, i.e. the orange vertical projection, represents the minimum price in $P(M, M)$.

2.8.3 Bankruptcy Nash equilibria prices

Let the *cohesive uniqueness Nash equilibria prices* be the regions $U(B, B) \subset P(B, B)$, $U(B, M) \subset P(B, M)$, $U(M, B) \subset P(M, B)$ and $U(M, M) \subset$

$P(M, M)$, where for every point in these regions, there is a unique cohesive Nash equilibrium. Let

$$\begin{aligned}\bar{\mathcal{H}} &= \max\{H(B, B), H(B, M), H(M, B), H(M, M)\} ; \\ \underline{\mathcal{H}} &= \min\{H(B, B), H(B, M), H(M, B), H(M, M)\} ; \\ \bar{\mathcal{V}} &= \max\{V(B, B), V(B, M), V(M, B), V(M, M)\} ; \\ \underline{\mathcal{V}} &= \min\{V(B, B), V(B, M), V(M, B), V(M, M)\} .\end{aligned}$$

Define the following subsets:

$$\begin{aligned}\mathcal{U}(B, B) &= \{p \in \mathbb{R} : p \leq x - \bar{\mathcal{H}} \text{ and } p \leq y - \bar{\mathcal{V}}\} \subseteq U(B, B) ; \\ \mathcal{U}(B, M) &= \{p \in \mathbb{R} : p \leq x - \bar{\mathcal{H}} \text{ and } p \geq y - \underline{\mathcal{V}}\} \subseteq U(B, M) ; \\ \mathcal{U}(M, B) &= \{p \in \mathbb{R} : p \geq x - \underline{\mathcal{H}} \text{ and } p \leq y - \bar{\mathcal{V}}\} \subseteq U(M, B) ; \\ \mathcal{U}(M, M) &= \{p \in \mathbb{R} : p \geq x - \underline{\mathcal{H}} \text{ and } p \geq y - \underline{\mathcal{V}}\} \subseteq U(M, M) .\end{aligned}$$

We call the prices in $\mathcal{U}(B, B)$ the *bankruptcy* Nash equilibria prices of the mountain resort M , because for every price in $\mathcal{U}(B, B)$, there are no tourists choosing the mountain resort M . Similarly, we call the prices in $\mathcal{U}(M, M)$ the *bankruptcy* Nash equilibria prices of the beach resort B , because for every price in $\mathcal{U}(M, M)$ there are no tourists choosing the beach resort B . We call the prices in $\mathcal{U}(B, M)$ and $\mathcal{U}(M, B)$ the *competitive business* Nash equilibria prices, because for every price in $\mathcal{U}(B, M)$ and in $\mathcal{U}(M, B)$, one type of tourists choose the beach resort B and the other type of tourists choose the mountain resort M .

2.9 Cohesive decision in an extended model

In this section, we introduce a more general framework for the decision model, where individuals will have to make decisions according to their preferences that also depend upon the other individuals' decisions. In this model, there are N types $t \in \{t_1, t_2, \dots, t_N\}$ of individuals and M possible decisions $d \in \{d_1, d_2, \dots, d_M\}$ that individuals can make. We fully characterize the space of all parameters where the Nash equilibria are cohesive, i.e. individuals with similar preferences make the same decisions. We prove that small changes in the parameters can imply the appearance or disappearance of cohesive Nash equilibria. These abrupt changes in the structure of the cohesive Nash equilibria can be used to characterize economical, political or social revolutions.

2.9.1 Extended decision model

The *decision model* has N types $\mathbf{T} = \{t_1, \dots, t_N\}$ of individuals. Let $\mathbf{I}_t = \{1, \dots, n_t\}$ be the set of all individuals with type $t \in \mathbf{T}$. Let

$$\mathbf{I} = \bigsqcup_{t=1}^N \mathbf{I}_t .$$

Each individual $i \in \mathbf{I}$ has to make one decision $d \in \mathbf{D} = \{d_1, \dots, d_M\}$. Let $t : \mathbf{I} \rightarrow \mathbf{T}$ be the *type function* that associates to each individual $i \in \mathbf{I}$ his type $t(i) \in \mathbf{T}$.

We describe the pure decision of the individuals by a *strategy map* $S : \mathbf{I} \rightarrow \mathbf{D}$ that associates to each individual $i \in \mathbf{I}$ his decision $S(i) \in \mathbf{D}$. Let $\mathbf{S}$

be the space of all strategies S .

Let Ω be the *preference decision matrix* whose coordinates ω_t^d indicate how much an individual i with type $t = t(i)$ likes ($\omega_t^d > 0$) or dislikes ($\omega_t^d < 0$), to make decision $d = S(i)$. The preference decision coordinates indicate for each type of individuals the decision that the individuals prefer, i.e. the taste type of the individuals (see [2, 15, 17, 36, 38]). Hence, the *preference decision matrix* $\Omega \in \mathbb{R}^{N \times M}$ is given by

$$\Omega = \begin{pmatrix} \omega_1^1 & \dots & \omega_1^M \\ \vdots & \ddots & \vdots \\ \omega_N^1 & \dots & \omega_N^M \end{pmatrix}.$$

Let $\Lambda(d) \in \mathbb{R}^{N \times N}$ be the *preference neighbors matrix* whose coordinates $\alpha_{tt'}^d$ indicate how much an individual with type t who decides d likes ($\alpha_{tt'}^d > 0$) or dislikes ($\alpha_{tt'}^d < 0$) that an individual with type t' makes decision d . The preference neighbors coordinates indicate, for each type of individuals who make decision d , whom they prefer, or do not prefer, to share that decision with, i.e. the crowding type of the individuals (see [2, 15, 17, 36, 38]).

Given a strategy $S \in \mathbf{S}$, let $L(S)$ be the *strategic decision matrix* whose coordinates $l_t^d = l_t^d(S)$ indicate the number of individuals with type t who make decision d

$$L(S) = \begin{pmatrix} l_1^1 & \dots & l_1^M \\ \vdots & \ddots & \vdots \\ l_N^1 & \dots & l_N^M \end{pmatrix}.$$

Let $\mathbf{L} = \{L(S) \in \mathbb{R}^{N \times M} : S \in \mathbf{S}\}$ be the set of all strategic decision matrices $L(S)$.

Let the *auxiliar utility function* $\mathcal{U} : \mathbf{T} \times \mathbf{D} \times \mathbf{L} \rightarrow \mathbb{R}$ be given by

$$\mathcal{U}(t; d, L) = \omega_t^d - \alpha_{tt}^d + \sum_{t'=1}^N \alpha_{tt'}^d l_{t'}^d .$$

The *utility function* $U : \mathbf{I} \times \mathbf{S} \rightarrow \mathbb{R}$ of an individual $i \in \mathbf{I}$, with type $t(i) \in \mathbf{T}$, making decision $S(i) \in \mathbf{D}$ is given by

$$U(i; S) = \mathcal{U}(t(i); S(i); L(S)) .$$

We observe that if two individuals i and j with the same type $t(i) = t(j) = t$ making the same decision $S(i) = S(j) = d$, then their utilities $U(i; S)$ and $U(j; S)$ are equal, i.e.

$$U(i; S) = U(j; S) = \mathcal{U}(t; d, L(S)) .$$

Given a strategy $S^* \in \mathbf{S}$. For every $i \in \mathbf{I}$ and $d \in \mathbf{D} \setminus \{S^*(i)\}$, define the strategy $S_{i \rightarrow d}^*$ by $S_{i \rightarrow d}^*(i) = d$ and $S_{i \rightarrow d}^*(j) = S^*(j)$ for every $j \in \mathbf{I} \setminus \{i\}$.

By Definition 2.1.3, a strategy $S^* : \mathbf{I} \rightarrow \mathbf{D}$ is a (*pure*) *Nash equilibrium* if

$$U(i; S^*) \geq U(i; S_{i \rightarrow d}^*)$$

for every $i \in \mathbf{I}$ and $d \in \mathbf{D} \setminus \{S^*(i)\}$. The *Nash domain* $\mathbf{N}(S)$ of a strategy $S \in \mathbf{S}$ is the set of all preference decision matrices $\Omega \in \mathbb{R}^{N \times M}$ with the property that S is a Nash equilibrium.

We define the *relative decision preference coordinate* $x(t; d, d')$ for an individual i with type $t = t(i)$ by

$$x(t; d, d') = \omega_t^d - \omega_t^{d'} \quad (2.10)$$

for every $d, d' \in \mathbf{D}$ with $d \neq d'$.

Given a strategy $S \in \mathbf{S}$. For every individual $i \in \mathbf{I}$ with type $t = t(i)$ and for every $d \in \mathbf{D} \setminus \{S(i)\}$, the *preference threshold* $T(i \rightarrow d)$ of the relative decision preference coordinate $x(t; d, S(i))$ is defined by

$$T(i \rightarrow d) = T(t(i); S(i), d) = -\alpha_{tt}^{S(i)} + \sum_{t'=1}^N \left(\alpha_{tt'}^{S(i)} l_{t'}^{S(i)} - \alpha_{tt'}^d l_{t'}^d \right). \quad (2.11)$$

2.9.2 Cohesive Nash equilibria

A *cohesive strategy*⁶ $S : \mathbf{I} \rightarrow \mathbf{D}$ is a strategy in which all individuals with the same type prefer to make the same decision. We observe that there are $N \times M$ cohesive strategies.

We now state the following main result.

Theorem 2.9.1 *A cohesive strategy S is a Nash equilibrium if, and only if,*

$$x(t(i); d, S(i)) \leq T(i \rightarrow d) \quad (2.12)$$

for every individual $i \in \mathbf{I}$ and every decision $d \in \mathbf{D} \setminus \{S(i)\}$.

Therefore, the Nash domain $\mathbf{N}(S)$ of a cohesive strategy S is non-empty and coincides with the half-hyper-plan consisting of all the preference deci-

⁶or equivalently *herd strategy*

sion matrices in the space $\mathbb{R}^{N \times M}$ satisfying inequalities (2.12) (see Figure 2.29).

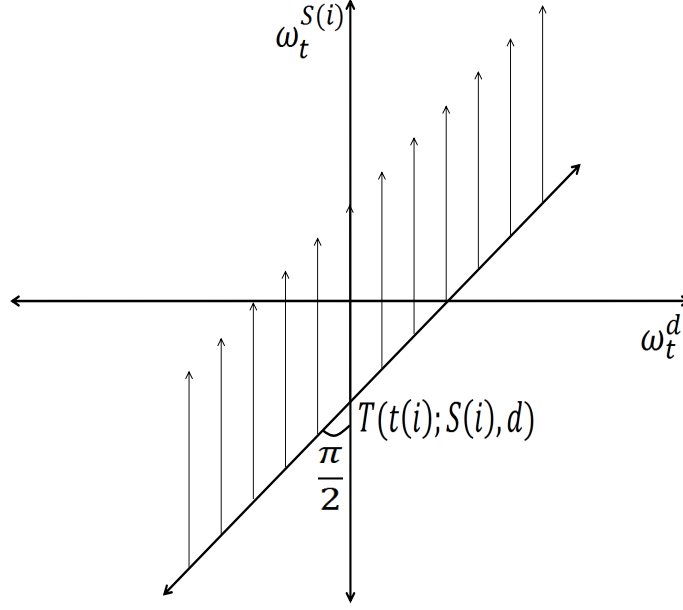


Figure 2.29: The cohesive Nash domain.

Proof. A cohesive strategy S is a Nash equilibrium if, and only if, for every individual $i \in \mathbf{I}$ and every decision $d \in \mathbf{D} \setminus \{S(i)\}$,

$$U(i; S) \geq U(i; S_{i \rightarrow d}).$$

Let $c = S(i)$ and $t = t(i)$ and note that $l_t^c = n_t$ and $l_t^d = 0$. Substituting the values in the utilities of the last inequality, we obtain

$$\omega_t^c + \alpha_{tt}^c (n_t - 1) + \sum_{\substack{t'=1 \\ t' \neq t}}^N \alpha_{tt'}^c l_{t'}^c \geq \omega_t^d + \sum_{\substack{t'=1 \\ t' \neq t}}^N \alpha_{tt'}^d l_{t'}^d.$$

Rearranging the terms of the last inequality, we get

$$\omega_t^d - \omega_t^c \leq -\alpha_{tt}^c + \sum_{t'=1}^N (\alpha_{tt',l_{t'}}^c - \alpha_{tt',l_{t'}}^d) ,$$

Combining the last inequality with (2.10) and (2.11), we conclude the proof.

■

2.10 Conclusions

The union of the equilibria $E(l_1, k_1, p; l_2, k_2, q)$ form hystereses. The equilibria $U(l_1, k_1, p; l_2, k_2, q)$ are the unstable part of the hystereses and the equilibria $S(l_1; l_2)$ are the stable part of the hystereses. The equilibria in $E(l_1, k_1, p; l_2, k_2, q) \setminus \{S(l_1, k_1, p; l_2, k_2, q), U(l_1, k_1, p; l_2, k_2, q)\}$ can be stable or unstable equilibria. Hence, small changes in the coordinates of the influence matrix that determines the equilibria sets $U(l_1, k_1, p; l_2, k_2, q)$ and $S(l_1; l_2)$ can create or annihilate cohesive and disparate Nash equilibria giving rise to abrupt and collective changes in the decisions of the individuals that are explained by the hystereses. Furthermore, we observed the appearance of periodic attracting cycles and so the individuals can keep changing their decisions with a periodic pattern. We studied all possible bifurcations in tilings which results in 289 of combinatorial classes of decision tilings, described by the decision Bussola which demonstrates the high complexity of making decision. We have presented an extended version of the decision model studied in [32] by determining all the cohesive Nash equilibria where individuals with the same type (belonging to the same group) make the same

decision. We found the cohesive thresholds that determine the space of all parameters where the cohesive strategies are Nash equilibria. Since small changes in the parameters might imply the appearance and disappearance of cohesive Nash equilibria, the decision model can be used to characterize and to understand better social, economical and political revolutions. We demonstrated how the characteristics of the leader can have a positive or negative influence over the decisions of the individuals. In particular, we show that an individualist leader might have to be more persuasive than an altruistic leader to convince the individuals to make a particular decision. We have presented a resort pricing model in which small changes in the the coordinates of the preference location matrix, which indicates the resort that the tourists prefer, and of the preference neighbors matrix, which indicates whom the tourists prefer to be with in each resort, can create and annihilate Nash equilibrium prices domain. We found the prices that lead to the bankruptcy of the resorts. We have characterized the cohesive strategies of the tourists that are Nash equilibria.

Chapter 3

A family of stochastic optimal control problems with multiple random time horizons

We consider a family of stochastic optimal control problems with the property that the objective functional depends on multiple random time horizons, assumed to be independent and identically distributed continuous random variables. The state variable follows a stochastic differential equation driven by a standard multi-dimensional Brownian motion. We resort to the concept of order statistics to restate the stochastic optimal control problem as one with a fixed planning horizon. For this class of stochastic optimal control problems, we derive a sequence of dynamic programming principles and the corresponding Hamilton-Jacobi-Bellman equations. We conclude with an application to a consumption-investment problem. In this

chapter, we survey in part, the work presented in [33].

3.1 Framework and main results

Let $K \in \mathbb{N}$ be fixed and $T > 0$ be a finite time horizon. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\}$ given by the $\mathbb{P}$ -augmentation of the filtration generated by the M -dimensional Brownian motion $W(t)$, $\sigma\{W(s), s \leq t\}$ for $t \geq 0$.

We denote by $L_{\mathcal{F}_t}^1([0, T]; \mathbb{R})$ the set of all $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted $\mathbb{R}$ -valued processes $x(\cdot)$ such that

$$E \left[\int_0^T |x(t)| \, dt \right] < \infty$$

and by $L_{\mathcal{F}_T}^1(\Omega; \mathbb{R})$ the set of $\mathbb{R}$ -valued $\mathcal{F}_T$ -measurable random variables X such that $E[|X|]$ is finite. The spaces $L_{\mathcal{F}_t}^1([0, T]; \mathbb{R})$ and $L_{\mathcal{F}_T}^1(\Omega; \mathbb{R})$ are defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

Assume that $\tau_1, \tau_2, \dots, \tau_K$ are non-negative independent and identically distributed continuous random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with distribution function $F : [0, \infty) \rightarrow [0, 1]$ and density $f : [0, \infty) \rightarrow \mathbb{R}^+$. Moreover, assume that the random variables τ_i are completely independent of the filtration $\mathbb{F}$ for all $i \in \{1, 2, \dots, K\}$.

Let $\tau_{(1)} \leq \tau_{(2)} \leq \dots \leq \tau_{(K)}$ be the order statistics associated with the random variables $\tau_1, \tau_2, \dots, \tau_K$. Define the following finite sequence of times:

- (i) the sequence $(\xi_i)_{i=1}^K$ is given by

$$\xi_i = \min\{\tau_i, T\}, \quad i = 1, 2, \dots, K.$$

(ii) the sequence $(t_i)_{i=0}^K$ is given by $t_0 = 0$ and

$$t_i = \min\{\tau_{(i)}, T\}, \quad i = 1, 2, \dots, K .$$

Note that the sequences (ξ_i) and (t_i) are both non-negative and bounded above by the deterministic horizon T . However, unlike the sequence of random variables (ξ_i) , the sequence (t_i) defined above is monotone increasing, i.e. we have that

$$0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_K \leq T .$$

Moreover, we note that either $t_K < T$ or there exist $k^* \in \{1, 2, \dots, K\}$ such that for every $k \in \{1, 2, \dots, K\}$ with the property that $k \geq k^*$, we have that $t_k = T$.

Using the sequence (t_i) , we introduce the sequence of random intervals Δ_k as being given by

$$\Delta_k = [t_{k-1}, t_k) , \quad k \in \{1, 2, \dots, K\} . \quad (3.1)$$

We remark that for every $k \in \{1, 2, \dots, K\}$ with the property that $t_{k-1} < T$, the random intervals Δ_k are non-empty with probability one.

We now introduce some technical assumptions:

(A1) (U, d) is a Polish space¹.

¹ (U, d) is Polish space if it is a separable completely metrizable topological space.

(A2) For all $k = 1, 2, \dots, K$, the maps

$$\begin{aligned} f_k & : [0, T] \times \mathbb{R}^N \times U \rightarrow \mathbb{R}^N , \\ g_k & : [0, T] \times \mathbb{R}^N \times U \rightarrow \mathbb{R}^{N \times M} , \\ \Psi_k & : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R} , \\ L_k & : [0, T] \times \mathbb{R}^N \times U \rightarrow \mathbb{R} \end{aligned}$$

are uniformly continuous with respect to all its variables, Lipschitz continuous with respect to the variable x , and bounded when restricted to $x = 0$.

Let $f : [0, t_K] \times \mathbb{R}^N \times U \rightarrow \mathbb{R}^N$ and $g : [0, t_K] \times \mathbb{R}^N \times U \rightarrow \mathbb{R}^{N \times M}$ be the functions defined through the relations

$$f(t, x, u) = f_k(t, x, u) \quad \text{and} \quad g(t, x, u) = g_k(t, x, u)$$

for every $(t, x, u) \in \Delta_k \times \mathbb{R}^N \times U$, extended by continuity to $t = t_K$. Consider the stochastic controlled system

$$\begin{aligned} dx(t) & = f(t, x(t), u(t)) dt + g(t, x(t), u(t)) dW(t) , \quad t \in [0, t_K] , \\ x(0) & = x_0 , \end{aligned} \tag{3.2}$$

where $x(t)$ is a state trajectory in $\mathbb{R}^N$ with the corresponding control $u(t) \in U$.

We consider an objective functional of the form

$$J(u(\cdot)) = E \left[\sum_{i=1}^K \left(\int_0^{\xi_i} L_i(t, x(t), u(t)) dt + \Psi_i(\xi_i, x(\xi_i)) \right) \right],$$

where $L_i(\cdot, x(\cdot), u(\cdot)) \in L^1_{\mathcal{F}_t}([0, t_K]; \mathbb{R})$ and $\Psi_i(\xi_i, x(\xi_i)) \in L^1_{\mathcal{F}_{\xi_i}}(\Omega; \mathbb{R})$, $i = 1, 2, \dots, K$. We remark that the Lagrangian functions L_i do not depend on any random variable $\xi_1, \xi_2, \dots, \xi_K$ but are integrated over random time intervals $[0, \xi_i]$, and that the terminal costs Ψ_i are evaluated at the instants of time $\xi_1, \xi_2, \dots, \xi_K$.

A function $u : [0, t_K] \times \Omega \rightarrow U$ is said to be a strong admissible control if:

- (i) u is measurable and $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted;
- (ii) the stochastic differential equation (3.2) has a strong solution.

We denote the set of all strong admissible controls by $\mathcal{U}^s[0, T]$.

The problem we are going to address in this chapter is to find a control $u \in \mathcal{U}^s[0, T]$ which maximizes the objective functional $J(u(\cdot))$ subject to the state equation (3.2) over the set of admissible controls $\mathcal{U}^s[0, T]$. We will use dynamic programming techniques to address the maximization problem described above. In order to proceed, we need to consider the weak formulation of the stochastic control problem under consideration as an auxiliary tool.

For each $t \in [0, t_K)$, let $I(t)$ be the set of indices given by

$$I(t) = \{i \in \{1, 2, \dots, K\} : \tau_i > t\}$$

and let $\kappa : [0, t_K) \rightarrow \{1, 2, \dots, K\}$ be the function assigning to each $t \in [0, t_K)$ the cardinality of the set $I(t)$. Clearly, $\kappa(t)$ is integer valued, non-increasing and such that $\kappa(0) = K$ a.s.. For any $(s, y) \in [0, t_K) \times \mathbb{R}^N$, consider the state equation:

$$\begin{aligned} dx(t) &= f(t, x(t), u(t)) dt + g(t, x(t), u(t)) dW(t), \quad t \in [s, t_K], \\ x(s) &= y, \end{aligned} \quad (3.3)$$

along with the objective functional

$$J(s, y; u(\cdot)) = E \left[\sum_{i \in I(s)} \int_s^{\xi_i} L_i(t, x(t), u(t)) dt + \Psi_i(\xi_i, x(\xi_i)) \right]. \quad (3.4)$$

Note that the objective functional $J(s, y; u(\cdot))$ in (3.4) depends only on the random variables $\tau_1, \tau_2, \dots, \tau_K$ with values strictly greater than the initial time s . Moreover, note that in (3.3) the initial state y is (almost surely) a deterministic variable under $(\Omega, \mathcal{F}, \mathbb{P})$ and that in (3.4) the mathematical expectation E is taken with respect to the probability measure $\mathbb{P}$.

For each $s \in [0, t_K)$, we denote by $\mathcal{U}^w[s, T]$ the set of all 5-tuples $(\Omega, \mathcal{F}, \mathbb{P}, W(\cdot), u(\cdot))$ for which the following conditions hold:

- (i) $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space;
- (ii) $\{W(t)\}_{t \geq s}$ is an M -dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$ over $[s, T]$, and $\mathcal{F}_t^s$ is the filtration generated by $\{W(t)\}_{t \geq s}$, $\sigma\{W(r) : s \leq r \leq t\}$, augmented by all the $\mathbb{P}$ -null sets in $\mathcal{F}$;
- (iii) $u : [s, t_K] \times \Omega \rightarrow U$ is an $\{\mathcal{F}_t^s\}_{t \geq s}$ -adapted process on $(\Omega, \mathcal{F}, \mathbb{P})$;

- (iv) under $u(\cdot)$, for any $y \in \mathbb{R}^N$ equation (3.3) admits a unique solution $x(\cdot)$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t^s\}_{t \geq s}, \mathbb{P})$.

We call $\mathcal{U}^w[s, T]$ the set of weak admissible controls.

The optimal control problem under consideration can be restated in dynamic programming form. For a given pair $(s, y) \in [0, t_K) \times \mathbb{R}^N$, find $\bar{u}(\cdot) \in \mathcal{U}^w[s, T]$ such that

$$J(s, y; \bar{u}(\cdot)) = \sup_{u(\cdot) \in \mathcal{U}^w[s, T]} J(s, y; u(\cdot)) . \quad (3.5)$$

Note that the assumptions (A1)-(A2) introduced above require U to be a complete metric space and the functions involved to be continuous in (t, x, u) . Moreover, under (A1)-(A2), for any $(s, y) \in [0, t_K) \times \mathbb{R}^N$ and $u(\cdot) \in \mathcal{U}^w[s, T]$, the SDE (3.3) admits a unique solution $x(\cdot) = x(\cdot; s, y, u(\cdot))$, which is a strong Markov process (see [27, 31, 35]). Thus, the *value function* is well-defined by

$$V(s, y) = \sup_{u(\cdot) \in \mathcal{U}^w[s, T]} J(s, y; u(\cdot)) , \quad (s, y) \in [0, t_K) \times \mathbb{R}^N . \quad (3.6)$$

Using the definitions of the sequence (t_i) , the index set $I(t)$ and the map $\kappa(t)$, the objective functional $J(s, y; u(\cdot))$ in (3.4) can be rewritten as

$$J(s, y; u(\cdot)) = E \left[\frac{1}{\kappa(s)} \sum_{i=K+1-\kappa(s)}^K \sum_{j \in I(s)} \int_s^{t_i} L_j(t, x(t), u(t)) dt + \Psi_j(t_i, x(t_i)) \right] . \quad (3.7)$$

The previous identity is the content of Lemma 3.2.4.

The solution to the optimal control problem (3.5) can be constructed in an iterative fashion. Note first that the functional (3.7) and the controlled system (3.2) change at the instants of time (t_i) . Moreover, recall that either $t_K < T$ or there exist $k^* \in \{1, 2, \dots, K\}$ such that for every $k \in \{1, 2, \dots, K\}$ with the property that $k \geq k^*$, we have that $t_k = T$. Then, for every $k \in \{1, 2, \dots, K\}$ with the property that $t_{k-1} < T$ and for every $s \in \Delta_k$, the functional (3.7) reduces to

$$J^k(s, y; u(\cdot)) = E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \left(\int_s^{t_i} L_j(t, x(t), u(t)) dt + \Psi_j(t_i, x(t_i)) \right) \middle| s \in \Delta_k \right].$$

We use the sequence of functionals J^k to obtain a dynamic programming principle for the value function defined in (3.6) in each set $\Delta_k \times \mathbb{R}^N$.

Before proceeding to the precise statement of the dynamic programming principle, we need to introduce some more notation. For every $0 \leq s \leq t$ and every $i, k \in \{1, 2, \dots, K\}$ such that $i \geq k$, let $G_{\tau(i)|\tau(k)}^\pm(t, s)$ denote the conditional probabilities

$$\begin{aligned} G_{\tau(i)|\tau(k)}^+(t, s) &= P\left(\{\tau(i) > t\} \mid \{\tau(k) > s\}\right) \\ G_{\tau(i)|\tau(k)}^-(t, s) &= P\left(\{\tau(i) \leq t\} \mid \{\tau(k) > s\}\right) \end{aligned}$$

and let $g_{\tau(i)|\tau(k)}^-(t, s)$ denote the density function of $G_{\tau(i)|\tau(k)}^-(t, s)$, given by

$$g_{\tau(i)|\tau(k)}^-(t, s) = \frac{d}{dt} G_{\tau(i)|\tau(k)}^-(t, s) .$$

Finally, for every $k \in \{1, 2, \dots, K\}$ such that $t_{k-1} < T$, $i \in \{1, 2, \dots, K\}$ such that $i \geq k$, $j \in I(t_{k-1})$ and $t_{k-1} \leq s \leq t$, define the *conditional Lagrangian function* to be

$$\mathcal{L}_{\tau(i)|\tau(k)}^j(t, s, x, u) = G_{\tau(i)|\tau(k)}^+(t, s)L_j(t, x, u) + g_{\tau(i)|\tau(k)}^-(t, s)\Psi_j(t, x) . \quad (3.8)$$

We are now able to state the dynamic programming principle.

Theorem 3.1.1 (Dynamic programming principle) *Assume that conditions (A1) and (A2) hold. Then, for every $k \in \{1, 2, \dots, K\}$ such that $t_{k-1} < T$, the restriction of the value function V to the set $\Delta_k \times \mathbb{R}^N$ is identically equal to the function determined by the recursive relation*

$$\begin{aligned} V(s, y) = & \sup_{u(\cdot) \in \mathcal{U}^w[s, T]} E \left[G_{\tau(k)|\tau(k)}^+(\hat{s}, s) V(\hat{s}, x(\hat{s}; s, y, u(\cdot))) \right. \\ & \left. + \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_s^{\hat{s}} \mathcal{L}_{\tau(i)|\tau(k)}^j(t, s, x(t; s, y, u(\cdot)), u(t)) dt \right] \Big| s \in \Delta_k \end{aligned}$$

for all $\hat{s} \in [s, T]$ and combined with the boundary condition

$$V(T, x) = \sum_{j \in I(t_{k-1})} \Psi_j(T, x) .$$

Under appropriate regularity assumptions, the dynamic programming

principle above is a key step in the derivation of a family of HJB equations, second-order partial differential equation whose concatenated “solutions” correspond to the value function $V(\cdot, \cdot)$ of the optimal control problem under consideration here.

Let $I \subseteq \mathbb{R}$ be an interval and denote by $C^{1,2}(I \times \mathbb{R}^N; \mathbb{R})$ the set of all continuous functions $V : I \times \mathbb{R}^N \rightarrow \mathbb{R}$ such that V_t , V_x , and V_{xx} are all continuous functions of (t, x) .

Theorem 3.1.2 *Suppose that conditions (A1) and (A2) hold. Additionally, assume that for every $i, j \in \{1, 2, \dots, K\}$ such that $i \geq j$ the conditional density functions $g_{\tau(i)|\tau(j)}^-(t, s)$ are uniformly continuous with respect to t . Let $k \in \{1, 2, \dots, K\}$ be such that $t_{k-1} < T$ and assume that the value function V is such that $V \in C^{1,2}(\Delta_k \times \mathbb{R}^N; \mathbb{R})$. Then, on each set $\Delta_k \times \mathbb{R}^N$ the value function V is identically equal to the solution of the Hamilton-Jacobi-Bellman equation*

$$\begin{cases} V_t - g_{\tau(k)|\tau(k)}^-(t, t)V + \sup_{u \in U} \mathcal{H}^k(t, x, u, V_x, V_{xx}) = 0 \\ V(T, x) = \sum_{j \in I(t_{k-1})} \Psi_j(T, x) \end{cases}, \quad (t, x) \in [t_{k-1}, T) \times \mathbb{R}^N,$$

where the Hamiltonian function $\mathcal{H}^k$ is given by

$$\begin{aligned} \mathcal{H}^k(t, x, u, p, B) &= \sum_{j \in I(t_{k-1})} \left(L_j(t, x, u) + G_k(t) \Psi_j(t, x) \right) \\ &\quad + \langle p, f_k(t, x, u) \rangle + \frac{1}{2} \text{tr} \left(g_k^T(t, x, u) B g_k(t, x, u) \right) \end{aligned}$$

for all $(t, x, u, p, B) \in [0, T] \times \mathbb{R}^N \times U \times \mathbb{R}^N \times \mathcal{S}^N$, $\mathcal{S}^N$ is the set of all $N \times N$

symmetric real matrices and $G_k(t)$ is given by

$$G_k(t) = \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K g_{\tau_{(i)}|\tau_{(k)}}^-(t, t) .$$

We prove Theorem 3.1.1 and Theorem 3.1.2 in Sections 3.2 and 3.3, respectively. Section 3.4 is devoted to the discussion of an application of these results.

3.2 A dynamic programming principle

The goal of this section is to obtain a dynamic programming principle for problem (3.6). In order to achieve this goal, we need to transform our problem to one with a deterministic time horizon. We use the concept of order statistics to help us restate the stochastic optimal control problem formulated in the preceding section as one with a fixed planning horizon.

3.2.1 Order statistics

We start by providing some auxiliary results concerning order statistics that will be useful in the sequel. For further details on the subject see [18]. Recall that we are assuming that $\tau_1, \tau_2, \dots, \tau_K$ are independent and identically distributed continuous random variables with distribution function $F : [0, \infty) \rightarrow [0, 1]$ and density $f : [0, \infty) \rightarrow \mathbb{R}^+$ such that

$$F(t) = \int_0^t f(s) \, ds .$$

Let $\tau_{(1)}, \tau_{(2)}, \dots, \tau_{(K)}$ be the order statistics of the random variables $\tau_1, \tau_2, \dots, \tau_K$. It is clear that

$$\tau_{(1)} \leq \tau_{(2)} \leq \dots \leq \tau_{(K)} ,$$

with strict inequalities almost surely.

It is well known that the distribution function of the order statistic $\tau_{(i)}$ is given by

$$G_{\tau_{(i)}}(t) = P(\{\tau_{(i)} \leq t\}) = \int_0^t g_{\tau_{(i)}}(s) \, ds , \quad i = 1, 2, \dots, K ,$$

where $g_{\tau_{(i)}}(t)$ is the corresponding probability density function given by

$$g_{\tau_{(i)}}(t) = \frac{K!}{(i-1)!(K-i)!} (F(t))^{i-1} (1-F(t))^{K-i} f(t) .$$

Similarly, it is known that the joint distribution function of any two order statistics $\tau_{(i)}$ and $\tau_{(j)}$, for which $1 \leq i < j \leq K$, is given by

$$G_{\tau_{(i)}, \tau_{(j)}}(t_1, t_2) = P(\{\tau_{(i)} \leq t_1 , \tau_{(j)} \leq t_2\}) = \int_0^{t_1} \int_0^{t_2} g_{\tau_{(i)}, \tau_{(j)}}(s, u) \, ds \, du ,$$

where $g_{\tau_{(i)}, \tau_{(j)}}(t_1, t_2)$ is the corresponding joint probability density function given by

$$\begin{aligned} & g_{\tau_{(i)}, \tau_{(j)}}(t_1, t_2) \\ &= K! \frac{(F(t_1))^{i-1}}{(i-1)!} \frac{(F(t_2) - F(t_1))^{j-i-1}}{(j-i-1)!} \frac{(1-F(t_2))^{K-j}}{(K-j)!} f(t_1) f(t_2) \end{aligned}$$

and $0 < t_1 \leq t_2 < \infty$.

Note that for every $i \neq j$ the order statistics $\tau_{(i)}$ and $\tau_{(j)}$ are not independent. For every $0 \leq r \leq t$ and every $i, j \in \{1, 2, \dots, K\}$ such that $i \geq j$, we define the conditional probability $G_{\tau_{(i)}|\tau_{(j)}}^+(t, r)$ by

$$G_{\tau_{(i)}|\tau_{(j)}}^+(t, r) = P\left(\{\tau_{(i)} > t\} \mid \{\tau_{(j)} > r\}\right) \quad (3.9)$$

and the conditional probability $G_{\tau_{(i)}|\tau_{(j)}}^-(t, r)$ by

$$G_{\tau_{(i)}|\tau_{(j)}}^-(t, r) = P\left(\{\tau_{(i)} \leq t\} \mid \{\tau_{(j)} > r\}\right). \quad (3.10)$$

Note that $G_{\tau_{(i)}|\tau_{(j)}}^-(t, r)$ is a (conditional) distribution function with density function given by

$$g_{\tau_{(i)}|\tau_{(j)}}^-(t, r) = \frac{d}{dt} G_{\tau_{(i)}|\tau_{(j)}}^-(t, r). \quad (3.11)$$

For each $i, j \in \{1, 2, \dots, K\}$ such that $i \geq j$ and $0 \leq t \leq t_1 \leq t_2$, we define the conditional probability $G_{\tau_{(i)}, \tau_{(i+1)}|\tau_{(j)}}^{-,+}(t_1, t_2, t)$ by

$$G_{\tau_{(i)}, \tau_{(i+1)}|\tau_{(j)}}^{-,+}(t_1, t_2, t) = P\left(\{\tau_{(i)} \leq t_1 \wedge \tau_{(i+1)} > t_2\} \mid \{\tau_{(j)} > t\}\right). \quad (3.12)$$

The conditional probability functions introduced above have the following properties.

Lemma 3.2.1 *For every $t \leq T$ and for every $i, j \in \{1, 2, \dots, K-1\}$ such that $i \geq j$, the following identities hold*

$$\begin{aligned} G_{\tau_{(i+1)}|\tau_{(j)}}^+(T, t) &= G_{\tau_{(i)}, \tau_{(i+1)}|\tau_{(j)}}^{-,+}(T, T, t) + G_{\tau_{(i)}|\tau_{(j)}}^+(T, t) \\ G_{\tau_{(i)}|\tau_{(j)}}^-(T, t) &= G_{\tau_{(i)}, \tau_{(i+1)}|\tau_{(j)}}^{-,+}(T, T, t) + G_{\tau_{(i+1)}|\tau_{(j)}}^-(T, t) . \end{aligned}$$

Proof. Let $t \leq T$ and let $i, j \in \{1, 2, \dots, K-1\}$ be such that $i \geq j$. Using the definition (3.12), we rewrite $G_{\tau_{(i)}, \tau_{(i+1)}|\tau_{(j)}}^{-,+}(T, T, t)$ in the form

$$G_{\tau_{(i)}, \tau_{(i+1)}|\tau_{(j)}}^{-,+}(T, T, t) = P(\{\tau_{(i)} \leq T \wedge \tau_{(i+1)} > T \mid \tau_{(j)} > t\}) .$$

Using basic properties of probability measures, the last identity becomes

$$\begin{aligned} G_{\tau_{(i)}, \tau_{(i+1)}|\tau_{(j)}}^{-,+}(T, T, t) &= P(\{\tau_{(i)} \leq T\} \mid \{\tau_{(j)} > t\}) - P(\{\tau_{(i+1)} \leq T\} \mid \{\tau_{(j)} > t\}) \\ &= G_{\tau_{(i)}|\tau_{(j)}}^-(T, t) - G_{\tau_{(i+1)}|\tau_{(j)}}^-(T, t) , \end{aligned}$$

concluding the proof of the second identity in the statement.

Considering the complementary events in the previous equality, it becomes clear that

$$G_{\tau_{(i)}, \tau_{(i+1)}|\tau_{(j)}}^{-,+}(T, T, t) = G_{\tau_{(i+1)}|\tau_{(j)}}^+(T, t) - G_{\tau_{(i)}|\tau_{(j)}}^+(T, t) ,$$

concluding the proof of the Lemma. ■

The following properties hold for the conditional probabilities introduced before.

Lemma 3.2.2 *For every $t \leq T$ and every $i, k \in \{1, 2, \dots, K\}$ such that $i \geq k$, the following identities hold*

$$G_{\tau(i)|\tau(k)}^+(T, t) = \sum_{j=k+1}^i G_{\tau(j-1), \tau(j)|\tau(k)}^{-,+}(T, T, t) + G_{\tau(k)|\tau(k)}^+(T, t) \quad (3.13)$$

$$G_{\tau(i)|\tau(k)}^-(T, t) = \sum_{j=i+1}^K G_{\tau(j-1), \tau(j)|\tau(k)}^{-,+}(T, T, t) + G_{\tau(K)|\tau(k)}^-(T, t) . \quad (3.14)$$

Proof. We will prove identity (3.13). The proof of identity (3.14) is analogous and we skip it. Let $t \leq T$ and fix $k \in \{1, 2, \dots, K\}$. We start by noting that if $k = K$ the equality (3.13) is trivial. Thus, we consider only the case where $k < K$.

We proceed by induction in $i \in \{k, \dots, K\}$. Note first that for $i = k$, identity (3.13) holds trivially. We now assume that identity (3.13) holds for every $i \in \{k, \dots, K'\}$, where $K' \leq K - 1$, and we will prove that identity (3.13) also holds for $i = K' + 1$. We will show that

$$G_{\tau(K'+1)|\tau(k)}^+(T, t) = \sum_{j=k+1}^{K'+1} G_{\tau(j-1), \tau(j)|\tau(k)}^{-,+}(T, T, t) + G_{\tau(k)|\tau(k)}^+(T, t) . \quad (3.15)$$

Applying Lemma 3.2.1, we get

$$G_{\tau(K'+1)|\tau(k)}^+(T, t) = G_{\tau(K'), \tau(K'+1)|\tau(k)}^{-,+}(T, T, t) + G_{\tau(K')| \tau(k)}^+(T, t) .$$

Since identity (3.13) holds for every $i \in \{k, \dots, K'\}$, replacing $G_{\tau(K')| \tau(k)}^+(T, t)$

in the previous equality, we obtain

$$\begin{aligned} G_{\tau_{(K'+1)}|\tau_{(k)}}^+(T, t) &= G_{\tau_{(K')}, \tau_{(K'+1)}|\tau_{(k)}}^{-,+}(T, T, t) \\ &\quad + \sum_{j=k+1}^{K'} G_{\tau_{(j-1)}, \tau_{(j)}|\tau_{(k)}}^{-,+}(T, T, t) + G_{\tau_{(k)}|\tau_{(k)}}^+(T, t) . \end{aligned}$$

Note that the last identity can be written in the form

$$G_{\tau_{(K'+1)}|\tau_{(k)}}^+(T, t) = \sum_{j=k+1}^{K'+1} G_{\tau_{(j-1)}, \tau_{(j)}|\tau_{(k)}}^{-,+}(T, T, t) + G_{\tau_{(k)}|\tau_{(k)}}^+(T, t) ,$$

thus proving that identity (3.15) is satisfied. ■

The following lemma plays a crucial role in the proof of the main results in this chapter.

Lemma 3.2.3 *Let $k \in \{1, 2, \dots, K\}$ be such that $t_{k-1} < T$ and let Δ_k be the random interval defined in (3.1). For every $s, \hat{s} \in \Delta_k$ such that $s \leq \hat{s} \leq t$ and every $i \in \{1, 2, \dots, K\}$ such that $i \geq k$, the following identities hold*

$$\begin{aligned} G_{\tau_{(i)}|\tau_{(k)}}^+(t, s) &= G_{\tau_{(i)}|\tau_{(k)}}^+(t, \hat{s}) G_{\tau_{(k)}|\tau_{(k)}}^+(\hat{s}, s) \\ g_{\tau_{(i)}|\tau_{(k)}}^-(t, s) &= g_{\tau_{(i)}|\tau_{(k)}}^-(t, \hat{s}) G_{\tau_{(k)}|\tau_{(k)}}^+(\hat{s}, s) . \end{aligned}$$

Proof. Let k be as given in the statement. Fix $s, \hat{s} \in \Delta_k$ and $t \geq 0$ such that $s \leq \hat{s} \leq t$ and $i \in \{1, 2, \dots, K\}$ such that $i \geq k$.

Using the definition of the conditional probability $G_{\tau_{(i)}|\tau_{(k)}}^+(t, s)$, we have

that

$$\begin{aligned}
 G_{\tau_{(i)}|\tau_{(k)}}^+(t, s) &= P(\{\tau_{(i)} > t\} \mid \{\tau_{(k)} > s\}) \\
 &= \frac{P(\{\tau_{(i)} > t \wedge \tau_{(k)} > s\})}{P(\{\tau_{(k)} > s\})} \\
 &= \frac{P(\{\tau_{(i)} > t \wedge \tau_{(k)} > s\})}{P(\{\tau_{(k)} > \hat{s}\})} \frac{P(\{\tau_{(k)} > \hat{s}\})}{P(\{\tau_{(k)} > s\})}.
 \end{aligned} \tag{3.16}$$

Since $s, \hat{s} \in \Delta_k$, we have that

$$\{\omega \in \Omega : \tau_{(i)} > t \wedge \tau_{(k)} > s\} = \{\omega \in \Omega : \tau_{(i)} > t \wedge \tau_{(k)} > \hat{s}\}. \tag{3.17}$$

Moreover, since $s \leq \hat{s}$, we obtain that

$$\{\omega \in \Omega : \tau_{(k)} > \hat{s}\} \subseteq \{\omega \in \Omega : \tau_{(k)} > s\}.$$

In particular, the previous inclusion implies that

$$\{\omega \in \Omega : \tau_{(k)} > \hat{s}\} = \{\omega \in \Omega : \tau_{(k)} > \hat{s} \wedge \tau_{(k)} > s\}. \tag{3.18}$$

Combining (3.16) with (3.17) and (3.18), we get

$$\begin{aligned}
 G_{\tau_{(i)}|\tau_{(k)}}^+(t, s) &= \frac{P(\{\tau_{(i)} > t \wedge \tau_{(k)} > \hat{s}\})}{P(\{\tau_{(k)} > \hat{s}\})} \frac{P(\{\tau_{(k)} > \hat{s} \wedge \tau_{(k)} > s\})}{P(\{\tau_{(k)} > s\})} \\
 &= G_{\tau_{(i)}|\tau_{(k)}}^+(t, \hat{s}) G_{\tau_{(k)}|\tau_{(k)}}^+(\hat{s}, s),
 \end{aligned}$$

concluding the proof of the first identity.

To prove the second identity, we consider the complementary events in

the first identity

$$1 - G_{\tau(i)|\tau(k)}^-(t, s) = \left(1 - G_{\tau(i)|\tau(k)}^-(t, \hat{s})\right) G_{\tau(k)|\tau(k)}^+(\hat{s}, s) .$$

Since $g_{\tau(i)}^-(t, s)$ is the density function of the conditional distribution function $G_{\tau(i)}^-(t, s)$, by differentiating the last identity with respect to t , we obtain

$$-g_{\tau(i)|\tau(k)}^-(t, s) = -g_{\tau(i)|\tau(k)}^-(t, \hat{s}) G_{\tau(k)|\tau(k)}^+(\hat{s}, s) .$$

Hence,

$$g_{\tau(i)|\tau(k)}^-(t, s) = g_{\tau(i)|\tau(k)}^-(t, \hat{s}) G_{\tau(k)|\tau(k)}^+(\hat{s}, s) ,$$

as required. ■

3.2.2 Transformation to a deterministic time horizon

We will now build on the results of section 3.2.1 to reduce the analysis of the optimal control problem under consideration here to the analysis of a finite number of optimal control problems, all with a fixed planning horizon.

Lemma 3.2.4 *The objective functional $J(s, y; u(\cdot))$ in (3.4) can be written as*

$$J(s, y; u(\cdot)) = E \left[\frac{1}{\kappa(s)} \sum_{i=K+1-\kappa(s)}^K \sum_{j \in I(s)} \int_s^{t_i} L_j(t, x(t), u(t)) dt + \Psi_j(t_i, x(t_i)) \right] .$$

Proof. Recall the definitions of the set $I(t)$, the map $\kappa(t)$ and the sequence (t_i) introduced in section 3.1. Let Λ be a discrete set and denote by S_Λ the symmetric group of permutations $\sigma : \Lambda \rightarrow \Lambda$ of Λ .

For each $t \in [0, t_K)$, the $\kappa(t)!$ elements of the set $S_{I(t)}$ provide all the possible orders for the values of the random variables τ_i with index such that $i \in I(t)$. Recalling that $\tau_1, \tau_2, \dots, \tau_K$ are independent and identically distributed, we conclude that each element $\sigma \in S_{I(t)}$ has the same probability of occurrence: $1/\kappa(t)!$. Thus, we obtain that

$$E \left[\sum_{i \in I(s)} \int_s^{\xi_i} L_i(t, x(t), u(t)) dt + \Psi_i(\xi_i, x(\xi_i)) \right] = \\ E \left[\sum_{i=K+1-\kappa(s)}^K \frac{1}{\kappa(s)!} \sum_{\sigma \in S_{I(s)}} \sum_{j \in I(s)} \int_s^{t_i} L_{\sigma(j)}(t, x(t), u(t)) dt + \Psi_{\sigma(j)}(t_i, x(t_i)) \right].$$

We now note that

$$\sum_{\sigma \in S_{I(s)}} \sum_{j \in I(s)} \int_s^{t_i} L_{\sigma(j)}(t, x(t), u(t)) dt + \Psi_{\sigma(j)}(t_i, x(t_i)) \\ = (\kappa(s) - 1)! \sum_{j \in I(s)} \int_s^{t_i} L_j(t, x(t), u(t)) dt + \Psi_j(t_i, x(t_i)).$$

The proof is complete after combining the two identities above and rearranging terms. ■

Using Lemma 3.2.4, we obtain that for every $k \in \{1, 2, \dots, K\}$ with the property that $t_{k-1} < T$ and every $s \in \Delta_k$ the functional (3.4) can be written as

$$J^k(s, y; u(\cdot)) = E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \left(\int_s^{t_i} L_j(t, x(t), u(t)) dt \right. \right. \\ \left. \left. + \Psi_j(t_i, x(t_i)) \right) \middle| s \in \Delta_k \right]. \quad (3.19)$$

For convenience of notation let us introduce deterministic variables $\tau_{(0)}$ and $\tau_{(K+1)}$ such that

$$\tau_{(0)} = 0 \quad \text{and} \quad \tau_{(K+1)} > T .$$

We will also use I_A to denote the indicator function of some set A . We will keep this notation in what remains of this chapter.

Lemma 3.2.5 *Let $k \in \{1, 2, \dots, K\}$ be such that $t_{k-1} < T$ and let Δ_k be the random interval defined in (3.1). The functional $J^k(s, y; u(\cdot))$ in (3.19) can be written as*

$$J^k(s, y; u(\cdot)) = E \left[\sum_{\ell=k}^{K+1} I_{\{\tau_{(\ell-1)} \leq T, \tau_{(\ell)} > T\}} Q_{\ell}^k(s, y; u(\cdot)) \middle| s \in \Delta_k \right], \quad (3.20)$$

where $Q_{\ell}^k(s, y; u(\cdot))$, $\ell \in \{k, \dots, K+1\}$, is given by

$$\begin{aligned} Q_{\ell}^k(s, y; u(\cdot)) &= \frac{1}{\kappa(t_{k-1})} \sum_{i=\ell}^K \sum_{j \in I(t_{k-1})} \left(\int_s^T L_j(t, x(t), u(t)) dt + \Psi_j(T, x(T)) \right) \\ &\quad + \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^{\ell-1} \sum_{j \in I(t_{k-1})} \left(\int_s^{\tau_{(i)}} L_j(t, x(t), u(t)) dt + \Psi_j(\tau_{(i)}, x(\tau_{(i)})) \right) \end{aligned}$$

and we are using the convention that a sum is zero whenever its lower index is strictly larger than its upper index.

Proof. The proof follows from equation (3.19) combined with the definition of the random times t_i and the definition of the order statistics $\tau_{(i)}$, $i \in \{1, 2, \dots, K\}$. ■

The following lemma provides a representation for the functional $J^k(s, y; u(\cdot))$ in (3.19) with a fixed planning horizon.

Lemma 3.2.6 *Let $k \in \{1, 2, \dots, K\}$ be such that $t_{k-1} < T$ and let Δ_k be the random interval defined in (3.1). The functional $J^k(s, y; u(\cdot))$ in (3.19) can be written as*

$$J^k(s, y; u(\cdot)) = E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \left(\int_s^T \mathcal{L}_{\tau_{(i)}|\tau_{(k)}}^j(t, s, x(t), u(t)) dt + G_{\tau_{(i)}|\tau_{(k)}}^+(T, s) \Psi_j(T, x(T)) \right) \middle| s \in \Delta_k \right],$$

where the conditional Lagrangian $\mathcal{L}_{\tau_{(i)}|\tau_{(k)}}^j$ is as given in (3.8) and the conditional probabilities $G_{\tau_{(i)}|\tau_{(k)}}^+(t, s)$ are as given in (3.9).

Proof. To simplify notation, throughout this proof we will denote by $G_{\tau_{(i)}}^\pm(T, s)$ the conditional probabilities $G_{\tau_{(i)}|\tau_{(k)}}^\pm(T, s)$ and by $g_{\tau_{(i)}}^-(T, s)$ the conditional densities $g_{\tau_{(i)}|\tau_{(k)}}^-(T, s)$.

Let k be as given in the statement. For each $\ell \in \{k, \dots, K+1\}$, define the auxiliary functionals $Q_\ell^{k,T} = Q_\ell^{k,T}(s, y; u(\cdot))$ and $Q_\ell^{k,\tau} = Q_\ell^{k,\tau}(s, y; u(\cdot))$ by

$$Q_\ell^{k,T} = \frac{1}{\kappa(t_{k-1})} \sum_{i=\ell}^K \sum_{j \in I(t_{k-1})} \left(\int_s^T L_j(t, x(t), u(t)) dt + \Psi_j(T, x(T)) \right)$$

$$Q_\ell^{k,\tau} = \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^{\ell-1} \sum_{j \in I(t_{k-1})} \left(\int_s^{\tau_{(i)}} L_j(t, x(t), u(t)) dt + \Psi_j(\tau_{(i)}, x(\tau_{(i)})) \right)$$

and note that the random variables $Q_\ell^k(s, y; u(\cdot))$ introduced in the statement of Lemma 3.2.5 can be written as

$$Q_\ell^k(s, y; u(\cdot)) = Q_\ell^{k,T}(s, y; u(\cdot)) + Q_\ell^{k,\tau}(s, y; u(\cdot)) .$$

Thus, a simple computation shows that the functional $J^k(s, y; u(\cdot))$ in (3.20) can be represented as

$$J^k(s, y; u(\cdot)) = J^{k,T}(s, y; u(\cdot)) + J^{k,\tau}(s, y; u(\cdot)) , \quad (3.21)$$

where $J^{k,T}(s, y; u(\cdot))$ and $J^{k,\tau}(s, y; u(\cdot))$ are the functionals given by

$$\begin{aligned} J^{k,T}(s, y; u(\cdot)) &= E \left[I_{\{\tau_{(k)} > T\}} Q_k^{k,T}(s, y; u(\cdot)) \right. \\ &\quad \left. + \sum_{\ell=k+1}^K I_{\{\tau_{(\ell-1)} \leq T, \tau_{(\ell)} > T\}} Q_\ell^{k,T}(s, y; u(\cdot)) \mid s \in \Delta_k \right] \end{aligned} \quad (3.22)$$

and

$$\begin{aligned} J^{k,\tau}(s, y; u(\cdot)) &= E \left[\sum_{\ell=k+1}^K I_{\{\tau_{(\ell-1)} \leq T, \tau_{(\ell)} > T\}} Q_\ell^{k,\tau}(s, y; u(\cdot)) \right. \\ &\quad \left. + I_{\{\tau_{(K)} \leq T\}} Q_{K+1}^{k,\tau}(s, y; u(\cdot)) \mid s \in \Delta_k \right] . \end{aligned} \quad (3.23)$$

We will now break the proof into two parts: we will compute $J^{k,T}(s, y; u(\cdot))$ first, and we will deal with $J^{k,\tau}(s, y; u(\cdot))$ afterwards.

For the computation of $J^{k,T}(s, y; u(\cdot))$, we start by recalling that the random variables $\tau_1, \tau_2, \dots, \tau_K$ are independent of the filtration $\mathbb{F}$. Thus, so

are the order statistics $\tau_{(1)}, \tau_{(2)}, \dots, \tau_{(K)}$. Furthermore, by definition of the conditional probability $G_{\tau_{(i)}}^+(T, s)$ for the i^{th} order statistic, as well as the conditional probability $G_{\tau_{(i)}, \tau_{(j)}}^{-,+}(T, T, s)$ for the order statistics $\tau_{(i)}$ and $\tau_{(j)}$, we have that

$$\begin{aligned} E \left[I_{\{\tau_{(i)} > T\}} \mid s \in \Delta_k \right] &= G_{\tau_{(i)}}^+(T, s) \\ E \left[I_{\{\tau_{(i)} \leq T, \tau_{(j)} > T\}} \mid s \in \Delta_k \right] &= G_{\tau_{(i)}, \tau_{(j)}}^{-,+}(T, T, s) . \end{aligned}$$

Combining the observations above with the linearity of the expected value, we rewrite (3.22) as

$$\begin{aligned} J^{k,T}(s, y; u(\cdot)) &= E \left[G_{\tau_{(k)}}^+(T, s) Q_k^{k,T}(s, y; u(\cdot)) \right. \\ &\quad \left. + \sum_{\ell=k+1}^K G_{\tau_{(\ell-1)}, \tau_{(\ell)}}^{-,+}(T, T, s) Q_\ell^{k,T}(s, y; u(\cdot)) \mid s \in \Delta_k \right] . \end{aligned}$$

Rearranging terms, we rewrite the previous identity as

$$J^{k,T}(s, y; u(\cdot)) = E \left[\frac{1}{\kappa(t_{k-1})} \sum_{j \in I(t_{k-1})} \sum_{i=k}^K a_k^i(T, s) \Upsilon_j^T(s, y; u(\cdot)) \mid s \in \Delta_k \right] ,$$

where $\Upsilon_j^T(s, y; u(\cdot))$ is given by

$$\Upsilon_j^T(s, y; u(\cdot)) = \int_s^T L_j(t, x(t), u(t)) dt + \Psi_j(T, x(T)) , \quad j \in I(t_{k-1})$$

and $a_k^i(T, s)$ is given by

$$a_k^i(T, s) = G_{\tau(k)}^+(T, s) + \sum_{\ell=k+1}^i G_{\tau(\ell-1), \tau(\ell)}^{-, +}(T, T, s), \quad i \in \{k, \dots, K\}.$$

Using Lemma 3.2.2 and rearranging terms again, we obtain that

$$J^{k, T}(s, y; u(\cdot)) = E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} G_{\tau(i)}^+(T, s) \Upsilon_j^T(s, y; u(\cdot)) \mid s \in \Delta_k \right]. \quad (3.24)$$

We will now compute $J^{k, \tau}(s, y; u(\cdot))$. Using the definition of $Q_\ell^{k, \tau}(s, y; u(\cdot))$, $\ell \in \{k, \dots, K+1\}$, we obtain that identity (3.23) may be rewritten as

$$J^{k, \tau}(s, y; u(\cdot)) = E \left[\frac{1}{\kappa(t_{k-1})} \sum_{j \in I(t_{k-1})} \sum_{\ell=k+1}^{K+1} I_{\{\tau(\ell-1) \leq T, \tau(\ell) > T\}} \sum_{i=k}^{\ell-1} \Upsilon_j^{\tau(i)}(s, y; u(\cdot)) \mid s \in \Delta_k \right], \quad (3.25)$$

where for each $j \in I(t_{k-1})$ and each $i \in \{k, \dots, K\}$, $\Upsilon_j^{\tau(i)}(s, y; u(\cdot))$ is given by

$$\Upsilon_j^{\tau(i)}(s, y; u(\cdot)) = \int_s^{\tau(i)} L_j(t, x(t), u(t)) dt + \Psi_j(\tau(i), x(\tau(i))). \quad (3.26)$$

Rearranging terms in identity (3.25), we get

$$J^{k, \tau}(s, y; u(\cdot)) = E \left[\frac{1}{\kappa(t_{k-1})} \sum_{j \in I(t_{k-1})} \sum_{i=k}^K \Upsilon_j^{\tau(i)}(s, y; u(\cdot)) \sum_{\ell=i+1}^{K+1} I_{\{\tau(\ell-1) \leq T, \tau(\ell) > T\}} \mid s \in \Delta_k \right]. \quad (3.27)$$

Noting that if $\tau_{(\ell-1)} > T$, then $\tau_{(\ell)} > T$, and one can check that

$$\{\omega \in \Omega : \tau_{(\ell-1)} \leq T, \tau_{(\ell)} > T\} \cup \{\omega \in \Omega : \tau_{(\ell)} \leq T\} = \{\omega \in \Omega : \tau_{(\ell-1)} \leq T\}.$$

As a consequence, we conclude that

$$\sum_{\ell=i+1}^{K+1} I_{\{\tau_{(\ell-1)} \leq T, \tau_{(\ell)} > T\}} = I_{\{\tau_{(i)} \leq T\}}, \quad i = 1, \dots, K.$$

Combining the identity above with (3.27), we obtain

$$J^{k,\tau}(s, y; u(\cdot)) = E \left[\frac{1}{\kappa(t_{k-1})} \sum_{j \in I(t_{k-1})} \sum_{i=k}^K I_{\{\tau_{(i)} \leq T\}} \Upsilon_j^{\tau_{(i)}}(s, y; u(\cdot)) \mid s \in \Delta_k \right].$$

Recalling the definition of $\Upsilon_j^{\tau_{(i)}}$ in (3.26) and taking the expected value with respect to each random variable $\tau_{(i)}$ on the right hand side of the identity above, we get

$$\begin{aligned} J^{k,\tau}(s, y; u(\cdot)) = & E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \left\{ \int_s^T g_{\tau_{(i)}}^-(t, s) \Psi_j(t, x(t)) dt \right. \right. \\ & \left. \left. + \int_s^T g_{\tau_{(i)}}^-(r, s) \int_s^r L_j(t, x(t), u(t)) dt dr \right\} \mid s \in \Delta_k \right]. \end{aligned} \quad (3.28)$$

For all $s \leq t \leq T$, by the definition of $G_{\tau_{(i)}}^+(t, s)$ and $G_{\tau_{(i)}}^-(t, s)$ in (3.9) and (3.10), we obtain that

$$G_{\tau_{(i)}}^+(t, s) - G_{\tau_{(i)}}^+(T, s) = G_{\tau_{(i)}}^-(T, s) - G_{\tau_{(i)}}^-(t, s).$$

Recall that $G_{\tau(i)}^-(T, s)$ and $G_{\tau(i)}^-(t, s)$ are conditional distribution functions.

Hence,

$$\begin{aligned} G_{\tau(i)}^+(t, s) - G_{\tau(i)}^+(T, s) &= \int_s^T g_{\tau(i)}^-(r, s) \, dr - \int_s^t g_{\tau(i)}^-(r, s) \, dr \\ &= \int_t^T g_{\tau(i)}^-(r, s) \, dr . \end{aligned} \quad (3.29)$$

Applying *Fubini-Tonelli Theorem* to the second term of identity (3.28), we obtain that the order of integration can be interchanged for all $i = 1, 2, \dots, K$. Hence, we get that

$$\begin{aligned} &\int_s^T \int_s^r g_{\tau(i)}^-(r, s) L_j(t, x(t), u(t)) \, dt \, dr \\ &= \int_s^T \int_t^T g_{\tau(i)}^-(r, s) L_j(t, x(t), u(t)) \, dr \, dt \\ &= \int_s^T \left(\int_t^T g_{\tau(i)}^-(r, s) \, dr \right) L_j(t, x(t), u(t)) \, dt . \end{aligned}$$

Substituting the value of $\int_t^T g_{\tau(i)}^-(r, s) \, dr$ from (3.29) in the last identity, we obtain

$$\begin{aligned} &\int_s^T \int_s^r g_{\tau(i)}^-(r, s) L_j(t, x(t), u(t)) \, dt \, dr \\ &= \int_s^T \left(G_{\tau(i)}^+(t, s) - G_{\tau(i)}^+(T, s) \right) L_j(t, x(t), u(t)) \, dt . \end{aligned}$$

Combining the identity above with (3.28) and rearranging terms, we obtain

$$\begin{aligned} J^{k,\tau}(s, y; u(\cdot)) = & E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \left\{ \int_s^T g_{\tau(i)}^-(t, s) \Psi_j(t, x(t)) dt \right. \right. \\ & \left. \left. + \int_s^T \left(G_{\tau(i)}^+(t, s) - G_{\tau(i)}^+(T, s) \right) L_j(t, x(t), u(t)) dt \right\} \mid s \in \Delta_k \right]. \quad (3.30) \end{aligned}$$

Putting together the identities (3.21), (3.24) and (3.30), we obtain

$$\begin{aligned} J^k(s, y; u(\cdot)) = & E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \left(\int_s^T G_{\tau(i)}^+(t, s) L_j(t, x(t), u(t)) dt \right. \right. \\ & \left. \left. + \int_s^T g_{\tau(i)}^-(t, s) \Psi_j(t, x(t)) dt + G_{\tau(i)}^+(T, s) \Psi_j(T, x(T)) \right) \mid s \in \Delta_k \right]. \end{aligned}$$

Rearranging terms in the equality above, we obtain the identity in the statement. ■

Let $\mathcal{F}_t^s$ be the filtration $\sigma\{W(r) : s \leq r \leq t\}$ augmented by all the $\mathbb{P}$ -null sets in $\mathcal{F}$ and let $u : [s, t_K] \times \Omega \rightarrow U$ be an $\{\mathcal{F}_t^s\}_{t \geq s}$ -adapted process on $(\Omega, \mathcal{F}, \mathbb{P})$. Clearly, under $u(\cdot)$, for any $y \in \mathbb{R}^N$ equation (3.3) admits a unique solution $x(\cdot)$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t^s\}_{t \geq s}, \mathbb{P})$.

The following result holds.

Lemma 3.2.7 *Let $k \in \{1, 2, \dots, K\}$ be such that $t_{k-1} < T$ and let Δ_k be the random interval defined in (3.1). For every $(s, y) \in \Delta_k \times \mathbb{R}^N$, $u(\cdot) \in \mathcal{U}^w[s, T]$*

and $\hat{s} \in [s, t_k)$, we have

$$\begin{aligned} & J^k(\hat{s}, x(\hat{s}; s, y, u(\cdot)); u(\cdot)) = \\ & E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \left(\int_{\hat{s}}^T \mathcal{L}_{\tau(i)|\tau(k)}^j(t, \hat{s}, x(t; \hat{s}, x(\hat{s}), u(\cdot)), u(t)) dt \right. \right. \\ & \left. \left. + G_{\tau(i)|\tau(k)}^+(T, \hat{s}) \Psi_j(T, x(T; \hat{s}, x(\hat{s}), u(\cdot))) \right) \middle| \mathcal{F}_{\hat{s}}^s, \hat{s} \in [s, t_k) \right] (\omega) \quad \mathbb{P} - \text{a.s. } \omega, \end{aligned}$$

where the conditional Lagrangian $\mathcal{L}_{\tau(i)|\tau(k)}^j$ is as given in (3.8).

Proof. The result follows from Lemma 3.2.6, the definition of the conditional probabilities $G_{\tau(i)|\tau(k)}^+(t, s)$ and densities $g_{\tau(i)|\tau(k)}^-(t, \hat{s})$, $i = 1, 2, \dots, K$, and the fact that $x(\hat{s}, s, y, u(\cdot))$ is a deterministic constant under the probability space $(\Omega, \mathcal{F}, \mathbb{P}(\cdot | \mathcal{F}_{\hat{s}}^s))$. ■

3.2.3 Dynamic programming principle

Let $k \in \{1, 2, \dots, K\}$ be such that $t_{k-1} < T$ and let Δ_k be the random interval defined in (3.1).

For the derivation of the dynamic programming principle, we will consider the case where $t_k = T$ and thus, the random interval Δ_k is simply given by $\Delta'_k = [t_{k-1}, T)$. For every $(s, y) \in [t_{k-1}, T) \times \mathbb{R}^N$, define the functional $\mathcal{J}^k$ to be given by

$$\begin{aligned} \mathcal{J}^k(s, y; u(\cdot)) &= E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \left(\int_s^T \mathcal{L}_{\tau(i)|\tau(k)}^j(t, s, x(t), u(t)) dt \right. \right. \\ &\quad \left. \left. + G_{\tau(i)|\tau(k)}^+(T, s) \Psi_j(T, x(T)) \right) \right]. \end{aligned} \quad (3.31)$$

Furthermore, define the k^{th} value function $\mathcal{V}^k$ to be given by

$$\mathcal{V}^k(s, y) = \sup_{u(\cdot) \in \mathcal{U}^w[s, T]} \mathcal{J}^k(s, y; u(\cdot)) \quad (3.32)$$

for every $(s, y) \in [t_{k-1}, T) \times \mathbb{R}^N$.

Before stating and proving the dynamic programming principle, we provide an auxiliary result which gives basic properties of the value function that will be useful in the sequel.

Proposition 3.2.1 *Assume that conditions (A1) and (A2) hold and let $k \in \{1, 2, \dots, K\}$ be such that $t_{k-1} < T$. Then, there exists a constant $C' > 0$ such that:*

(i) *for every $(s, y) \in [t_{k-1}, T) \times \mathbb{R}^N$, we have that*

$$|\mathcal{V}^k(s, y)| \leq C' (1 + \|y\|_N^2),$$

where $\|\cdot\|_N$ denotes the Euclidean norm in $\mathbb{R}^N$;

(ii) *for every $s, \hat{s} \in [t_{k-1}, T)$ and $y, \hat{y} \in \mathbb{R}^N$, we have that*

$$|\mathcal{V}^k(s, y) - \mathcal{V}^k(\hat{s}, \hat{y})| \leq C' \left(\|y - \hat{y}\|_N^2 + (1 + \|y\|_N^2 \vee \|\hat{y}\|_N^2) |s - \hat{s}|^{\frac{1}{2}} \right),$$

where $a \vee b \triangleq \max\{a, b\}$.

Proof. To simplify notation, throughout this proof we will denote the conditional probabilities $G_{\tau_{(i)}|\tau_{(k)}}^+(T, s)$ by $G_{\tau_{(i)}}^+(T, s)$ and the conditional densities $g_{\tau_{(i)}|\tau_{(k)}}^-(t, s)$ by $g_{\tau_{(i)}}^-(t, s)$.

Let $(s, y) \in [t_{k-1}, T) \times \mathbb{R}^N$ be fixed. Standard results on stochastic differential equations (see e.g. [27, 31]) guarantee that for any $u(\cdot) \in \mathcal{U}^w[s, T]$, we have

$$E \left[\sup_{t \in [s, T]} \|x(t)\|_N^2 \right] \leq C'(1 + \|y\|_N^2). \quad (3.33)$$

Combining Lemma 3.2.6 with the triangle inequality and Riemann integral properties, we get

$$\begin{aligned} |\mathcal{J}^k(s, y; u(\cdot))| \leq E & \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \left(\int_s^T G_{\tau(i)}^+(t, s) |L_j(t, x(t), u(t))| dt \right. \right. \\ & \left. \left. + \int_s^T g_{\tau(i)}^-(t, s) |\Psi_j(t, x(t))| dt + G_{\tau(i)}^+(T, s) |\Psi_j(T, x(T))| \right) \right]. \end{aligned}$$

Rearranging the previous inequality and applying again the triangle inequality, we obtain

$$\begin{aligned} |\mathcal{J}^k(s, y; u(\cdot))| \leq & E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \left(\int_s^T G_{\tau(i)}^+(t, s) |L_j(t, x(t), u(t)) - L_j(t, 0, u(t))| dt \right. \right. \\ & + \int_s^T G_{\tau(i)}^+(t, s) |L_j(t, 0, u(t))| dt \\ & + \int_s^T g_{\tau(i)}^-(t, s) |\Psi_j(t, x(t)) - \Psi_j(t, 0)| dt + \int_s^T g_{\tau(i)}^-(t, s) |\Psi_j(t, 0)| dt \\ & \left. \left. + G_{\tau(i)}^+(T, s) |\Psi_j(T, x(T)) - \Psi_j(T, 0)| + G_{\tau(i)}^+(T, s) |\Psi_j(T, 0)| \right) \right]. \end{aligned}$$

Using assumption (A2), we obtain that

$$|\mathcal{J}^k(s, y; u(\cdot))| \leq CE \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \left(G_{\tau(i)}^+(T, s)(1 + \|x(t)\|_N^2) \right. \right. \\ \left. \left. + \int_s^T \left(G_{\tau(i)}^+(t, s) + g_{\tau(i)}^-(t, s) \right) (1 + \|x(t)\|_N^2) dt \right) \right].$$

Resorting to inequality (3.33) and recalling that $G_{\tau(i)}^+(t, s)$ and $g_{\tau(i)}^-(t, s)$ are bounded above for all $i = 1, 2, \dots, K$, we obtain

$$|\mathcal{J}^k(s, y; u(\cdot))| \leq C'(1 + \|y\|_N^2)$$

for every $u(\cdot) \in \mathcal{U}^w[s, T]$ and we conclude the proof of (i).

We now prove (ii). Let $t_{k-1} \leq s \leq \hat{s} < T$ and $y, \hat{y} \in \mathbb{R}^N$. For any w -admissible control $u(\cdot) \in \mathcal{U}^w[s, T]$, let $x(\cdot)$ and $\hat{x}(\cdot)$ be the states corresponding to $(s, y, u(\cdot))$ and $(\hat{s}, \hat{y}, u(\cdot))$, respectively. Standard results on stochastic differential equations (see [27, 31]) insure that

$$E \left[\sup_{t \in [\hat{s}, T]} \|x(t) - \hat{x}(t)\|_N^2 \right] \leq C' \left(\|y - \hat{y}\|_N^2 + (1 + \|y\|_N^2 \vee \|\hat{y}\|_N^2) |s - \hat{s}|^{\frac{1}{2}} \right).$$

Proceeding similarly to what we have done in the proof of (i), we obtain

$$|\mathcal{J}^k(s, y; u(\cdot)) - \mathcal{J}^k(\hat{s}, \hat{y}; u(\cdot))| \leq C' \left(\|y - \hat{y}\|_N^2 + (1 + \|y\|_N^2 \vee \|\hat{y}\|_N^2) |s - \hat{s}|^{\frac{1}{2}} \right).$$

Taking the supremum in $u(\cdot) \in \mathcal{U}^w[s, T]$, we conclude the proof of (ii). ■

Combining Lemma 3.2.6 and Lemma 3.2.7 with Proposition 3.2.1 one can now state the following dynamic programming principle, obtaining a recursive relationship for the maximum expected utility.

Theorem 3.2.1 (Dynamic programming principle) *Assume that conditions (A1) and (A2) hold and let $k \in \{1, 2, \dots, K\}$ be such that $t_{k-1} < T$. Then, for any $(s, y) \in [t_{k-1}, T) \times \mathbb{R}^N$ and every $\hat{s} \in [s, T)$, we have that*

$$\begin{aligned} \mathcal{V}^k(s, y) = & \sup_{u(\cdot) \in \mathcal{U}^w[s, T]} E \left[G_{\tau_{(k)} | \tau_{(k)}}^+(\hat{s}, s) \mathcal{V}^k(\hat{s}, x(\hat{s}; s, y, u(\cdot))) \right. \\ & \left. + \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_s^{\hat{s}} \mathcal{L}_{\tau_{(i)} | \tau_{(k)}}^j(t, s, x(t; s, y, u(\cdot)), u(t)) dt \right], \end{aligned} \quad (3.34)$$

where the conditional Lagrangian $\mathcal{L}_{\tau_{(i)} | \tau_{(k)}}^j$ is as given in (3.8).

Proof. To simplify notation, throughout this proof we will denote the conditional probabilities $G_{\tau_{(i)} | \tau_{(k)}}^+(T, s)$ by $G_{\tau_{(i)}}^+(T, s)$ and the conditional densities $g_{\tau_{(i)} | \tau_{(k)}}^-(t, s)$ by $g_{\tau_{(i)}}^-(t, s)$. Moreover, we will denote the conditional Lagrangian functions $\mathcal{L}_{\tau_{(i)} | \tau_{(k)}}^j$ by $\mathcal{L}_{\tau_{(i)}}^j$ and will drop the use of the index k in the functional $\mathcal{J}^k$ and the function $\mathcal{V}^k$.

Denote the right-hand side of (3.34) by $\bar{\mathcal{V}}(s, y)$. For any $\epsilon > 0$ there exists $u(\cdot) \in \mathcal{U}^w[s, T]$ such that

$$\mathcal{V}(s, y) - \epsilon < \mathcal{J}(s, y; u(\cdot)) .$$

Using Lemma 3.2.6, we obtain

$$\mathcal{V}(s, y) - \epsilon < E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \left(\int_s^T \mathcal{L}_{\tau(i)}^j(t, s, x(t; s, y, u(\cdot)), u(t)) dt + G_{\tau(i)}^+(T, s) \Psi_j(T, x(T; s, y, u(\cdot))) \right) \right].$$

Let $\hat{s} \in [s, T)$ and rewrite the last inequality as

$$\begin{aligned} \mathcal{V}(s, y) - \epsilon < E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_s^{\hat{s}} \mathcal{L}_{\tau(i)}^j(t, s, x(t; s, y, u(\cdot)), u(t)) dt \right. \\ \left. + E \left\{ \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_{\hat{s}}^T \mathcal{L}_{\tau(i)}^j(t, s, x(t; s, y, u(\cdot)), u(t)) dt \right. \right. \\ \left. \left. + G_{\tau(i)}^+(T, s) \Psi_j(T, x(T; s, y, u(\cdot))) \right\} \middle| \mathcal{F}_{\hat{s}}^s \right]. \end{aligned} \quad (3.35)$$

Combining the definition of the conditional Lagrangian function $\mathcal{L}_{\tau(i)}^j$ in (3.8) with Lemma 3.2.3, we obtain

$$\begin{aligned} & \mathcal{L}_{\tau(i)}^j(t, s, x(t; s, y, u(\cdot)), u(t)) \\ &= G_{\tau(k)}^+(\hat{s}, s) \mathcal{L}_{\tau(i)}^j(t, \hat{s}, x(t; s, y, u(\cdot)), u(t)) \end{aligned} \quad (3.36)$$

Using (3.36), we rewrite inequality (3.35) as

$$\begin{aligned} \mathcal{V}(s, y) - \epsilon &< E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_s^{\hat{s}} \mathcal{L}_{\tau(i)}^j(t, s, x(t; s, y, u(\cdot)), u(t)) dt \right. \\ &+ G_{\tau(k)}^+(\hat{s}, s) E \left\{ \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \left(\int_{\hat{s}}^T \mathcal{L}_{\tau(i)}^j(t, \hat{s}, x(t; \hat{s}, y, u(\cdot)), u(t)) dt \right. \right. \\ &\quad \left. \left. + G_{\tau(i)}^+(T, \hat{s}) \Psi_j(T, x(T; s, y, u(\cdot))) \right) \mid \mathcal{F}_{\hat{s}}^s \right\} \Bigg]. \end{aligned}$$

Using the Markov property of the solutions of (3.3), we obtain

$$\begin{aligned} \mathcal{V}(s, y) - \epsilon &< E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_s^{\hat{s}} \mathcal{L}_{\tau(i)}^j(t, s, x(t; s, y, u(\cdot)), u(t)) dt \right. \\ &+ G_{\tau(k)}^+(\hat{s}, s) E \left\{ \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \left(\int_{\hat{s}}^T \mathcal{L}_{\tau(i)}^j(t, \hat{s}, x(t; \hat{s}, x(\hat{s}), u(\cdot)), u(t)) dt \right. \right. \\ &\quad \left. \left. + G_{\tau(i)}^+(T, \hat{s}) \Psi_j(T, x(T; \hat{s}, x(\hat{s}), u(\cdot))) \right) \mid \mathcal{F}_{\hat{s}}^s \right\} \Bigg]. \end{aligned}$$

By Lemma 3.2.7, we obtain

$$\begin{aligned} \mathcal{V}(s, y) - \epsilon &< E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_s^{\hat{s}} \mathcal{L}_{\tau(i)}^j(t, s, x(t; s, y, u(\cdot)), u(t)) dt \right. \\ &\quad \left. + G_{\tau(k)}^+(\hat{s}, s) \mathcal{J}(\hat{s}, x(\hat{s}; s, y, u(\cdot)); u(\cdot)) \right]. \end{aligned}$$

Hence, from the previous inequality we get

$$\begin{aligned} \mathcal{V}(s, y) - \epsilon &< E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_s^{\hat{s}} \mathcal{L}_{\tau(i)}^j(t, s, x(t; s, y, u(\cdot)), u(t)) dt \right. \\ &\quad \left. + G_{\tau(k)}^+(\hat{s}, s) \mathcal{V}(\hat{s}, x(\hat{s}; s, y, u(\cdot))) \right] \leq \bar{\mathcal{V}}(s, y). \end{aligned}$$

We will now prove the converse. Using Proposition 3.2.1 and its proof, for any $\epsilon > 0$ there is $\delta = \delta(\epsilon)$ such that if $|y - \hat{y}| < \delta$ we have

$$|\mathcal{J}(\hat{s}, y; u(\cdot)) - \mathcal{J}(\hat{s}, \hat{y}; u(\cdot))| + |\mathcal{V}(\hat{s}, y) - \mathcal{V}(\hat{s}, \hat{y})| \leq \epsilon \quad (3.37)$$

for every $u(\cdot) \in \mathcal{U}^w[\hat{s}, T]$. Let $\{D_j\}_{j \geq 1}$ be a Borel partition of $\mathbb{R}^N$ with diameter $\text{diam}(D_j) < \delta$. For each $j \in \mathbb{N}$ there is $u_j(\cdot) \in \mathcal{U}^w[\hat{s}, T]$ such that

$$\mathcal{J}(\hat{s}, x_j; u_j(\cdot)) \geq \mathcal{V}(\hat{s}, x_j) - \epsilon. \quad (3.38)$$

Hence for any $x \in D_j$, resorting to inequalities (3.37) and (3.38), we obtain

$$\mathcal{J}(\hat{s}, x; u_j(\cdot)) \geq \mathcal{J}(\hat{s}, x_j; u_j(\cdot)) - \epsilon \geq \mathcal{V}(\hat{s}, x_j) - 2\epsilon \geq \mathcal{V}(\hat{s}, x) - 3\epsilon. \quad (3.39)$$

Note that by the definition of weak admissible control $u_j(\cdot)$ there exist $(\Omega_j, \mathcal{F}_j, \mathbb{P}_j, W_j(\cdot), u_j(\cdot))$ and $v_j : [0, T] \times C([0, T]; \mathbb{R}^M) \rightarrow U$ such that v_j is a progressively measurable process with respect to the filtration $\mathcal{G}_{\hat{s}}$ given by

$$\mathcal{G}_{\hat{s}} = \bigcap_{t \geq \hat{s}} \sigma \left(\{h(\cdot \wedge t) : h(\cdot) \in C([0, T]; \mathbb{R}^M)\} \right)$$

and

$$u_j(t, \omega) = v_j(t, W_j(\cdot \wedge t, \omega)) \quad \mathbb{P}_j - \text{a.s. } \omega \in \Omega_j$$

for every $t \in [\hat{s}, T]$.

For any $u(\cdot) \in \mathcal{U}^w[s, T]$, let $x(\cdot) := x(\cdot; s, y, u(\cdot))$ denote the corresponding state trajectory for equation (3.3). Define a new control

$$\tilde{u}(t, \omega) = \begin{cases} u(t, \omega), & \text{if } t \in [s, \hat{s}); \\ v_j(t, W(\cdot \wedge t, \omega) - W(\hat{s}, \omega)), & \text{if } t \in [\hat{s}, T] \text{ and } x(t, \omega) \in D_j. \end{cases}$$

Clearly, we have that $\tilde{u}(\cdot) \in \mathcal{U}^w[s, T]$. Moreover, we have that the following inequality holds

$$\mathcal{V}(s, y) \geq \mathcal{J}(s, y; \tilde{u}(\cdot)).$$

Using Lemma 3.2.6, we obtain

$$\begin{aligned} \mathcal{V}(s, y) \geq E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_s^T \mathcal{L}_{\tau(i)}^j(t, s, x(t; s, y, \tilde{u}(\cdot)), \tilde{u}(t)) dt \right. \\ \left. + G_{\tau(i)}^+(T, s) \Psi_j(T, x(T; s, y, \tilde{u}(\cdot))) \right]. \end{aligned}$$

Take $s \leq \hat{s} < T$. Combining Lemma 3.2.3 and identity (3.36) with the

Markov property of the solutions of (3.3), we rewrite the last inequality as

$$\begin{aligned} \mathcal{V}(s, y) \geq E & \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_s^{\hat{s}} \mathcal{L}_{\tau(i)}^j(t, s, x(t; s, y, u(\cdot)), u(t)) dt \right. \\ & + G_{\tau(k)}^+(\hat{s}, s) E \left\{ \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \left(\int_{\hat{s}}^T \mathcal{L}_{\tau(i)}^j(t, \hat{s}, x(t; \hat{s}, x(\hat{s}), u(\cdot)), \tilde{u}(t)) dt \right. \right. \\ & \quad \left. \left. + G_{\tau(i)}^+(T, \hat{s}) \Psi_j(T, x(T; \hat{s}, x(\hat{s}), \tilde{u}(\cdot))) \right) \mid \mathcal{F}_{\hat{s}}^s \right\} \Big]. \end{aligned}$$

Using Lemma 3.2.7, the inequality above becomes

$$\begin{aligned} \mathcal{V}(s, y) \geq E & \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_s^{\hat{s}} \mathcal{L}_{\tau(i)}^j(t, s, x(t; s, y, u(\cdot)), u(t)) dt \right. \\ & \quad \left. + G_{\tau(k)}^+(\hat{s}, s) \mathcal{J}(\hat{s}, x(\hat{s}; s, y, u(\cdot)); \tilde{u}(\cdot)) \right]. \end{aligned}$$

Hence, using (3.39) we obtain that

$$\begin{aligned} \mathcal{V}(s, y) \geq E & \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_s^{\hat{s}} \mathcal{L}_{\tau(i)}^j(t, s, x(t; s, y, u(\cdot)), u(t)) dt \right. \\ & \quad \left. + G_{\tau(k)}^+(\hat{s}, s) \mathcal{V}(\hat{s}, x(\hat{s}; s, y, u(\cdot))) - 3\epsilon \right]. \end{aligned}$$

We conclude the proof by taking the supremum over $u(\cdot) \in \mathcal{U}^w[s, T]$. ■

Theorem 3.1.1 follows as a consequence of Theorem 3.2.1.

Proof of Theorem 3.1.1. Recall from Lemma 3.2.4 that for every $k \in \{1, 2, \dots, K\}$ with the property that $t_{k-1} < T$ and every $s \in \Delta_k$ the functional J in (3.4) can be written as the functional J^k defined in (3.19). Note also that the functional J^k in (3.19) is identically equal to the restriction of the functional $\mathcal{J}^k$ defined in (3.31) to the set of initial conditions $(s, y) \in \Delta_k \times \mathbb{R}^N$. Therefore, the value function V in (3.6) agrees with the k^{th} value function $\mathcal{V}^k$ in (3.32), which satisfies the dynamic programming principle stated in Theorem 3.2.1. ■

Proposition 3.2.2 *Let $k \in \{1, 2, \dots, K\}$ be such that $t_{k-1} < T$ and assume that conditions (A1) and (A2) hold. For every $s \in [t_{k-1}, T)$, if $(\bar{x}(\cdot), \bar{u}(\cdot))$ is optimal for (3.32) then the following identity holds $\mathbb{P} - \text{a.s.}$*

$$\begin{aligned} \mathcal{V}^k(t, \bar{x}(t)) = E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \left(\int_t^T \mathcal{L}_{\tau(i)|\tau(k)}^j(v, t, \bar{x}(v), \bar{u}(v)) dv \right. \right. \\ \left. \left. + G_{\tau(i)|\tau(k)}^+(T, t) \Psi_j(T, \bar{x}(T)) \right) \middle| \mathcal{F}_t^s \right] \end{aligned}$$

for every $t \in [s, T)$.

Proof. Throughout this proof, we will use the simplified notation convention introduced for the proof of Theorem 3.2.1.

Fix $s \in [t_{k-1}, T)$ and let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be an optimal solution for problem (3.32). Using the proof of Theorem 3.2.1, we have

$$\mathcal{V}(s, y) = \mathcal{J}(s, y; \bar{u}(\cdot)) .$$

Using Lemma 3.2.6, we rewrite the last identity as

$$\mathcal{V}(s, y) = E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \left(\int_s^T \mathcal{L}_{\tau(i)}^j(v, s, \bar{x}(v), \bar{u}(v)) dv + G_{\tau(i)}^+(T, s) \Psi_j(T, \bar{x}(T)) \right) \right].$$

Letting $t \in [s, T)$ and combining Lemma 3.2.3 and identity (3.36) with the Markov property of the solutions of (3.3), we obtain

$$\begin{aligned} \mathcal{V}(s, y) &= E \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_s^t \mathcal{L}_{\tau(i)}^j(v, s, \bar{x}(v), \bar{u}(v)) dv \right. \\ &\quad + G_{\tau(k)}^+(t, s) E \left\{ \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_t^T \mathcal{L}_{\tau(i)}^j(v, t, \bar{x}(v), \bar{u}(v)) dv \right. \\ &\quad \left. \left. + G_{\tau(i)}^+(T, t) \Psi_j(T, \bar{x}(T)) \mid \mathcal{F}_t^s \right\} \right]. \end{aligned}$$

Using Lemma 3.2.7, the equality above reduces to

$$\begin{aligned} \mathcal{V}(s, y) &= E \left[G_{\tau(k)}^+(t, s) \mathcal{J}(t, \bar{x}(t); \bar{u}(t)) \right. \\ &\quad \left. + \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_s^t \mathcal{L}_{\tau(i)}^j(v, s, \bar{x}(v), \bar{u}(v)) dv \right]. \end{aligned}$$

Therefore, combining the equality above with the dynamic programming

principle (3.34), we obtain that

$$\begin{aligned}
\mathcal{V}(s, y) &= \mathcal{J}(s, y; \bar{u}(\cdot)) = E \left[G_{\tau(k)}^+(t, s) \mathcal{J}(t, \bar{x}(t); \bar{u}(t)) \right. \\
&\quad \left. + \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_s^t \mathcal{L}_{\tau(i)}^j(v, s, \bar{x}(v), \bar{u}(v)) dv \right] \\
&\leq E \left[G_{\tau(k)}^+(t, s) \mathcal{V}(t, \bar{x}(t)) \right. \\
&\quad \left. + \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_s^t \mathcal{L}_{\tau(i)}^j(v, s, \bar{x}(v), \bar{u}(v)) dv \right] \\
&\leq \mathcal{V}(s, y) .
\end{aligned} \tag{3.40}$$

Hence, all the inequalities in (3.40) are indeed equalities and we conclude that

$$E[\mathcal{J}(t, \bar{x}(t); \bar{u}(t))] = E[\mathcal{V}(t, \bar{x}(t))] .$$

The result then follows by observing that by definition of the value function, we must have that $\mathcal{V}(t, \bar{x}(t)) \geq \mathcal{J}(t, \bar{x}(t); \bar{u}(t))$, $\mathbb{P}$ - a.s.. ■

3.3 Hamilton-Jacobi-Bellman equation

In this section we use the dynamic programming principle obtained in the previous section to derive a sequence of HJB equations describing the value function V on each non-empty set $\Delta_k \times \mathbb{R}^N$. Recall that the value function V agrees with the k^{th} value function $\mathcal{V}^k$ in the set $\Delta_k \times \mathbb{R}^N$. The statement below provides a HJB for the k^{th} value function $\mathcal{V}^k$ introduced in (3.32). Theorem 3.1.2 follows as a consequence.

Theorem 3.3.1 *Assume that conditions (A1) and (A2) hold and that for every $i, j \in \{1, 2, \dots, K\}$ such that $i \geq j$ the conditional density functions $g_{\tau(i)|\tau(j)}^-(t, s)$ are uniformly continuous with respect to its first argument t . Let $k \in \{1, 2, \dots, K\}$ be such that $t_{k-1} < T$ and assume that the k^{th} value function $\mathcal{V}^k$ is such that $\mathcal{V}^k \in C^{1,2}([t_{k-1}, T) \times \mathbb{R}^N; \mathbb{R})$. Then, the k^{th} value function $\mathcal{V}^k$ is the solution of the Hamilton-Jacobi-Bellman equation*

$$\begin{cases} V_t - g_{\tau(k)|\tau(k)}^-(t, t)V + \sup_{u \in U} \mathcal{H}^k(t, x, u, V_x, V_{xx}) = 0 \\ V(T, x) = \sum_{j \in I(t_{k-1})} \Psi_j(T, x), \quad (t, x) \in [t_{k-1}, T) \times \mathbb{R}^N, \end{cases} \quad (3.41)$$

where the Hamiltonian function $\mathcal{H}^k$ is given by

$$\begin{aligned} \mathcal{H}^k(t, x, u, p, B) = & \sum_{j \in I(t_{k-1})} \left(L_j(t, x, u) + G_k(t) \Psi_j(t, x) \right) \\ & + \langle p, f_k(t, x, u) \rangle + \frac{1}{2} \text{tr} (g_k^T(t, x, u) B g_k(t, x, u)) \end{aligned} \quad (3.42)$$

for all $(t, x, u, p, B) \in [0, T] \times \mathbb{R}^N \times U \times \mathbb{R}^N \times \mathcal{S}^N$, $\mathcal{S}^N$ is the set of all $N \times N$ symmetric real matrices and $G_k(t)$ is given by

$$G_k(t) = \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K g_{\tau(i)|\tau(k)}^-(t, t) .$$

Proof. Throughout this proof, we will use the simplified notation convention introduced for the proof of Theorem 3.2.1. Additionally, we will denote $\mathcal{H}^k$, f_k and g_k by $\mathcal{H}$, f and g , respectively.

Fix $k \in \{1, 2, \dots, K\}$ such that $t_{k-1} < T$, $(s, y) \in [t_{k-1}, T) \times \mathbb{R}^N$ and $u \in U$. Let $x(\cdot)$ be the state trajectory corresponding to the control $u(\cdot) \in \mathcal{U}^w[s, T]$ for equation (3.3), where $u(t)$ is constant and given by $u(t) \equiv u$. Take $\hat{s} \in [s, T)$. Using Itô's formula, we obtain

$$\mathcal{V}(\hat{s}, x(\hat{s})) - \mathcal{V}(s, y) = \int_s^{\hat{s}} a(t, x(t), u(t)) dt + \int_s^{\hat{s}} b(t, x(t), u(t)) dW(t),$$

where

$$\begin{aligned} a(t, x, u) &= \mathcal{V}_t(t, x) + \langle \mathcal{V}_x(t, x), f(t, x, u) \rangle + \frac{1}{2} \text{tr}(g^T(t, x, u) \mathcal{V}_{xx}(t, x) g(t, x, u)) \\ b(t, x, u) &= (\mathcal{V}_x(t, x))^T g(t, x, u). \end{aligned}$$

Dividing the previous identity by $\hat{s} - s$ and taking the expectation in both sides, we obtain

$$\frac{E[\mathcal{V}(\hat{s}, x(\hat{s})) - \mathcal{V}(s, y)]}{\hat{s} - s} = \frac{1}{\hat{s} - s} E \left[\int_s^{\hat{s}} a(t, x(t), u(t)) dt \right]. \quad (3.43)$$

Using the dynamic programming principle, Theorem 3.2.1, we obtain

$$\begin{aligned} -\frac{E[\mathcal{V}(\hat{s}, x(\hat{s})) - \mathcal{V}(s, y)]}{\hat{s} - s} &\geq \frac{1}{\hat{s} - s} E \left[\left(G_{\tau(k)}^+(\hat{s}, s) - 1 \right) \mathcal{V}(\hat{s}, x(\hat{s})) \right. \\ &\quad \left. + \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_s^{\hat{s}} \mathcal{L}_{\tau(i)}^j(t, s, x(t), u(t)) dt \right]. \end{aligned}$$

Recalling that $G_{\tau(k)}^+(s, s) = 1$ and combining identity (3.43) with the last inequality, the later becomes

$$\begin{aligned} \frac{1}{\hat{s} - s} E \left[\int_s^{\hat{s}} a(t, x(t), u(t)) + \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \mathcal{L}_{\tau(i)}^j(t, s, x(t), u(t)) dt \right. \\ \left. + \left(G_{\tau(k)}^+(\hat{s}, s) - G_{\tau(k)}^+(s, s) \right) \mathcal{V}(\hat{s}, x(\hat{s})) \right] \leq 0. \quad (3.44) \end{aligned}$$

Recall the definition of the conditional Lagrangian function $\mathcal{L}_{\tau(i)}^j$ in (3.8) and note that

$$\begin{aligned} \mathcal{L}_{\tau(i)}^j(t, t, x(t), u(t)) &= G_{\tau(i)}^+(t, t) L_j(t, x(\cdot), u(\cdot)) + g_{\tau(i)}^-(t, t) \Psi_j(t, x(\cdot)) \\ &= L_j(t, x(\cdot), u(\cdot)) + g_{\tau(i)}^-(t, t) \Psi_j(t, x(\cdot)). \end{aligned}$$

From the equality above, we obtain that

$$\begin{aligned} \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \mathcal{L}_{\tau(i)}^j(t, t, x(t), u(t)) \\ = \sum_{j \in I(t_{k-1})} L_j(t, x(t), u(t)) + G_k(t) \Psi_j(t, x(t)), \quad (3.45) \end{aligned}$$

where $G_k(t)$ is as given in the statement of the theorem.

Letting $\hat{s} \downarrow s$ in (3.44) and using (3.45), we obtain

$$\mathcal{V}_t(s, y) - g_{\tau(k)}^-(s, s) \mathcal{V}(s, y) + \mathcal{H}(s, y, u, \mathcal{V}_x(s, y), \mathcal{V}_{xx}(s, y)) \leq 0$$

for every $u \in U$, where $\mathcal{H}$ is as given in (3.42). This results in

$$\mathcal{V}_t(s, y) - g_{\tau(k)}^-(s, s)\mathcal{V}(s, y) + \sup_{u \in U} \mathcal{H}(s, y, u, \mathcal{V}_x(s, y), \mathcal{V}_{xx}(s, y)) \leq 0.$$

On the other hand, for any $\epsilon > 0$ and any $\hat{s} \in (s, T)$ with $\hat{s} - s$ small enough, there exists a $\tilde{u}(\cdot) := u_{\epsilon, \hat{s}}(\cdot) \in \mathcal{U}^w[s, T]$ such that

$$\begin{aligned} \mathcal{V}(s, y) - \epsilon(\hat{s} - s) &\leq E \left[G_{\tau(k)}^+(\hat{s}, s) \mathcal{V}(\hat{s}, x(\hat{s})) \right. \\ &\quad \left. + \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_s^{\hat{s}} \mathcal{L}_{\tau(i)}^j(t, s, x(t), \tilde{u}(t)) dt \right]. \end{aligned}$$

Rearranging terms in the previous inequality we get

$$\begin{aligned} \epsilon &\geq -\frac{E[\mathcal{V}(\hat{s}, x(\hat{s})) - \mathcal{V}(s, y)]}{\hat{s} - s} - \frac{1}{\hat{s} - s} E \left[\left(G_{\tau(k)}^+(\hat{s}, s) - 1 \right) \mathcal{V}(\hat{s}, x(\hat{s})) \right. \\ &\quad \left. + \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \int_s^{\hat{s}} \mathcal{L}_{\tau(i)}^j(t, s, x(t), \tilde{u}(t)) dt \right]. \end{aligned}$$

Combining identity (3.43) with the previous inequality, the latter becomes

$$\begin{aligned} \epsilon &\geq -\frac{1}{\hat{s} - s} E \left[\left(G_{\tau(k)}^+(\hat{s}, s) - 1 \right) \mathcal{V}(\hat{s}, x(\hat{s})) \right. \\ &\quad \left. + \int_s^{\hat{s}} a(t, x(t), \tilde{u}(t)) + \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \mathcal{L}_{\tau(i)}^j(t, s, x(t), \tilde{u}(t)) dt \right]. \end{aligned}$$

Noting that $G_{\tau_{(k)}}^+(s, s) = 1$ and using the definitions (3.9), (3.10) and (3.11), we get

$$\begin{aligned} \epsilon \geq & -\frac{1}{\hat{s} - s} E \left[\int_s^{\hat{s}} a(t, x(t), \tilde{u}(t)) - g_{\tau_{(k)}}^-(t, s) \mathcal{V}(\hat{s}, x(\hat{s})) \right. \\ & \left. + \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \mathcal{L}_{\tau_{(i)}}^j(t, s, x(t), \tilde{u}(t)) dt \right]. \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} \epsilon \geq & -\frac{1}{\hat{s} - s} E \left[\int_s^{\hat{s}} \left(\mathcal{V}_t(t, x(t)) - g_{\tau_{(k)}}^-(t, s) \mathcal{V}(\hat{s}, x(\hat{s})) \right. \right. \\ & + \sup_{\tilde{u} \in U} \left\{ \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \mathcal{L}_{\tau_{(i)}}^j(t, s, x(t), \tilde{u}) \right. \\ & + \langle \mathcal{V}_x(t, x(t)), f(t, x(t), \tilde{u}) \rangle \\ & \left. \left. + \frac{1}{2} \text{tr} (g^T(t, x(t), \tilde{u}) \mathcal{V}_{xx}(t, x(t)) g(t, x(t), \tilde{u})) \right\} \right) dt \right]. \end{aligned}$$

By assumption (A2), we have that the functions L_i , ψ_i , f and g are uniformly continuous. By hypothesis we have that the conditional densities $g_{\tau_{(k)}}^-$ are also uniformly continuous. Taking the limit $\hat{s} \rightarrow s$, we conclude that

$$-\epsilon \leq \mathcal{V}_t(s, y) - g_{\tau_{(k)}}^-(s, s) \mathcal{V}(s, y) + \sup_{u \in U} \mathcal{H}(s, y, u, \mathcal{V}_x(s, y), \mathcal{V}_{xx}(s, y)),$$

completing the proof. ■

We will now state a verification theorem, providing a rigorous connection between the solution of the HJB equation and the original stochastic control problem.

Theorem 3.3.2 (Verification Theorem) *Assume that conditions (A1) and (A2) hold and let $k \in \{1, 2, \dots, K\}$ be such that $t_{k-1} < T$. Let $\mathcal{V}^k \in C^{1,2}([t_{k-1}, T) \times \mathbb{R}^N; \mathbb{R})$ be a solution of the HJB equation (3.41). Then, the inequality*

$$\mathcal{V}^k(s, y) \geq \mathcal{J}^k(s, y; u(\cdot))$$

holds for every $u(\cdot) \in \mathcal{U}^w[s, T]$ and every $(s, y) \in [t_{k-1}, T) \times \mathbb{R}^N$. Furthermore, an admissible pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ is optimal for (3.32) if and only if

$$\begin{aligned} & \mathcal{V}_t^k(t, \bar{x}(t)) - g_{\tau_{(k)} | \tau_{(k)}}^-(t, t) \mathcal{V}(t, \bar{x}(t)) \\ & + \mathcal{H}^k(t, \bar{x}(t), \bar{u}(t), \mathcal{V}_x^k(t, \bar{x}(t)), \mathcal{V}_{xx}^k(t, \bar{x}(t))) = 0, \end{aligned}$$

for a.e. $t \in [s, T)$, $\mathbb{P}$ -a.s..

Proof. Throughout this proof, we will use the simplified notation convention introduced for the proof of Theorem 3.2.1. Additionally, we will denote $\mathcal{H}^k$, f_k and g_k by $\mathcal{H}$, f and g , respectively.

For any $u(\cdot) \in \mathcal{U}^w[s, T]$ and corresponding state trajectory $x(\cdot)$, applying Itô's formula to the function

$$G_{\tau_{(k)}}^+(t, s) \mathcal{V}(t, x(t))$$

we get

$$\begin{aligned} \mathcal{V}(s, y) = E & \left[G_{\tau(k)}^+(T, s) \mathcal{V}(T, x(T)) \right. \\ & \left. - \int_s^T G_{\tau(k)}^+(t, s) a(t, x(t), u(t)) - g_{\tau(k)}^-(t, s) \mathcal{V}(t, x(t)) dt \right]. \end{aligned}$$

Using the terminal condition of the HJB equation (3.41), we obtain

$$\begin{aligned} \mathcal{V}(s, y) = E & \left[G_{\tau(k)}^+(T, s) \sum_{j \in I(t_{k-1})} \Psi_j(T, x(T)) \right. \\ & \left. - \int_s^T G_{\tau(k)}^+(t, s) a(t, x(t), u(t)) - g_{\tau(k)}^-(t, s) \mathcal{V}(t, x(t)) dt \right]. \end{aligned} \quad (3.46)$$

Using Lemma 3.2.3, for every $i \geq k$ and every $t \geq s$ we have that

$$G_{\tau(i)}^+(t, s) = G_{\tau(k)}^+(t, s) \quad \text{and} \quad g_{\tau(i)}^-(t, s) = g_{\tau(i)}^-(t, t) G_{\tau(k)}^+(t, s). \quad (3.47)$$

Combining (3.46) with (3.47), we get

$$\begin{aligned} \mathcal{V}(s, y) = E & \left[\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} G_{\tau(i)}^+(T, s) \Psi_j(T, x(T)) \right. \\ & \left. - \int_s^T G_{\tau(k)}^+(t, s) \left(a(t, x(t), u(t)) - g_{\tau(k)}^-(t, t) \mathcal{V}(t, x(t)) \right) dt \right]. \end{aligned}$$

Rearranging terms and using Lemma 3.2.6, the equality above becomes

$$\begin{aligned} \mathcal{V}(s, y) &= \mathcal{J}(s, y; u(\cdot)) \\ &\quad - E \left[\int_s^T G_{\tau(k)}^+(t, s) \left(a(t, x(t), u(t)) - g_{\tau(k)}^-(t, t) \mathcal{V}(t, x(t)) \right) \right. \\ &\quad \left. + \frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \mathcal{L}_{\tau(i)}^j(t, s, x(t), u(t)) \, dt \right]. \end{aligned} \quad (3.48)$$

Recalling the definition of the conditional Lagrangian function $\mathcal{L}_{\tau(i)}^j$ in (3.8) and using Lemma 3.2.3, we obtain that

$$\begin{aligned} &\frac{1}{\kappa(t_{k-1})} \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \mathcal{L}_{\tau(i)}^j(t, s, x(t), u(t)) \\ &= \frac{1}{\kappa(t_{k-1})} G_{\tau(k)}^+(t, s) \sum_{i=k}^K \sum_{j \in I(t_{k-1})} \mathcal{L}_{\tau(i)}^j(t, t, x(t), u(t)) \\ &= G_{\tau(k)}^+(t, s) \sum_{j \in I(t_{k-1})} L_j(t, x(t), u(t)) + G_k(t) \Psi_j(t, x(t)) . \end{aligned} \quad (3.49)$$

Combining (3.48) with (3.49), we get

$$\begin{aligned} \mathcal{V}(s, y) &= \mathcal{J}(s, y; u(\cdot)) - E \left[\int_s^T G_{\tau(k)}^+(t, s) \left(\mathcal{V}_t(t, x(t)) \right. \right. \\ &\quad \left. \left. - g_{\tau(k)}^-(t, t) \mathcal{V}(t, x(t)) + \mathcal{H}^k(t, x(t), u(t), \mathcal{V}_x(t, x(t)), \mathcal{V}_{xx}(t, x(t))) \right) dt \right], \end{aligned} \quad (3.50)$$

where $\mathcal{H}^k$ is as given in (3.42).

Since $G_{\tau_{(k)}}^+(t, s) \geq 0$, using the HJB equation (3.41), we obtain that

$$\mathcal{V}(s, y) \geq \mathcal{J}(s, y; u(\cdot)) ,$$

concluding the proof of the first part of the theorem

We will now prove the second part of the theorem. Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be an optimal solution for (3.32). From (3.50) and the fact that $G_{\tau_{(k)}}^+(t, s) \geq 0$, we get

$$\mathcal{V}_t(t, \bar{x}(t)) - g_{\tau_{(k)}}^-(t, t) \mathcal{V}(t, \bar{x}(t)) + \mathcal{H}^k(t, \bar{x}(t), \bar{u}(t), \mathcal{V}_x(t, \bar{x}(t)), \mathcal{V}_{xx}(t, \bar{x}(t))) \geq 0 .$$

The result then follows immediately from realizing that the opposite inequality is a consequence of the HJB equation (3.41). ■

3.4 A consumption-investment problem with multiple random time horizons

Consider a financial market consisting of one risk-free security and a fixed number $N \geq 1$ of risky securities. The asset prices are $(S_0(t))_{0 \leq t \leq T}$ and $(S_n(t))_{0 \leq t \leq T}$ for $n = 1, \dots, N$ evolve according to the equations:

$$\begin{aligned} dS_0(t) &= r(t)S_0(t)dt , & S_0(0) &= s_0 > 0 , \\ dS_n(t) &= \mu_n(t)S_n(t)dt + S_n(t) \sum_{m=1}^M \sigma_{nm}(t)dW_m(t) , & S_n(0) &= s_n > 0 , \end{aligned}$$

where $W(t) = (W_1(t), \dots, W_M(t))^T$ is a standard M -dimensional Brownian motion on a filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $r(t)$ is the riskless interest rate, $\mu(t) = (\mu_1(t), \dots, \mu_N(t)) \in \mathbb{R}^N$ is the vector of the risky-assets appreciation rates and $\sigma(t) = (\sigma_{nm}(t))_{1 \leq n \leq N, 1 \leq m \leq M}$ is the matrix of risky-assets volatilities. Each sub- σ -algebra $\mathcal{F}_t$ represents the information available to any financial market observer during the time span $[0, t]$.

For simplicity of exposition, we assume that the coefficients $r(t)$, $\mu(t)$ and $\sigma(t)$ are deterministic continuous functions on the interval $[0, T]$. We also assume that the interest rate $r(t)$ is positive for all $t \in [0, T]$ and the matrix $\sigma(t)$ is such that $\sigma\sigma^T$ is non-singular for Lebesgue almost all $t \in [0, T]$ and satisfies the following integrability condition

$$\sum_{n=1}^N \sum_{m=1}^M \int_0^T \sigma_{nm}^2(t) dt < \infty .$$

Furthermore, we suppose that there exists an $(\mathcal{F}_t)_{0 \leq t \leq T}$ -progressively measurable process $\pi(t) \in \mathbb{R}^M$, called the market price of risk, such that for Lebesgue-almost-every $t \in [0, T]$ the risk premium

$$\alpha(t) = (\mu_1(t) - r(t), \dots, \mu_N(t) - r(t)) \in \mathbb{R}^N$$

is related to $\pi(t)$ by the equation

$$\alpha^T(t) = \sigma(t)\pi^T(t) \quad \text{a.s.}$$

and the following two conditions hold

$$\int_0^T \|\pi(t)\|_M^2 dt < \infty \quad \text{a.s.}$$

$$E \left[\exp \left(- \int_0^T \pi(s) dW(s) - \frac{1}{2} \int_0^T \|\pi(s)\|_M^2 ds \right) \right] = 1 ,$$

where $\|\cdot\|_M$ denotes the Euclidean norm in $\mathbb{R}^M$. The existence of $\pi(t)$ ensures the absence of arbitrage opportunities in the financial market defined above. See [26] for further details on market viability.

Let $\tau_1, \tau_2, \dots, \tau_K$ be non-negative independent and identically distributed continuous random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, assume that $\tau_1, \tau_2, \dots, \tau_K$ are independent of the filtration $\mathbb{F}$ generated by the Brownian motion $W(t)$. Let (ξ_i) and (t_i) be as defined in Section 3.1.

We will consider the problem faced by an agent investing in the financial market described above and consuming from a basket of K goods that may become unavailable for consumption from some time τ_i onwards. The agent goal is to jointly maximize:

- i) the utility derived from consumption of the goods available at each instant of time;
- ii) the utility derived from wealth at the multiple random instants of time $\xi_1, \xi_2, \dots, \xi_K$, representing either the moment at which a given good becomes unavailable or the deterministic horizon T .

For each $i = 1, 2, \dots, K$ and $t \in [0, t_K)$, let $c_i(t)$ denote the agent consumption of the i^{th} good at the instant of time t . The *consumption process*

is given by

$$c(t) = (c_1(t), \dots, c_K(t)) .$$

We assume that the consumption process is a $(\mathcal{F}_t)$ -progressively measurable non-negative process with the property that each component $c_i(\cdot)$ is non-negative and satisfying the following integrability condition

$$\int_0^{\xi_i} \|c(t)\|_K^2 dt < \infty \quad \text{a.s. .}$$

For each $n = 0, 1, \dots, N$ and $t \in [0, t_K]$, let $\theta_n(t)$ denote the fraction of the agent wealth allocated to the asset S_n at time t . The *portfolio process* is given by

$$\Theta(t) = (\theta_0(t), \theta_1(t), \dots, \theta_N(t)) \in \mathbb{R}^{N+1} ,$$

where

$$\sum_{n=0}^N \theta_n(t) = 1 , \quad 0 \leq t \leq t_K . \quad (3.51)$$

We assume that the portfolio process is $(\mathcal{F}_t)$ -progressively measurable and that

$$\int_0^{t_K} \|\Theta(t)\|_{N+1}^2 dt < \infty \quad \text{a.s. .}$$

Using relation (3.51), we can always write $\theta_0(t)$ in terms of $\theta_1(t), \dots, \theta_N(t)$. From now on, we will define the portfolio process in terms of the *reduced portfolio process* $\theta(t) \in \mathbb{R}^N$ given by

$$\theta(t) = (\theta_1(t), \theta_2(t), \dots, \theta_N(t)) \in \mathbb{R}^N .$$

We define the *wealth process* $X(t)$, for $t \in [0, t_K]$ by

$$X(t) = x - \sum_{i=1}^K \int_0^{\xi_i \wedge t} c_i(s) \, ds + \sum_{n=0}^N \int_0^t \frac{\theta_n(s)X(s)}{S_n(s)} \, dS_n(s) ,$$

where x is the agent initial wealth. The last equation can be rewritten in the differential form

$$\begin{aligned} dX(t) = & \left(- \sum_{i \in I(t)} c_i(t) + \left(\theta_0(t)r(t) + \sum_{n=1}^N \theta_n(t)\mu_n(t) \right) X(t) \right) dt \\ & + \sum_{n=1}^N \theta_n(t)X(t) \sum_{m=1}^M \sigma_{nm}(t) dW_m(t) , \end{aligned}$$

where $I(t)$ is the index set defined in Section 3.1.

Let us denote by $\mathcal{A}(x)$ the set of all admissible decision strategies, i.e. all admissible choices for the control variables $(c, \theta) \in \mathbb{R}^{K+N}$. The dependence of $\mathcal{A}(x)$ on x denotes the restriction imposed on the wealth process by the boundary condition $X(0) = x$. Similarly, let us denote by $\mathcal{A}(t, x)$ the set of all admissible decision strategies (c, θ) for the dynamics of the wealth process with boundary condition $X(t) = x$.

The agent problem is to find an investment strategy $\nu \in \mathcal{A}(x)$ which maximizes the expected utility

$$V(x) = \sup_{(c, \theta) \in \mathcal{A}(x)} E \left[\sum_{i=1}^K \int_0^{\xi_i} L_i(t, c_i(t)) \, dt + \Psi_i(\xi_i, X(\xi_i)) \right] , \quad (3.52)$$

where $L_i(t, c_i(t))$, $i = 1, 2, \dots, K$, is the utility functions describing the agent preferences regarding the i^{th} good consumption over the random time

interval $[0, \xi_i]$, and $\Psi_i(\xi_i, X(\xi_i))$, $i = 1, 2, \dots, K$, are the utility functions describing the agent preferences concerning the amount of wealth held at the instants of time $\xi_1, \xi_2, \dots, \xi_K$.

Assume that the conditions of Theorem 3.1.2 hold. Then, for every $k \in \{1, 2, \dots, K\}$ such that $t_{k-1} < T$ the value function V restricted to the set $\Delta_k \times \mathbb{R}^N$ is identically equal to the solution of the Hamilton-Jacobi-Bellman equation:

$$\begin{cases} V_t - g_{\tau(k)|\tau(k)}^-(t, t)V + \sup_{(c, \theta) \in \mathcal{A}(t, x)} \mathcal{H}^k(t, x, c, \theta, V_x, V_{xx}) = 0 \\ V(T, x) = \sum_{i \in I(t_{k-1})} \Psi_i(T, x), \quad (t, x) \in [t_{k-1}, T) \times \mathbb{R}^N, \end{cases} \quad (3.53)$$

where the Hamiltonian function $\mathcal{H}^k$ is given by

$$\begin{aligned} \mathcal{H}^k(t, x, c, \theta, p, B) = & \sum_{i \in I(t_{k-1})} (L_i(t, c_i) + G_k(t) \Psi_i(t, x)) \\ & + \left(- \sum_{i \in I(t_{k-1})} c_i(t) + \left(r(t) + \sum_{n=1}^N \theta_n (\mu_n(t) - r(t)) \right) x \right) p \\ & + \frac{x^2}{2} \sum_{m=1}^M \left(\sum_{n=1}^N \theta_n \sigma_{nm}(t) \right)^2 B \end{aligned}$$

and $G_k(t)$ is as given in the statement of Theorem 3.1.2.

3.4.1 Constant relative risk aversion utility functions

We will now focus our attention on the class of discounted CRRA utility functions, given by

$$L_i(t, c_i) = e^{-\rho t} \frac{c_i^{\gamma_i}}{\gamma_i}, \quad \Psi_i(t, x) = e^{-\rho t} \frac{x^{\beta_i}}{\beta_i}, \quad i = 1, 2, \dots, K, \quad (3.54)$$

where the risk aversion parameters γ_i and β_i are different from zero and strictly less than one for every $i \in \{1, 2, \dots, K\}$, and the discount rate ρ is positive.

Computing the first-order conditions for a regular interior maximum of $\mathcal{H}^k$ with respect to (c, θ) , we obtain the following two equalities:

$$\begin{aligned} -V_x(t, x) + \frac{dL_i}{dc_i}(t, c_i^*) &= 0, \quad i \in I(t_{k-1}) \\ xV_x(t, x)\alpha + x^2V_{xx}(t, x)\theta^*\sigma\sigma^T &= 0_{\mathbb{R}^N}. \end{aligned}$$

Thus, solving with respect to c and θ , we are able to express the optimal strategies in terms of the value function derivatives as

$$\begin{aligned} c_i^*(t, x) &= \left(e^{\rho t} V_x(t, x)\right)^{-1/(1-\gamma_i)}, \quad i \in I(t_{k-1}) \\ \theta^*(t) &= -\frac{V_x(t, x)}{xV_{xx}(t, x)}\alpha(t)\xi, \end{aligned} \quad (3.55)$$

where ξ denotes the non-singular square matrix $(\sigma\sigma^T)^{-1}$. Using the utility functions (3.54) and substituting c_i^* and θ^* in the HJB equation (3.53), we

arrive at the following partial differential equation

$$\begin{aligned}
 & V_t(t, x) - g_{\tau_{(k)}|\tau_{(k)}}^-(t, t)V(t, x) + r(t)xV_x(t, x) \\
 & - \Sigma(t)\frac{(V_x(t, x))^2}{V_{xx}(t, x)} + G_k(t)e^{-\rho t} \sum_{i \in I(t_{k-1})} \frac{x^{\beta_i}}{\beta_i} \\
 & + \sum_{i \in I(t_{k-1})} \left(\frac{1 - \gamma_i}{\gamma_i} \right) e^{-\rho t/(1-\gamma_i)} (V_x(t, x))^{-\gamma_i/(1-\gamma_i)} = 0,
 \end{aligned} \tag{3.56}$$

where $\Sigma(t)$ is given by

$$\Sigma(t) = \alpha(t)\xi\alpha^T(t) - \frac{1}{2} \|\sigma^T(t)\xi\alpha^T(t)\|_M^2 \tag{3.57}$$

and the terminal condition is given by

$$V(T, x) = \sum_{i \in I(t_{k-1})} \Psi_i(T, x). \tag{3.58}$$

The following result provides the optimal strategies for discounted CRRA utility functions in case where the risk aversion parameters are all equal. Quite naturally, we obtain that the optimal strategy consists in consuming identical amounts of each good available for consumption at a given time.

Corollary 3.4.1 *Let ξ denote the non-singular square matrix given by $(\sigma\sigma^T)^{-1}$. Assume that the risk aversion parameters γ_i and β_i are all equal. Then, for every $k \in \{1, 2, \dots, K\}$ such that $t_{k-1} < T$ the value function V restricted to the set $\Delta_k \times \mathbb{R}$ is of the form*

$$V(t, x) = \kappa(t_{k-1})a_k(t)\frac{x^\gamma}{\gamma},$$

where $a_k(t)$ is the solution of the boundary value problem

$$\begin{aligned} \frac{da_k(t)}{dt} + \Phi_k(t) a_k(t) + \eta(t) (\kappa(t_{k-1}) a_k(t))^{-\gamma/(1-\gamma)} + G_k(t) e^{-\rho t} &= 0 \\ a_k(T) &= e^{-\rho T} , \end{aligned}$$

the function $\Sigma(t)$ is as given in (3.57), $G_k(t)$ is as given in the statement of Theorem 3.1.2 and $\Phi_k(t)$ and $\eta(t)$ are given by

$$\begin{aligned} \Phi_k(t) &= \gamma \left(r(t) + \frac{\Sigma(t)}{1-\gamma} \right) - g_{\tau(k)|\tau(k)}^-(t, t) , \\ \eta(t) &= \gamma (1-\gamma) e^{-\rho t/(1-\gamma)} . \end{aligned}$$

Furthermore, on each set $\Delta_k \times \mathbb{R}$, the optimal strategies are given by

$$\begin{aligned} c_i^*(t, x) &= x (\kappa(t_{k-1}) e^{\rho t} a_k(t))^{-1/(1-\gamma)} , \quad i \in I(t_{k-1}) \\ \theta^*(t, x) &= \frac{1}{1-\gamma} \alpha(t) \xi . \end{aligned}$$

Proof. Start by noting that in the case where the risk aversion parameters γ_i and β_i are all equal (we denote such common value by γ), the partial differential equation (3.56) becomes

$$\begin{aligned} V_t(t, x) - g_{\tau(k)|\tau(k)}^-(t, t) V(t, x) + r(t) x V_x(t, x) - \Sigma(t) \frac{(V_x(t, x))^2}{V_{xx}(t, x)} \\ + \sum_{i \in I(t_{k-1})} \frac{1}{\gamma} \left(G_k(t) e^{-\rho t} x^\lambda + (1-\gamma) e^{-\rho t/(1-\gamma)} (V_x(t, x))^{-\gamma/(1-\gamma)} \right) &= 0 , \end{aligned} \quad (3.59)$$

where $\Sigma(t)$ is as given in (3.57), $G_k(t)$ is as given in the statement of Theorem 3.1.2 and the terminal condition is as given in (3.58).

We consider an ansatz of the form

$$V(t, x) = \sum_{i \in I(t_{k-1})} a_k(t) \frac{x^\gamma}{\gamma} = \kappa(t_{k-1}) a_k(t) \frac{x^\gamma}{\gamma}$$

and substitute it in (3.59). A simple computation implies that $a_k(t)$ is determined by the following boundary value problem

$$\begin{aligned} \frac{da_k(t)}{dt} + \Phi_k(t) a_k(t) + \eta(t) (\kappa(t_{k-1}) a_k(t))^{-\gamma/(1-\gamma)} + G_k(t) e^{-\rho t} &= 0 \\ a_k(T) &= e^{-\rho T}, \end{aligned}$$

where $\Phi_k(t)$ and $\eta(t)$ are as given in the statement of this corollary. Furthermore, using (3.55) we obtain that optimal strategies on $\Delta_k \times \mathbb{R}$ are given by

$$\begin{aligned} c_i^*(t, x) &= x (\kappa(t_{k-1}) e^{\rho t} a_k(t))^{-1/(1-\gamma)}, \quad i \in I(t_{k-1}) \\ \theta^*(t, x) &= \frac{1}{1-\gamma} \alpha(t) \xi, \end{aligned}$$

concluding the proof. ■

3.4.2 Numerical solution

Before concluding, we provide a numerical example where the risk aversion parameters do not coincide. We take the deterministic fixed horizon to be $T = 20$ and consider a financial market with one risk-free security and one risky asset ($N = 1$) driven by a one-dimensional Brownian motion ($M = 1$). The riskless interest rate is $r = 0.04$, the risky asset appreciation rate is

$\mu = 0.08$ and its volatility is $\sigma = 0.19$, whereas the discount rate is taken to be $\rho = 0.04$. We assume that there is a basket of three goods available for consumption ($K = 3$) up until some random times τ_1, τ_2, τ_3 , which are assumed to be independent and identically distributed non-negative random variables with distribution function given by

$$F(t) = 1 - \exp(-0.0010748t - 0.0000035t^2) \ , \quad t \geq 0 \ .$$

Finally, the agent preferences are described by CRRA utilities of the form (3.54) with risk aversion parameters $\gamma_1 = 0.5$, $\gamma_2 = 0.45$, $\gamma_3 = 0.4$ and $\beta_1 = \beta_2 = \beta_3 = 0.5$.

In Figure 3.1, we plot the value function $V(t, x)$ for a realization where $\tau_1 = 10$, $\tau_2 = 15$ and $\tau_3 \geq 20$. The value function is obtained by solving three HJB equations of the form (3.56), restricting its solutions to the sets $\Delta_i \times \mathbb{R}^+$, $i \in \{1, 2, 3\}$, and finally, combining the corresponding graphs into a single one. Note the existence of discontinuities of the value function V on the sets $t_1 = 10$ and $t_2 = 15$. Such behaviour is due to the strong dependence of the functional J on the random variables τ_1, τ_2, τ_3 .

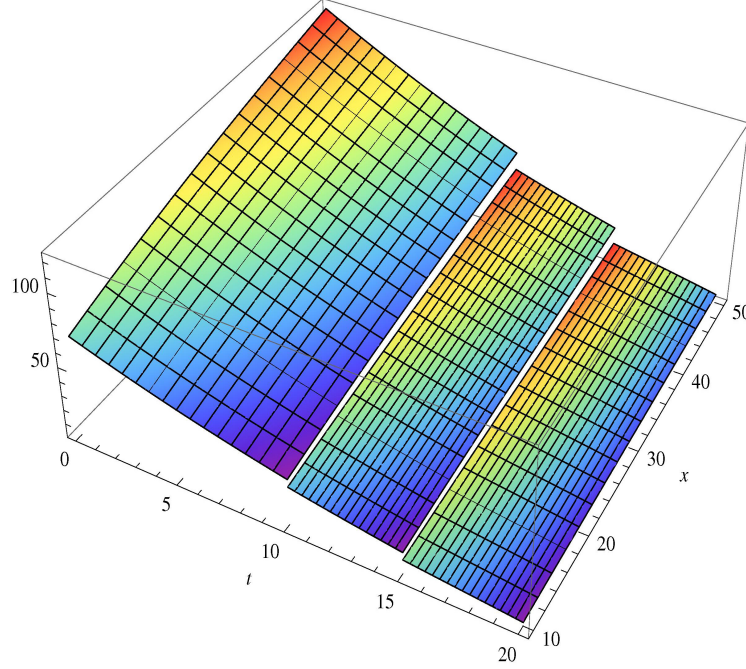
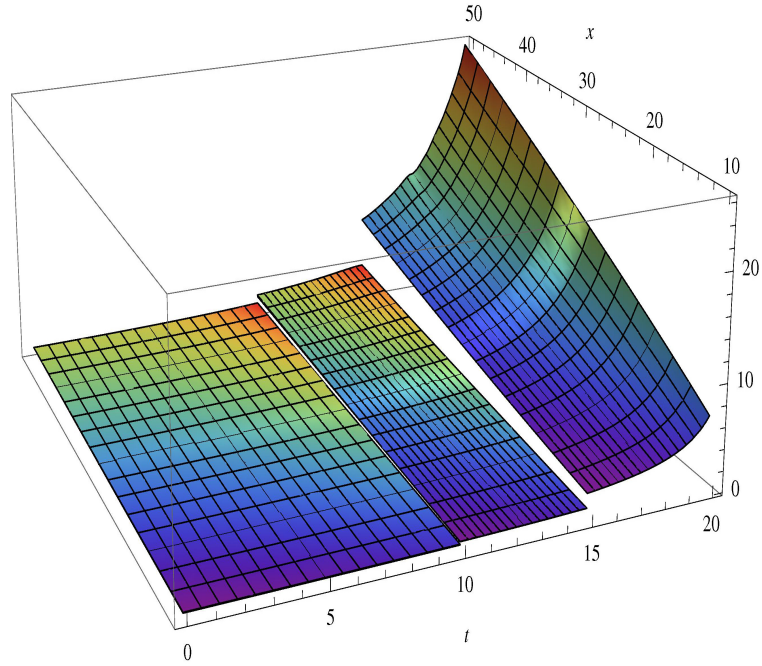
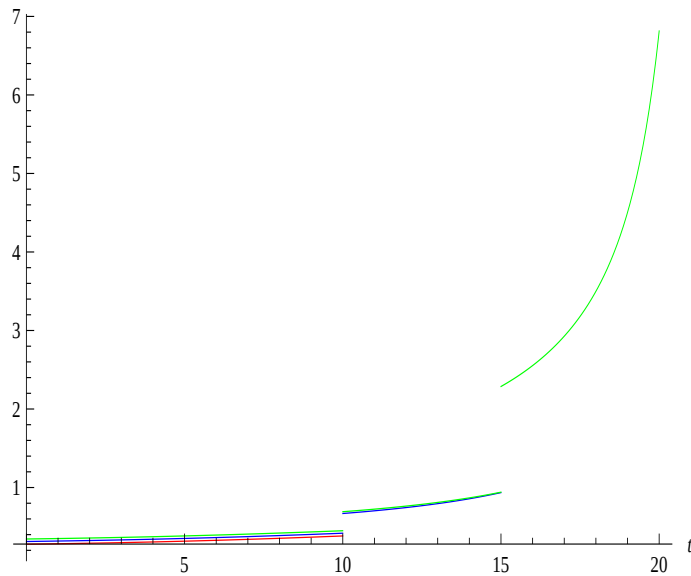


Figure 3.1: Plot of the value function $V(t, x)$ for values $(t, x) \in [0, 20] \times [10, 50]$ and a realization where $\tau_1 = 10$, $\tau_2 = 15$ and $\tau_3 \geq 20$.

In Figure 3.2a, we plot the optimal consumptions $c_i^*(t, x)$, $i \in \{1, 2, 3\}$, for a realization where $\tau_1 = 10$, $\tau_2 = 15$ and $\tau_3 \geq 20$. The optimal consumptions $c_i^*(t, x)$ are obtained directly from the value function according to (3.55). Note that on the set $\Delta_1 \times \mathbb{R}^+$ there exist three overlapping graphs, whereas on $\Delta_2 \times \mathbb{R}^+$ there are only two, and on $\Delta_3 \times \mathbb{R}^+$ only one. This is made clear in Figures 3.2b and 3.3, where a section of Figure 3.2a is presented for a fixed value $x = 10$. Note that the optimal consumptions are increasing functions of time t .



(a) Plot of the optimal consumptions $c^*(t, x)$ for values $(t, x) \in [0, 20] \times [10, 50]$ and a realization where $\tau_1 = 10$, $\tau_2 = 15$ and $\tau_3 \geq 20$.



(b) Optimal consumptions c^* for $x = 10$.

Figure 3.2: Optimal consumptions $c^*(t, x)$.

Moreover, as should be expected, the optimal consumptions of the remaining goods jump up when one of the goods becomes unavailable, since the same amount of wealth can be distributed among a smaller number of goods.

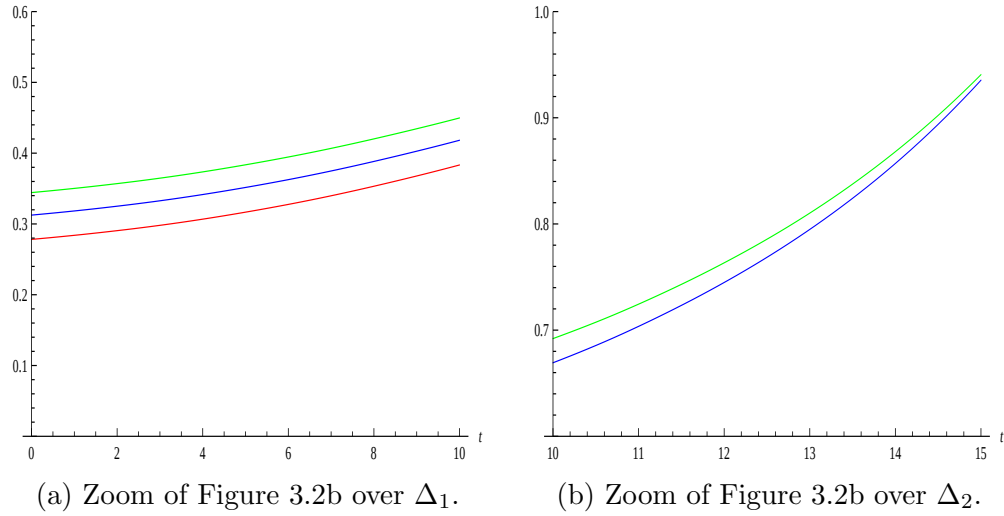


Figure 3.3: A section of the optimal consumptions for $x = 10$. The optimal consumption c_1^* is plotted in red, while c_2^* is plotted in blue and c_3^* is plotted in green.

3.5 Conclusions

We have studied a special class of stochastic optimal control problems with multiple random time horizons, for which the state variable follows a stochastic differential equation driven by a standard multi-dimensional Brownian motion, and the objective functional depends on non-negative continuous random variables $\tau_1, \tau_2, \dots, \tau_K$ that are assumed to be inde-

pendent and identically distributed. We have used the concept of order statistics and considered all the possible permutations for the order of the random variables $\tau_1, \tau_2, \dots, \tau_K$ to formulate the stochastic optimal control problem under consideration as one with a fixed planning horizon. We have then used dynamic programming techniques to derive a family of Hamilton-Jacobi-Bellman equations that describe the value function associated with the stochastic optimal control problem under consideration. We conclude with an application to the Merton's consumption-investment type problem.

Bibliography

- [1] I. Ajzen. Perceived behavioral control, self-efficacy, locus of control, and the theory of planned behavior. *Journal of Applied Social Psychology*, 32:665–683, 2002.
- [2] L. Almeida, J. Cruz, H. Ferreira, and A. A. Pinto. Bayesian-Nash equilibria in theory of planned behavior. *Journal of Difference Equations and Applications*, 17:1085–1093, 2011.
- [3] L. Almeida, J. Cruz, H. Ferreira, and A. A. Pinto. Leadership model. In M. Peixoto, A. A. Pinto and D. Rand, editor, *Dynamics, Games and Science I*, chapter 5, pages 53–59. Proceedings in Mathematics series, Springer-Verlag, 2011.
- [4] S. Baker, B. Beadnell, M. Gillmore, D. Morrison, B. Huang, and S. Stielstra. The theory of reasoned action and the role of external factors on heterosexual mens monogamy and condom use. *Journal of Applied Social Psychology*, 38:97–134, 2008.
- [5] R. E. Bellman. On the theory of dynamic programming. *Proc. Nat. Acad. Sci. USA*, 38:716–719, 1952.

-
- [6] R. E. Bellman. An introduction to the theory of dynamic programming. *Rand Corporation Report*, pages R-245, 1953.
 - [7] R. E. Bellman. Dynamic programming and a new formalism in the calculus of variations. *Proc. Nat. Acad. Sci. USA*, 40:231–235, 1954.
 - [8] R. E. Bellman. Dynamic programming and stochastic control process. *Information & Control*, 1:228–239, 1958.
 - [9] J. M. Bismut. *Analyse convexe et probabilités, Thèse*. PhD thesis, Faculté des Sciences de Paris, Paris, 1973.
 - [10] J. M. Bismut. Théorie probabiliste du contrôle des diffusions. *Mem. Amer. Math. Soc.*, 4:1–130, 1976.
 - [11] J. M. Bismut. An introductory approach to duality in optimal stochastic control. *SIAM Rev.*, 20:62–78, 1978.
 - [12] C. Blanchet-Scalliet, N. El Karoui, M. Jeanblanc, and L. Martellini. Optimal investment decisions when time-horizon is uncertain. *Journal of Mathematical Economics*, 44(11):1100–1113, 2008.
 - [13] V. G. Boltyanski, R. V. Gamkrelidze, and L. S. Pontryagin. On the theory of optimal processes. *Dokl. Akad. Nauk SSSR*, 10:7–10, 1956.
 - [14] V. G. Boltyanski, R. V. Gamkrelidze, and L. S. Pontryagin. The theory of optimal processes, I: The maximum principle. *Amer. Math. Soc. Transl.*, 2:341–382, 1961.

-
- [15] J. Brida, M. Defesa, M. Faias, and A. A. Pinto. Strategic choice in tourism with differentiated crowding types. *Economics Bulletin*, 30:1509–1515, 2010.
- [16] J. Brida, M. Defesa, M. Faias, and A. A. Pinto. A tourist’s choice model. In M. Peixoto, A. A. Pinto and D. Rand, editor, *Dynamics, Games and Science I*, chapter 10, pages 159–167. Proceedings in Mathematics series, Springer-Verlag, 2011.
- [17] J. P. Conley and M. H. Wooders. Tiebout economies with differential genetic types and endogenously chosen crowding characteristics. *Journal of Economic Theory*, 98:261–294, 2001.
- [18] H. A. David and H. N. Nagaraja. *Order Statistics*. INC. Wiley Series in Probability and Statistics. A John Wiley & Sons, third edition edition, 2003.
- [19] I. Duarte, D. Pinheiro, A. A. Pinto, and S. R. Pliska. Optimal life insurance purchase, consumption and investment on a financial market with multi-dimensional diffusive terms. *Optimization*, to appear (already published online) 2012.
- [20] F. Ferreira, F. A. Ferreira, and A. A. Pinto. Price-setting dynamical duopoly with incomplete information. In J. A. Tenreiro Machado, Albert C. J. Luo, Ramiro S. Barbosa, Manuel F. Silva and Lino B. Figueiredo, editor, *Nonlinear Science and Complexity*, pages 397–404. Springer, 2011.

-
- [21] W. H. Fleming and R. W. Rishel. *Deterministic and Stochastic Optimal Control*. Springer-Verlag, New York, 1975.
- [22] W. H. Fleming and H. M. Soner. *Controlled Markov Processes and Viscosity Solutions*. Springer-Verlag, New York, second edition edition, 2006.
- [23] J. J. Florentin. Optimal control of continuous time, markov, stochastic systems. *J. Electronics Control*, 10:473–488, 1961.
- [24] J. J. Florentin. Partial observability and optimal control. *J. of Electronics and Control*, 13:263–279, 1962.
- [25] U. G. Haussmann. General necessary conditions for optimal control of stochastic systems. *Math. Prog. Study*, 6:34–48, 1976.
- [26] I. Karatzas and S. Shreve. *Methods of Mathematical Finance*. Springer, 1998.
- [27] I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus*. Springer, 2000.
- [28] H. J. Kushner. Optimal stochastic control. *IRE Trans. Auto. Control*, AC-7:120–122, 1962.
- [29] H. J. Kushner. Necessary conditions for continuous parameter stochastic optimization problems. *SIAM J. Control*, 10:550–565, 1972.

-
- [30] Z. Liu, J. A. Siguaw, and C. A. Enz. Using tourist travel habits and preferences to assess strategic destination positioning: The case of costa rica. *Cornell Hospitality Quarterly*, 49:258–281, 2008.
- [31] X. Mao. *Stochastic differential equations and applications*. Ellis Horwood, 1997.
- [32] A. S. Mousa, T. Oliveira, A. A. Pinto, and R. Soeiro. Dynamics of human decisions. *Journal of Dynamics and Games*, 1(1):1–31, 2012.
- [33] A. S. Mousa, D. Pinheiro, and A. A. Pinto. A family of stochastic optimal control problems with multiple random time horizons. Submitted.
- [34] A. S. Mousa, A. A. Pinto, and R. Soeiro. Cohesive decisions. In Lluís Alsedà, Jim Cushing, Saber Elaydi, and Alberto Pinto, editors, *Discrete Dynamical Systems and Applications*. Springer Proceedings in Mathematics. Accepted for publication.
- [35] B. Oksendal. *Stochastic Differential Equations: An Introduction with Applications*. Springer, fifth edition edition, 1995.
- [36] A. A. Pinto. *Game Theory and Duopoly Models*. Interdisciplinary Applied Mathematics Series. Springer, New York, 2013.
- [37] A. A. Pinto, M. Faias, and A. S. Mousa. Resort pricing and bankruptcy. In M. Peixoto, A. A. Pinto and D. Rand, editor, *Dynamics, Games and Science II*, volume 2, chapter 40, pages 567–573. Proceedings in Mathematics series, Springer-Verlag, 2011.

-
- [38] A. A. Pinto, A. S. Mousa, M. S. Mousa, and R. M. Samarah. Tilings and bussola for making decisions. In M. Peixoto, A. A. Pinto and D. Rand, editor, *Dynamics, Games and Science I*, volume 1, chapter 44, pages 689–708. Proceedings in Mathematics series, Springer-Verlag, 2011.
- [39] A. A. Pinto, D. R. Rand, and F. Ferreira. *Fine Structures of Hyperbolic Diffeomorphisms*. Springer-Verlag Monograph, 2010.
- [40] S. R. Pliska and J. Ye. Optimal life insurance purchase and consumption/investment under uncertain lifetime. *Journal of Banking and Finance*, 31:1307–1319, 2007.
- [41] L. S. Pontryagin. The mathematical theory of optimal control processes and differential games. *Proc. Steklov Inst. Math.*, 4:123–159, 1986.
- [42] R. Soeiro, A. S. Mousa, and A. A. Pinto. Influência das decisões individuais num mercado competitivo. In J. Buescu F. Costa and J. T. Pinto, editors, *Matemática do Planeta Terra*. Editora Universitária do Instituto Superior Técnico. IST Press, Lisboa. Accepted for publication.
- [43] J. Yong and X. Y. Zhou. *Stochastic Controls: Hamiltonian Systems and HJB Equations*. Springer-Verlag, New York, 1999.

