Cosmological Dynamics of a Nonminimally Coupled Massive Vector Field

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Todas as correções determinadas pelo júri, e só essas, foram efetuadas.

O Presidente do Júri,

Porto, ______/______/_________

“What I have done is to show that it is possible for the way the universe began to be determined by the laws of science. In that case, it would not be necessary to appeal to God to decide how the universe began. This doesn’t prove that there is no God, only that God is not necessary."

Stephen Hawking

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Resumo

O propósito desta tese é estudar a possibilidade de haver inflação provocada por um campo vectorial massivo com simetria global $SO(3)$ acoplado não minimamente à gravidade. Com recurso à metrica E^3-invariante de Robertson-Walker construiu-se um *Ansatz* para o campo vectorial, permitindo-nos assim estudar a evolução do sistema. Estudamos o comportamento das equações de movimento usando os métodos da teoria de sistemas dinamicos e encontramos regimes de inflação exponencial, seguindo a Ref. [1].

Considerou-se a proposta feita na Ref. [2], incluido um acoplamento não-mínimo entre o campo vectorial e a curvatura - mais precisamente, com o escalar de Ricci e com o tensor de Ricci. Este primeiro acoplamento já foi examinado no contexto de geração de campos magnéticos primordiais [3, 4]. O segundo tipo de acoplamento tem sido considerado em modelos de gravidade com quebra espontânea da simetria de Lorentz [5–9]. Constrangimentos nos parametros do modelo são derivados através da interpretação dos pontos fixos obtidos. Como consequência, mostrou-se que com um campo vectorial acoplado não-minimamente com a gravidade é um candidato viável para proporcionar um regime inflacionário.

Abstract

The purpose of this thesis is to study the possibility that inflation is driven by a massive vector field with $SO(3)$ global symmetry non-minimally coupled to gravity. From an E^3-invariant Robertson-Walker metric we propose an *Ansatz* for the vector field, allowing us to study the evolution of the system. We study the behavior of the equations of motion using methods of the theory of dynamical systems and find exponential inflationary regimes as in Ref. [1].

We shall consider the proposal put forward in Ref. [2], by further including a nonminimal coupling between the vector field and curvature - more precisely, with the Ricci scalar and Ricci tensor. The first coupling has been examined in the context of primordial magnetic field generation [3, 4]. The second type of coupling has been considered in models of gravity with spontaneous breaking of Lorentz symmetry [5–9]. Constraints on the model parameters are derived from the physical interpretation of the resulting fixed points. As a consequence, we show that the inflation with a vector field non-minimally coupled to gravity is a viable candidate for driving an inflationary regime.

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Chapter 1

Introduction

The composition of our Universe has always intrigued the scientific community. The study of visible matter shows that this kind of matter is insufficient to describe the dynamics of the cosmos. So what are the constituents of the Universe? The most suitable solution proposed thus far encompasses Dark Matter and Dark Energy. These two components make up approximately 95% of the entire Universe (69% Dark Energy and 26 %Dark Matter), with the other 5% belonging to ordinary matter [10, 11].

Dark Matter

The existence of dark matter is suggested, for instance, by galactic rotation curves. In 1937, Fritz Zwicky analyzed the dynamics of the Coma galaxy cluster [12]. He considered a system of mutually interacting masses and with the help of the virial theorem he was able to find the mass of the coma Cluster through dynamical measurements. However, these measurements were several orders of magnitude greater (800 solar units approximately) than the ones obtained by measuring the luminosities of the visible objects. One possible explanation for this discrepancy is non-visible matter, which contributes to mass without increasing the galactic luminosity [12]. The term "Dark Matter" only appeared four decades after Zwicky's initial observations, due to the pioneering work of V. Rubin and W. Ford [13]: the authors used spectroscopic techniques to analyze the rotation curve of galaxies, and found out that the galactic rotation curve did not obey the Keplerian prediction, *i.e.*, the rotation curve of the galaxy did not decrease with increasing distance from the center; on the contrary, it remained approximately constant [14]. Again, this discrepancy can be accounted for by assuming that a dark matter halo extends well beyond the visible matter distribution, with the former providing the overwhelming contribution to the total gravitational mass of a galaxy.

To this day, the dark matter composition remains a mystery; however, some observations offer some clues to unravel its nature. For instance, observations of structures at different scales shows that dark matter should be "cold", *i.e*, composed of non-relativistic particles, favoring the formation of small scale structures like galaxies. They are also non-baryonic. Unlike normal matter, dark matter does not interact with electromagnetic forces: this means it does not emit or absorb any type of electromagnetic radiation [15].The best explanation found so far for this phenomenon is that dark matter is composed of Weakly Interacting Massive Particles (WIMPs) [16] that only interact through gravity and the weak force.Their existence and properties can be inferred from its gravitational effects [17].

Dark Energy

Dark Energy is a form of energy that permeates all space and acts as a repulsive force counteracting the attractive nature of gravity. It was the simplest solution found by Albert Einstein to establish a static universe, as he believed was the case after formulating General Relativity [18].

However, in 1927 Lemaitre showed that General Relativity admitted an expanding Universe [19], which was experimentally supported by Hubble two years later [20]. Furthermore, a static solution were a cosmological constant exactly balanced the overall gravitational attraction between matter was found to be unstable, as first noted by Eddington [21] and later expanded by Lemaitre [22]. As such, Einstein abandoned such idea and deemed it “my greatest blunder” after the discovery of Hubble that the Universe is indeed expanding.

The detection of the current phase of accelerated expansion in 1998 by the High-z Supernova Search Team [23] and by the Supernova Cosmology Project [24] lent new life to the cosmological constant Λ, which acts as the simplest mechanism to drive it. It is read directly in the Einstein-Hilbert action below:

$$\int d^4x\sqrt{-g}\left(\frac{1}{2}R+\mathcal{L}-\Lambda\right), \tag{1.1}$$

where g is the determinant of the metric and $\mathcal{L}$ is the Lagrangian density of all other fields. Varying the action with respect to the metric yields the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \quad , \tag{1.2}$$

Recalling that a perfect fluid has the energy momentum tensor [25]

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \quad , \tag{1.3}$$

the cosmological constant can be interpreted as an fluid with an equation of state parameter given by $w = p/\rho = -1$, such that $\rho = -p = \Lambda/(8\pi G)$ [26].

The fact that the vacuum has a non-vanishing energy density should provide a natural mechanism for generating a cosmological constant. However, this leads to the eponymous cosmological constant problem [27], as described here. From Lorentz invariance we obtain

$$< T_{\mu\nu} >=< \rho > g_{\mu\nu}, \tag{1.4}$$

allowing us to define an effective cosmological constant

$$\Lambda_{eff} = \Lambda + 8\pi G < \rho > \quad . \tag{1.5}$$

The above mentioned observations indicate that $\Lambda_{eff} \approx 10^{-47} GeV^4$.

Summing the zero-point energies for all normal modes of a field of mass m up to a high energy cutoff, $M >> m$, yields a vacuum energy density (with $\hbar = c = 1$)

$$< \rho >= \int_0^M \frac{4\pi k^2 dk}{(2\pi)^3} \frac{1}{2} \sqrt{k^2 + m^2} \approx \frac{M^4}{16\pi^2} \quad . \tag{1.6}$$

If we believe that General Relativity is valid up to the Planck scale, than we can take $M \approx (8\pi G)^{-1/2}$, which leads to

$$< \rho > \approx 2^{-10} \pi^4 G^{-2} = 2 \times 10^{71} GeV^4 \tag{1.7}$$

Comparing this with the observational value of Λ_{eff} implies that the two terms in Eq. (1.5) must cancel to 118 decimal places [27]. In the absence of this cancellation, the Universe would expand so rapidly that galaxies would have no time to form [28, 29]. This issue was found long before the accelerated expansion was discovered and requires an extreme fine tuning of the cosmological constant.

1.1 Inflation

The purpose of this thesis its to find out that if its possible to have inflation driven by a vector field nonminimally coupled to gravity, so our introduction would not be complete without a discussion about inflation. Two major problems in cosmology [30] are found when considering the early times after the Big Bang: the Horizon Problem and the Flatness Problem.

In the early 1980's, Alan Guth solved these two problems by proposing the inflationary hypothesis, an extremely fast period of accelerated expansion. This period lasts from 10^{-36} seconds until approximately 10^{-32} seconds after the Big-Bang.

Horizon Problem

The Horizon Problem was identified by Charles Misner in the 1960's. This problem appears when we look at different areas of the universe separated by vast distances and they present identical physical properties, such as temperature of the microwave background radiation (with a reported anisotropy of only 10^{-5} [11]). However, the exchange of information is limited by the speed of light: if the regions in the opposite sides of the Universe have not been in causal contact with one another during its lifetime, this would be impossible. To solve this problem, lets consider a photon moving along a radial trajectory in a flat Universe.

Assuming the Cosmological Principle, *i.e.* that the Universe is homogeneous and isotropic at large scales, implies that it is described by a Robertson-Walker metric

$$ds^2 = dt^2 - a(t)^2\left(dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2\right) \tag{1.8}$$

where we can normalize the scale factor $a(t)$ to $a(t_0) = 1$ without loss of generality. A radial null path obeys

$$ds^2 = -dt^2 + a^2dr^2 = 0 \quad , \tag{1.9}$$

so the comoving distance traveled by such a photon between times t_1 and t_2 is

$$\Delta r = \int_{t1}^{t2} \frac{dt}{a(t)} \quad . \tag{1.10}$$

For simplicity, let us imagine that we are in a matter-dominated universe, for which

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3}}. \tag{1.11}$$

The Hubble parameter is therefore given by

$$H = \frac{2}{3t} = a^{-\frac{3}{2}}H_0. \tag{1.12}$$

Photons travels a comoving distance

$$\Delta r = 2H_0^{-1}(\sqrt{a_2} - \sqrt{a_1}). \tag{1.13}$$

When $a = a_*$, the comoving horizon is the distance that a photon travels since the Big Bang,

$$r_{hor}(a_*) = 2H_0^{-1}\sqrt{a_*}, \tag{1.14}$$

so

$$d_{hor} = a_* r_{hor}(a_*) = 2H_*^{-1}. \tag{1.15}$$

The horizon problem can be simply stated by the fact that the Cosmic Microwave Background (CMB) is isotropic to a high degree of precision, even after considering widely separated points in the Universe. When we look at the CMB, we are observing the universe at a scale factor $a_{CMB} \approx 1/1200$; meanwhile, the comoving distance between a point on the CMB and an observer on Earth is

$$\Delta r = 2H_0^{-1}(1 - \sqrt{a_{CMB}}) \approx 2H_0^{-1}. \tag{1.16}$$

However, the comoving horizon distance for such a point is

$$r_{Hor}(a_{CMB}) = 2H_0^{-1}\sqrt{a_{CMB}} \approx 6 \times 10^{-2} H_0^{-1}. \tag{1.17}$$

Hence, if we observe two widely-separated parts of the CMB, they will have non-overlapping horizons; distinct patches of the CMB sky were causally disconnected at recombination. Nevertheless, they are observed to be at the same temperature to high precision.

Flatness Problem

The Flatness Problem is similar: the universe appears to be nearly flat on large scales - *i.e* the density of matter and energy of the universe appears to be fine-tuned. This is important because it allowed the universe not to collapse back on itself shortly after the Big Bang.

The inflationary model solves this issue by allowing the Universe to expand so rapidly that it flattened out any large-scale inhomogeneities in temperature and density.

1.2 Inflationary Dynamics

We are now able to study the dynamic underlying in the inflationary hypothesis.

Let us start from the Friedman equation for a flat universe and with no cosmological constant,

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho, \tag{1.18}$$

Applying a covariant derivative to both sides of the Einstein equation, and since metric compatibility implies that $\nabla_\alpha g^{\mu\nu} = 0$ and $\nabla_\alpha G^{\mu\nu} = 0$, we conclude that the energy-momentum tensor of matter is covariantly conserved, $\nabla_\mu T^{\mu\nu} = 0$. This conservation leads to $\dot{\rho} = -3H(\rho + p)$ which, together with the first law of thermodynamics, shows that the expansion of the Universe is adiabatic [25]. Taking the derivative of Eq. (1.18) we obtain

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \ . \tag{1.19}$$

Knowing that for an Universe in expansion $\dot{a} > 0$, in order to have an accelerated expansion we need

$$\rho + 3p < 0 \ \ , \tag{1.20}$$

Thus, in the Inflationary period the pressure was negative and smaller than $-\rho/3$. The question now is: which type of matter could present these properties? At first sight a cosmological constant could be a good candidate, since it presents $p = -\rho$. During the Λ-dominated stage, the Hubble parameter remains constant and presents a De Sitter (exponential) inflation. However, the cosmological constant never decays, leading to an inflation that never ends.

The simplest way to introduce a matter component with the required negative pressure is to consider a scalar field with a slow-roll condition. In this case, the energy of the field is diluted very slowly and $p \approx -\rho$.

Let us now consider the equations of motion for an homogeneous scalar field in a flat Friedman-Robertson-Walker (FRW) universe. The scalar field is assumed to have a canonical kinetic term and be driven by a potential $V(\phi)$, as depicted in the action below:

$$S = \int d^4x \sqrt{-g}\left(\frac{1}{2}R + \frac{1}{2}\dot{\phi}^2 - V(\phi)\right). \tag{1.21}$$

Varying eq (1.21) with respect to the scalar field, we obtain the Klein-Gordon equation:

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V(\phi) = 0 \ \ . \tag{1.22}$$

This equation is similar to the one describing the movement of a body subjected to a conservative force against a frictional force due to the expansion of the Universe.

We now impose the so-called slow-roll regime, when the scalar field rolls slowly down a shallow part of the potential: the kinetic term $\dot{\phi}^2/2$ is thus negligible when compared with the potential in Eq. (1.19) and the term $\ddot{\phi}$ is negligible in Eq. (1.22), so both equations can be approximated as

$$\ddot{a} = \frac{8\pi G}{3} GV(\phi)a \ \ , \tag{1.23}$$

and

$$3H\dot{\phi} = -V'(\phi) \ . \tag{1.24}$$

The consistency conditions that the potential has to meet in order to use this approach are

$$\left(\frac{V'}{V}\right)^2 << 48\pi G \quad , \quad \left|\frac{V''}{V}\right| << 9H^2, \tag{1.25}$$

leading to the so-called slow-roll parameters :

$$\epsilon(\phi) = \frac{M_p^2}{48\pi}\left(\frac{V'}{V}\right)^2 << 1 \quad , \quad \eta(\phi) = \frac{M_p^2}{24\pi}\left(\frac{V''}{V}\right) << 1. \tag{1.26}$$

The Friedmann equation now reads

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 \approx \frac{8\pi G}{3}V(\phi) \approx \mathrm{const.} \ , \tag{1.27}$$

which corresponds to the solution $a(t) = e^{H(t-t_0)}$.

1.3 Vector Field Inflation

Despite the success of scalar field inflation, vector fields have an interesting impact on the cosmology of the early universe. In particular, it is natural to examine whether inflation can be driven by vector fields, given that these fields are present in the Standard model of the fundamental interactions. We briefly discuss two proposals which posit the existence of a vector field: Einstein-Æther and the Bumblebee model.

1.3.1 Einstein-AEther Model

The Einstein-AEther theory is a modification of general relativity that contains a vector field named Æther. In this theory the Lorentz Symmetry is broken by the Higgs Mechanism.

This theory involves nonminimal coupling between the field and curvature plus an external potential [31]:

$$S = \int d^4x\sqrt{-g}\Big[\frac{R}{16\pi G} - \frac{\beta_1}{2}F_{\mu\nu}F^{\mu\nu} - \beta(\nabla_\mu A^\mu)^2 + \beta_{13}R_{\mu\nu}A^\mu A^\nu \\ +\beta_4 R A_\mu A^\mu - V(A_\mu A^\mu)\Big] \tag{1.28}$$

where β_i is the nonminimal coupling, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, A_μ is the vector field, $R_{\mu\nu}$ and R are the Ricci tensor and the Ricci Scalar, respectively, and $V(A_\mu A^\mu) = \lambda(A_\mu A^\mu + M^2)^2$.

The significance of Æther theories is reflected in the appearance of Nambu-Golstone Boson (NGB), massless and spinless particles associated with spontaneous symmetry breaking of global symmetries. In Ref. [32], it was shown that NGBs from Æther theories lead to, among other things, new kinds of Cherenkov radiation (see also refs. [33–35]).

1.3.2 Bumblebee Model

Another distinct proposal is the Bumblebee model, where the vector field has a vacuum expectation value that spontaneously breaks Lorentz's symmetry. This model can arise in the context of string theory, more specifically as a phenomenological implementation of the Standard Model Extension (SME) [5–9].

Standard Model Extension

The SME is a field theory that contains the Standard Model, General Relativity and operators that break Lorentz symmetry, together with terms that can also break CPT symmetry. Experimental investigations of symmetry breaking as Lorentz and CPT are facilitated by SME, which lead to a theoretical motivation of these symmetries [36].

In 2003 Alan Kostelecký studied the SME for a Riemann-Cartan spacetime [37]. The action S_{SME} for the full SME in the Riemann-Cartan Spacetime can be expressed as a sum of partial actions:

$$S_{SME} = S_{SM} + S_{LV} + S_{gravity} + ... \tag{1.29}$$

where the S_{SM} is the SM action, modified by the addition of gravitational couplings appropriate for a background Rienmann-Cartan spacetime. The S_{LV} contains the CPT- and Lorentz-violating terms that involve SM fields and dominate at low energy. The $S_{gravity}$ represents the pure gravity sector.

It is also important to notice the diference between explicit and spontaneous Lorentz breaking in the SME theory. Explicit Lorentz breaking clashes with the geometry of Riemann-Cartan spacetime, but spontaneous violation encounters no difficulty [37].

In the Bumblebee model, the Lorentz violation arises from the dynamics of a single vector B_μ called Bumblebee field, with a non-zero expectation value $< B_\mu >= b_\mu$. These models have great interest in SME study. Despite its simple form they present rotation, boost, and CPT violations. [36–40]

This model contains a potential that induces the Lorentz symmetry breaking, $V(B_\mu B^\mu \mp b^2)$, having a minimum value when $B_\mu B^\mu = \pm b^2$.

Some cosmological implications of this model were studied in Ref. [9], where the authors used a coupling between the vector field B_μ and the Ricci tensor $R_{\mu\nu}$. The action of this model is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}(R + \xi B^\mu B^\nu R_{\mu\nu}) - \frac{1}{4}B^{\mu\nu}B_{\mu\nu} - V(B^\mu B_\mu \pm b^2) + \mathcal{L}_M\right] \,, \tag{1.30}$$

where ξ is a coupling constant (with units M^{-2}), B_μ is the Bumblebee field, $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ is the field strength tensor, $b^2 =< B_\mu B^\mu >\neq 0$ and $\mathcal{L}_M$ is the Lagrangian density for the matter fields.

In Ref. [9], a time-like Bumblebee field $B_\mu = (B(t), 0)$ is assumed to possess only a time dependence. Through the study of the ensuing dynamical system, four points of equilibrium were found: two of them are unstable regardless of the value of the model's parameters, corresponding to the static and matter-dominated cases. The other two points yield an accelerated expansion. The study of this accelerated expansion showed a inflationary exponential regime, more precisely the De Sitter regime.

The simplistic assumption of a vector field with only a non-vanishing temporal component can be overcome by assuming that the latter obeys an appropriate symmetry which leads to non-vanishing spatial components, and also by considering the impact of coupling it with both the Ricci scalar as well as with the Ricci tensor.

Chapter 2

The Model and Cosmological Dynamics

In what following we shall study the dynamics inherent in our model (the Euler-Lagrange equation) and the possibility of existence of a de Sitter phase as well as some particular cases.

The action that we consider for an $SO(3)$-invariant gauge group with a massive vector field nonminimally coupled to the curvature reads

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{k^2}R + \mathcal{L}\right) \quad , \tag{2.1}$$

with

$$\mathcal{L} = \frac{1}{8e^2}Tr[F_{\mu\nu}F^{\mu\nu}] + \frac{1}{2}m^2Tr[A_\mu A^\mu] + \frac{1}{3}\alpha R A_\mu A^\mu + \beta R_{\mu\nu}A^\mu A^\nu \quad , \tag{2.2}$$

where $k^2 = 8\pi G$, e is the gauge coupling, α and β are the strengths of the nonminimal couplings between the gauge field and the Ricci scalar and Ricci tensor, respectively [1]. The gauge field strength is given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. Other kinetic terms could also be considered, such as $(\nabla A^\mu)^2$.

2.1 Field Equations

The variation of the action with respect to the metric yields:

$$\begin{aligned}\frac{1}{2k}G_{\mu\nu} &= -\frac{1}{k^2}R_{\mu\nu} + g_{\mu\nu}\mathcal{L} - m^2 A_\mu A_\nu - \frac{1}{2e^2}Tr[F_{\mu\nu}F^{\mu\nu}] - \\ &\frac{2}{3}\alpha\left[R_{\mu\nu}(A^\rho_\rho) + RA_\mu A^\nu - \nabla_\mu\nabla_\nu(A_\rho A^\rho) + g_{\mu\nu}(A_\rho A^\rho)\right] + \\ &\beta\left[2\nabla_{\beta(\mu}A_{\nu)}A^\beta - g_{\mu\nu}(\nabla_\alpha\nabla_\beta A^\alpha A^\beta) - \Box(A_\mu A_\nu) - 4A^\alpha R_{\alpha(\mu}A_{\nu)}\right] \quad (2.3)\end{aligned}$$

where $G_{\mu\nu}$ is the Einstein tensor, $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R.$

Variation with respect to the gauge field yields the vector field equations of motion:

$$\frac{1}{8e^2}\nabla_\mu(\nabla_\nu A^\nu) + \left(\frac{1}{2}m^2 + \frac{1}{3}\alpha R\right)A_\mu + \beta R_{\mu\nu}A^\nu = 0 \ . \quad (2.4)$$

Considering the 00 component and the trace of Eq. (2.3), we obtain the Friedmann and the Raychaudhuri equations. From Eq. (2.4) we obtain the vector field equation.

In this work we use the $SO(3)$-invariant *Ansatz* discussed in Ref. [2], which is briefly reviewed here. The geometry associated with the flat Friedmann-Robertson-Walker (FRW) universe has the form $M^4 = R^4 = R \times E^3/SO(3)$, where E^3 represents a six-dimensional Euclidean group of spacial hypersurfaces. Compatibility with the FRW geometry requires that the vector field is an $SO(3)$-invariant multiplet A^a_μ, $a = 1, ..., N$, where a is an internal field space index.

There are some geometric properties of this group to be considered. The generators of the isometry group $G = E^3$ satisfy the following commutation relations:

$$[T_i, T_j] = 0 \quad , \quad [Q_i, Q_j] = \epsilon_{ijk}Q_k \quad , \quad [Q_i, T_j] = \epsilon_{ijk}T_k, \quad (2.5)$$

where T_i and Q_i are the generators of translations and rotations, respectively. The corresponding Killing vector fields are

$$X_i = \frac{\partial}{\partial x^i} \quad , \quad Y_i = -\epsilon_{ijk}x^j\frac{\partial}{\partial x^k}, \quad (2.6)$$

so that the Lie derivatives obey

$$\mathcal{L}_{X_i} X_j = 0 \quad , \quad \mathcal{L}_{Y_i} X_j = -\mathcal{L}_{x_j} Y_i = \epsilon_{ijk} X_k \quad , \quad \mathcal{L}_{Y_i} Y_j = \epsilon_{ijk} Y_k, \tag{2.7}$$

where $\mathcal{L}_A B = [A, B]$ is the Lie bracket.

Imposing spatial homogeneity and isotropy,

$$\mathcal{L}_{X_i} A = 0 \quad , \quad \mathcal{L}_{Y_i} A = -[L_i, A] \ , \tag{2.8}$$

it is found that the vector field must have the following form [2]:

$$A_0 = 0 \quad , \quad A_i = A_i^a L_a = \chi_0(t) \delta_i^a L_a \ , \tag{2.9}$$

where $\chi_0(t)$ is an arbitrary function of the time and L_a are the generators of the internal $SO(3)$ group. The zeroth component of our vector field is taken by using the Schur's lemma . As seen in Ref. [2], A connects between two representations $\underline{1} \oplus \underline{3}$ and $\underline{3}$ of $SO(3)$. Knowing that both of this representations are non-equivalent, Schur's lemma implies that *A* must vanish between these two subspaces.

The Robertson-Walker metric has the form:

$$ds^2 = -N(t)^2 (dx^0)^2 + a(t)^2 \sum_{i=1}^{3} (dx^i)^2 \ , \tag{2.10}$$

The parameters $N(t)$ and $a(t)$ are the lapse function and the scale factor, respectively. Setting $N(t) = 1$, the Ricci tensor is diagonal and the scalar curvature is given by:

$$R_{tt} = -3\frac{\ddot{a}}{a} \quad , \quad R_{ii} = 2(\dot{a})^2 + a\ddot{a} \quad \rightarrow \quad R = 6\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right] \ . \tag{2.11}$$

It is possible to find the field equations for this model by substitution of Eqs. (2.9)-(2.10) into Eqs. (2.3) and (2.4). However, for the sake of simplicity, we prefer to replace the *Ansatz* Eqs. (2.9) and (2.10) into action Eqs. (2.1)-(2.2). This radical process of reducing the degrees of freedom is often called the Minisuperspace approach, and represents a great tool in the treatment of complex systems [**?**] and is consistent if and only if the constraints of the system are suitably considered. Discarding the infinite volume of the spatial hypersurface, we obtain:

$$S_{eff} = 3\int dt \left[-\frac{a\dot{a}^2}{k^2 N} + \frac{a}{4Ne^2}\left(\frac{\dot{\chi}_0^2}{2} - \frac{N^2}{a^2}V(\chi_0)\right) + \left(\frac{1}{4}Nm^2 + \gamma\frac{\dot{a}^2}{Na}\right)\chi_0^2\right], \tag{2.12}$$

where the quartic potential and the composite coupling have the form $V(\chi_0) = \chi_0^4/8$ and $\gamma \equiv \alpha - \beta$, respectively; notice that the $\ddot{a}$ term that appears in the scalar curvature has been integrated by parts. It can be seen that the two couplings between the vector field and the curvature have similar dynamical impact; in particular, it is important to notice that setting $\beta = \alpha$ cancels their contribution. In this thesis, we thus consider $\gamma \neq 0$.

To obtain the Friedmann and Raychaudhuri equations, together with the equation of motion for the vector field for this model, we vary the above action with respect to $N(t)$, $a(t)$ and $\chi_0(t)$ and set the gauge $N = 1$, obtaining

$$4(a^2 - k^2\gamma\chi_0^2)H^2 = \frac{k^2}{e^2}\left(\frac{\dot{\chi}_0^2}{2} + \frac{V}{a^2}\right) + k^2 m^2 \chi_0^2 \ , \tag{2.13}$$

$$(a^2 - k^2\gamma\chi_0^2)(\dot{H} + H^2) = -H^2 a^2 + k^2\left(2\gamma\dot{\chi}_0\chi_0 H + \frac{m^2\chi_0^2}{4}\right) \ , \tag{2.14}$$

$$\ddot{\chi}_0 + H\dot{\chi}_0 = -\frac{\chi_0^3}{2a^2} + 8e^2H^2\gamma\chi_0 - 2e^2m^2\chi_0 \ , \tag{2.15}$$

where $H = \dot{a}(t)/a(t)$ is the expansion rate.

2.2 De Sitter Phase

A de Sitter universe is a cosmological solution to Einstein's fields equation of General Relativity with a cosmological constant. In this model the universe is spatially flat and the ordinary mass is negligible, where the dark energy component dominates the dynamic of the universe.

Prior to doing a more detailed study of the dynamical system from the above Eqs. (2.13)-(2.15), its interesting to check if a exponential scale factor solution is allowed:

$$a(t) \sim e^{H_0 t} \ , \tag{2.16}$$

where H_0 is the Hubble parameter and t the cosmic time. Since $H(t) = H_0$, we can set $\dot{H} = 0$, in Eqs. (2.13)-(2.15). This, together with the *Ansatz* $\chi_0(t) = Aa(t)$, yields

$$
\begin{aligned}
4(1-k^2\gamma A^2)H_0^2 &= k^2A^2\left[\frac{H_0^2}{2}+\frac{A^2}{8}+m^2\right] \quad , \\
4(2-3k^2\gamma A^2)H_0^2 &= k^2A^2m^2 \quad , \\
4(1-4\gamma)H_0^2 &= -A^2-4m^2 \quad , \qquad (2.17)
\end{aligned}
$$

where to simplify our calculation, we have fixed $e = 1$. The equations above have the solutions

$$
H_{0\pm}^2 = \frac{2+(mk)^2(1+8\gamma)\pm\sqrt{4+4(1+8\gamma)(mk)^2+(1-16\gamma)^2(mk)^4}}{24k^2\gamma(4\gamma-1)} \quad , \qquad (2.18)
$$

and

$$
A^2 = 4\left[(4\gamma-1)H_0^2-m^2\right] = \frac{8}{k^2\left[12\gamma+\left(\frac{m}{H_0}\right)^2\right]} \quad . \qquad (2.19)
$$

Since $\chi_0(t)$ must be real, the following condition is required

$$
(4\gamma-1)H_0^2-m^2>0 \quad , \quad 12\gamma H_0^2+m^2>0 \quad . \qquad (2.20)
$$

These conditions, together with requirement of a real expansion rate $H_0^2 > 0$, imply that only the positive branch H_{0+} should be considered, and that the coupling strength must obey the restriction $\gamma > 1/4$.

2.2.1 Massless case

When $m = 0$, the above equations turn to:

$$
H_0^2 = \frac{1}{6k^2\gamma(4\gamma-1)} \quad , \quad \chi_0(t) = \sqrt{\frac{2}{3\gamma}}\frac{a(t)}{k} \quad . \qquad (2.21)
$$

2.2.2 Strong coupling limit

The strong coupling limit appears when $(mk)^2\gamma \gg 1$. Performing a first order expansion of the Eqs. (2.13)-(2.15), we obtain:

$$H_0 = \sqrt{\frac{1 \pm 2}{12\gamma}} m = \begin{cases} \frac{m}{2\sqrt{\gamma}} & , \quad \gamma > 0 \\ \frac{m}{2\sqrt{-3\gamma}} & , \quad \gamma < 0 \end{cases} \quad . \tag{2.22}$$

A positive coupling $\gamma > 0$ leads to a real valued vector field with

$$\chi_0(t) = \frac{1}{\sqrt{2\gamma}} \frac{a(t)}{k} \quad . \tag{2.23}$$

When $\gamma < 0$, the vector field is an imaginary function

$$\chi_0(t) = \frac{4i}{\sqrt{3}} m a(t) \quad . \tag{2.24}$$

This indicates that a strong coupling can only be possible if the Ricci scalar is stronger than to the Ricci tensor, $\alpha > \beta$.

2.2.3 Weak coupling limit

Expanding around $(mk)^2\gamma \ll 1$ and $m \neq 0$, we get only one real solution of Eqs. (2.13)-(2.15),

$$H_0 = \frac{1}{2k} \sqrt{\frac{2 + (mk)^2}{-3\gamma}} \quad , \quad \gamma < 0 \ , \tag{2.25}$$

requiring that $\gamma < 0$. However, this also leads to

$$\chi_0(t) = \sqrt{\frac{2 + (mk)^2}{3\gamma}} \frac{a(t)}{k} \quad , \tag{2.26}$$

which is thus imaginary – as expected, since the previously condition $\gamma > 1/4$ is not respected. We thus conclude that a weak coupling regime is not possible in the presence of a massive vector field.

Before further study, its also important to check if a power-law behavior for the scale factor and vector field is viable: setting $a(t) \sim t^p$ and $\chi_0 \sim t^n$, we obtain $H(t) \sim t^{-1}$, and from Eqs. (2.13)-(2.15), the relationships:

$$\begin{aligned} 0 &= At^{2p-2} + Bt^{2n-2} + Ct^{4n-2p} + Dt^{2n} \ , \\ 0 &= Et^{2p-2} + Ft^{2n-2} + Gt^{2n} \ , \\ 0 &= Ht^{n-2} + It^{3n-2p} + Jt^{n} \ , \end{aligned} \tag{2.27}$$

where non-vanishing constants are represented by capital letters. Thus, it is clear that this monomial solution for the dynamical system ensued by Eqs. (2.13)-(2.15) is not possible, further motivating a more rigorous study of its critical points.

Chapter 3

Cosmological Dynamics

3.1 Dynamical System Analysis

Dynamical systems are a mathematical tool that relates a function and its time dependence to a point in the phase space. The idea has its origins in the work of Henri Poincaré, who first considered them when approaching Newtonian mechanics and formulated the Poincaré recurrence theorem (1890), which states that some systems, after a sufficiently long but finite time span, will return to a state very close to the initial state.

3.1.1 Example of linear dynamical system

To see some properties of linear dynamical systems, consider the set of linear differential equations with the form

$$\frac{dX}{dt} = AX \quad , \tag{3.1}$$

where A is an $n \times n$ matrix and X is a n-vector. For illustration we set $n = 2$.

Its easy to notice that $X = 0$ is the only fixed point of this system. Linearity implies that the superposition principle is valid, *i.e.*, if $X_1(t)$ and $X_2(t)$ are solutions to the system, then so is $X_1(t) + X_2(t)$.

For concreteness, let A be an already diagonalized 2×2 matrix

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} ,$$

that is,

$$\dot{x} = ax \quad , \tag{3.2}$$

$$\dot{y} = by \quad . \tag{3.3}$$

If we set the initial condition $y(0) = 0$, then $y(t) = 0$. This means that the problem reduces to a one-dimensional case, with the solution

$$x(t) = x(0)e^{at} \quad , \tag{3.4}$$

so that, if $a > 0$, the solution moves away from the origin as time increases; conversely, if $a < 0$, the solution approaches the origin. The same happens for $y(t)$ if $y(0) \neq 0$, the sign of b controlling if it moves away ($b > 0$) or towards ($b < 0$) the origin. Due to the superposition principle, and in the case $x(0) \neq 0$ and $y(0) \neq 0$, the solution is combination of these two behaviors.

Thus, the fixed point of the dynamical system (*i.e.* the origin) can be now classified into three types,

- if $a > 0$ and $b > 0$ we have a source; in the case of a and b complex if $\mathrm{Re}\, a > 0$ and $\mathrm{Re}\, b > 0$, we have a spiral source;
- if $a < 0$ and $b < 0$ we have a sink; in the case of a and b complex if $\mathrm{Re}\, a < 0$ and $\mathrm{Re}\, b < 0$, we have a spiral sink;
- if $a < 0 < b$ we have a saddle point; in the case of a and b complex if $\mathrm{Re}\, a = 0$ and $\mathrm{Re}\, b = 0$ we have a center.

In the case above, the matrix A is already diagonal. In the general case of n independent variables and a non-diagonal matrix A, we may still extract useful properties of the dynamical systems by inspecting the eigenvalues $\{\lambda_i\}$ ($i = 1, ..., n$) of A evaluated at each fixed point: thus, the fixed points can be:

- source, if all of the eigenvalues have a positive real part;
- sink, if all of the eigenvalues have a negative real part;
- saddle point, if some eigenvalues have positive real part and others negative.

More complicated dynamical systems can arise and so we are forced to resort to stronger results in analysis of dynamic systems: The Hartman-Grobman theorem, which states that the behavior of a dynamical system near a hyperbolic equilibrium point is approximately the same as the behavior of its linearization near the equilibrium point — provided that no eigenvalue of the linearization has real part equal to zero.

3.1.2 Cosmological Dynamics

To find general inflationary solutions to our cosmological model we need to solve the dynamical system associated with Eqs. (2.13)-(2.15). To accomplish our purpose we introduce the following dimensionless variables,

$$
\begin{aligned}
x = \frac{k\chi_0(t)}{a(t)\sqrt{1-w^2}} \quad &, \quad y = \frac{k^2\dot{\chi}_0(t)}{2\sqrt{2(1-w^2)}} \ , \\
z = kH \quad &, \quad \tau = \frac{t}{k} \ ,
\end{aligned}
\tag{3.5}
$$

with the auxiliary function

$$
w = \sqrt{\gamma}\frac{k\chi_0(t)}{a(t)} = x\sqrt{\frac{\gamma}{1+\gamma x^2}} \ . \tag{3.6}
$$

The algebraic constraint obtained by the Friedmann Eq. (2.13) reads:

$$
z^2 = y^2 + \frac{1}{32}\frac{x^4}{1+\gamma x^2} + \frac{1}{4}\mu^2 x^2 \ , \tag{3.7}
$$

where we define the reduce mass as $\mu = mk$.

Using the above constraint, only two degrees of freedom remain. Deriving the variables (x, y) with respect to the dimensionless time τ we obtain

$$
\begin{aligned}
x_\tau \equiv \frac{dx}{d\tau} &= (1+\gamma x^2)\left[y - x\sqrt{y^2 + \frac{1-w^2}{32}x^4 + \frac{1}{4}\mu^2 x^2}\right] \ , \\
y_\tau \equiv \frac{dy}{d\tau} &= -\frac{1-w^2}{4\sqrt{2}}x^3 + \frac{4\gamma}{\sqrt{2}}(y^2 + \frac{1-w^2}{32}x^4 + \frac{1}{4}\mu^2 x^2)x - \\
&\quad \frac{\mu^2}{\sqrt{2}}x + 2\sqrt{2}\gamma y^2 x - (\gamma x^2 + 2)y\sqrt{y^2 + \frac{1-w^2}{32}x^4 + \frac{1}{4}\mu^2 x^2} \ .
\end{aligned}
\tag{3.8}
$$

In Fig. 3.1 the first quadrant of the phase space of our dynamical system is represented, clearly depicting the corresponding two finite critical points

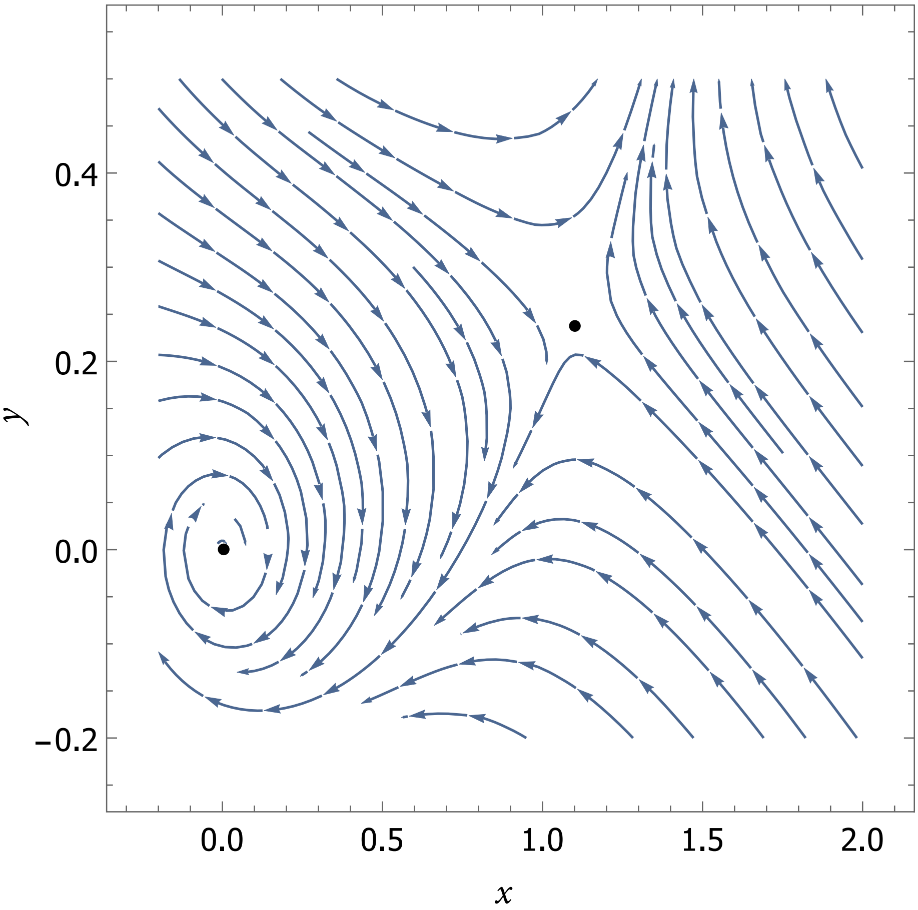

FIGURE 3.1: Phase space of the dynamical system.

($\gamma = \mu = 1$ were considered). The origin is a stable critical point, as the eigenvalues of the Jacobian matrix are purely imaginary; the other critical point presented its a saddle point. These and the remaining (finite) critical points are characterized below.

3.2 Finite critical points

Looking for the dynamical system Eq. (3.8), the first thing to remark is that the origin, $F(0,0)$, is a trivial critical point. With the help of the Jacobian matrix derived from Eq. (3.8) and their respective eigenvalues, $\lambda_{\pm} = \pm 2i\sqrt{2}\mu$, we see that we are in the presence of a stable critical point; this result is not surprising, as the vector field vanishes and so the nonminimal couplings have no impact, collapsing to the case studied in Ref. [2].

Beyond the trivial critical point, eight non-trivial critical points arise, as shown in Table 1. For convenience we define:

$$
\begin{aligned}
X_{\pm}(\mu,\gamma) &= \frac{2+\mu^2 \pm \sqrt{(1-16\gamma)^2\mu^4 + 4(8\gamma+1)\mu^2 + 4}}{2\gamma\left[1+(8\gamma-1)\mu^2\right]} \quad , \\
Y_{\pm} &= \frac{1}{12\gamma(4\gamma-1)}\left[\mu^2 + \frac{2+(1+4\gamma)\mu^2}{8} X_{\pm}\right] \quad .
\end{aligned}
\tag{3.9}
$$

As one can see, the dynamical system is invariant under reflections $(x, y) \to (-x, -y)$, so the first four critical points suffice to complete our analysis (first column).

The critical points (C, D, G, H), are unphysical, leading to an imaginary value H_- for the expansion rate. This can be seen in the value for the expansion rate, read from the algebraic constraint Eq. (3.7) that coincides with those discussed in the previous section, as can be seen from Eq. (2.18).

Point	(x, y)	H	Point	(x, y, z)	H
A	$(\sqrt{X_+}, \sqrt{Y_+})$	H_+	E	$(-\sqrt{X_+}, -\sqrt{Y_+})$	H_+
B	$(\sqrt{X_+}, -\sqrt{Y_+})$	H_+	F	$(-\sqrt{X_+}, \sqrt{Y_+})$	H_+
C	$(\sqrt{X_-}, \sqrt{Y_-})$	H_-	G	$(-\sqrt{X_-}, -\sqrt{Y_-})$	H_-
D	$(\sqrt{X_-}, -\sqrt{Y_+})$	H_-	H	$(-\sqrt{X_-}, \sqrt{Y_-})$	H_-

TABLE 3.1: Non-trivial, finite critical points.

Since the four points above lead to unphysical solution, we only study the remaining four critical points (A, B, C, D). Due to the complexity of the expressions for the eigenvalues of the Jacobian matrix, we used a numerical procedure to show that this critical points are in fact saddle points.

This can be accomplished by assigning a range of values for the reduced mass μ and the coupling γ and numerically computing the value of the two eigenvalues λ_1 and λ_2 of the Jacobian matrix. The evaluation of the four critical points lead us to conclude that the critical points (A, B) have the same two eigenvalues λ_1 and λ_2, with real parts that are almost symmetric; the same occurs for the pair (C, D). This behavior is graphically shown in Figs. 3.2 and 3.3, where it is clear that these critical points fall neatly in the line $Re(\lambda_1) = -Re(\lambda_2)$.

However, this is not sufficient to ensure that (x, y) are real-valued. For the points (A, B) to have physical meaning we must also consider the definition of the dimensionless variables Eq. (3.5). Knowing that (x, y) are real and $1 - w^2 > 0 \to 1 + \gamma x^2 > 0$ leads to the condition $\gamma > 1/4$ — precisely the constraint obtained in the previous section.

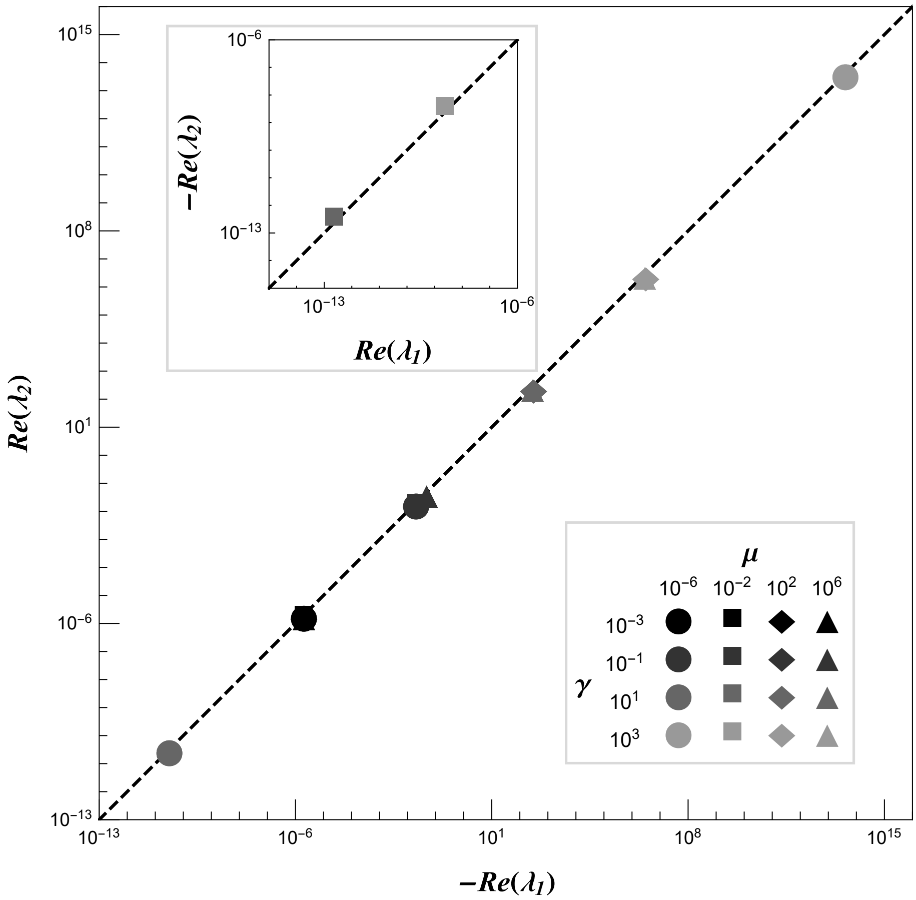

FIGURE 3.2: Real part of the eigenvalues of critical points (A, B).

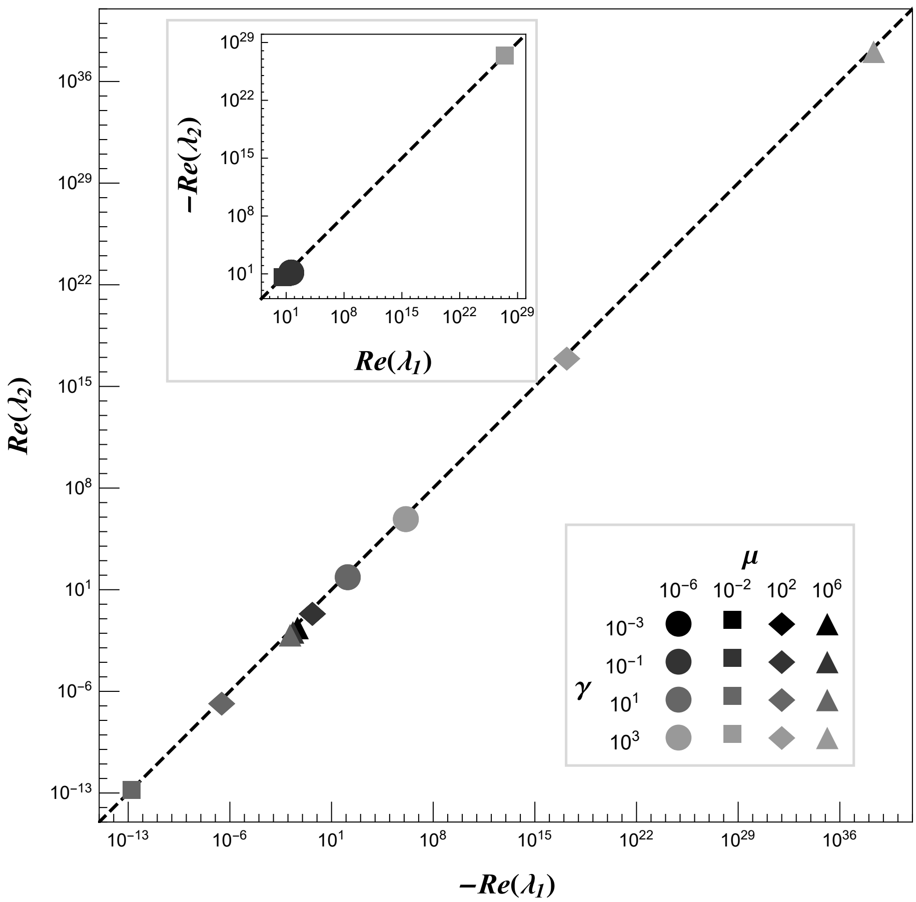

FIGURE 3.3: Real part of the eigenvalues of critical points (C, D).

3.3 Critical points at infinity

The putative critical points found at infinity are now analyzed, by resorting to a boundary at infinity, $x^2 + y^2 = \infty$, which is then compactified to a circle of unit radius.

In order to do so, using the usual definition of polar angle and redefining the radial coordinate together with the new time variable, we get:

$$x = \frac{\rho}{1-\rho}\cos\theta \quad , \quad y = \frac{\rho}{1-\rho}\sin\theta \quad , \quad \frac{d\zeta}{d\tau} = \frac{1}{(1-\rho)^2} \quad , \tag{3.10}$$

where $0 \leq \rho \leq 1$.

We can rewrite the dynamical system Eq. (3.8) as

$$\begin{aligned} &\rho_\zeta \equiv \frac{d\rho}{d\zeta} = \Pi(\rho,\theta) = \frac{\sqrt{2}}{16}\frac{\rho g(\rho,\theta)\sin\theta\cos\theta}{(1-\rho)^2 + \gamma\rho^2\cos^2\theta} + \\ &\frac{1}{2}\big[-3 + 6\rho - (3+\gamma)\rho^2 + [1 - 2\rho + (1-\gamma)\rho^2]\cos 2\theta\big]\rho^2 f(\rho,\theta) \quad , \\ &\theta_\zeta \equiv \frac{d\theta}{d\zeta} = \Psi(\rho,\theta) = -\rho f(\rho,\theta)\sin\theta\cos\theta + \frac{\sqrt{2}}{16}\frac{h(\rho,\theta)}{(1-\rho)^2 + \gamma\rho^2\cos^2\theta} \quad , \end{aligned} \tag{3.11}$$

with

$$\begin{aligned} f^2(\rho,\theta) &= (1-\rho)\left(\frac{\rho^2\cos^4\theta}{32\,[(1-\rho)^2 + \gamma\rho^2\cos^2\theta]} + \frac{1}{4}\mu^2\cos^2\theta + \sin^2\theta\right) \quad , \\ g(\rho,\theta) &= 8(\mu^2-4)(1-5\rho) + \big(16[5\mu^2 - 4\gamma - 10] + 2\cos^2\theta\big)\rho^2 + \\ &\quad 2\big(160 + 96\gamma - 40\mu^2 - 3\cos^2\theta\big)\rho^3 + \\ &\quad \big(8[5\mu^2 - 24\gamma - 20] + 2[3 - 32\gamma^2]\cos^2\theta + [8\gamma(4-\mu^2) - 1]\cos^4\theta\big)\rho^4 + \\ &\quad \big(8[4 + 8\gamma - \mu^2] - 2[1 - 32\gamma^2]\cos^2\theta - [8\gamma(4-\mu^2) - 1]\cos^4\theta\big)\rho^5 \quad , \\ h(\rho,\theta) &= 8(1-\rho)^4[(4-\mu^2)\cos^2\theta - 4] - 2\rho^2[1 - 2\rho + (1-16\gamma^2)\rho^2]\cos^4\theta + \\ &\quad \gamma[1 + 8\gamma(\mu^2-4)]\rho^4\cos^6\theta \quad . \end{aligned} \tag{3.12}$$

Finding the solution of the equations $\Pi(1,\theta) = 0$ and $\Psi(1,\theta) = 0$ allows us to obtain the critical points at an infinitely distant boundary. To facilitate our work we can do a symmetry study of Eq. (3.11), as one can see the latter equation is invariant under the transformation $\theta \to \theta + \pi$, so we can just consider the critical points lying on the region $[0, \pi]$. As $f(1, \theta) = g(1, \theta) = 0$ and $h(1, \theta)$, the critical points are given by

$$\left[32\gamma + (1 + 8\gamma[\mu^2 - 4])\cos^2\theta\right]\cos^2\theta = 0 \quad , \tag{3.13}$$

with solutions

$$N\left(1, \frac{\pi}{2}\right) \quad , \quad S_{\pm}\left(1, \arccos\left(\pm\sqrt{\frac{1}{1 - \frac{1}{32\gamma} - \frac{1}{4}\mu^2}}\right)\right) \quad . \tag{3.14}$$

The critical point $S_{\pm}$ is real under the condition

$$\left[\mu < 2 \wedge \left(\gamma < 0 \vee \gamma > \frac{1}{8(4 - \mu^2)}\right)\right] \vee \left[\mu > 2 \wedge -\frac{1}{8(\mu^2 - 4)} < \gamma < 0\right] \quad , \tag{3.15}$$

where the symbols $\wedge$ (and) and $\vee$ (or) have been used.

The derivation of the eigenvalues of the Jacobian around critical points $S_{\pm}$ (which are degenerate) and N is enabled by the linearization of the system Eq. (3.11); in the case of fixed point N, this requires a change in the time variable $\zeta \to \hat{\zeta}$, such that $d\hat{\zeta}/d\zeta = \rho - 1$. The ensuing results are presented in Table 2.

Point	Eigenvalues
S_+	$\frac{8\gamma\sqrt{-\gamma(8\gamma\mu^2+1)}}{1+8\gamma(\mu^2-4)}$
S_-	$-\frac{8\gamma\sqrt{-\gamma(8\gamma\mu^2+1)}}{1+8\gamma(\mu^2-4)}$
N	$\frac{3\pm\sqrt{1-64\gamma}}{2}$

TABLE 3.2: Eigenvalues of the critical points at infinity $S_{\pm}$ and N for the dynamical system Eq. (3.11).

In order to extract the expansion rate, we resort again to the definition Eq. (3.5) and the algebraic constraint Eq. (3.7) which, in the compactified polar coordinates, reads

$$(kH)^2 = z^2(\rho, \theta) = \frac{\rho^2}{32(1 - \rho)^2}\left(\frac{\rho^2\cos^4\theta}{\gamma\rho^2\cos^2\theta + (1 - \rho)^2} + 8\mu^2\cos^2\theta + 32\sin^2\theta\right) \quad . \tag{3.16}$$

Critical Point N

Inspection of Table 2 shows that the critical point $N(1, \pi/2)$ is

- a saddle point, if $\gamma \leq -1/8$;
- unstable, if $-1/8 < \gamma < 1/64$;
- a focus, if $\gamma > 64$.

Replacing $\theta = \pi/2$ in Eq. (3.16), we see that $z \sim 1/(1-\rho) \to \infty$ for all values of the coupling γ and reduced mass μ, thus yielding the possibility of a Big Rip scenario (if N is a focus), *i.e.* the Universe evolves towards an infinite expansion rate.

Critical Points $S_\pm$

By analyzing Table 2, we can ascertain the behavior of the critical points $S_\pm$: imposing the condition Eq. (3.15) for real critical points and vary the coupling γ and reduced mass μ to determine the behavior of the corresponding degenerate eigenvalues, as depicted in Fig. 3.4. We find that the latter are never positive, yielding:

- $\gamma < 0$: $-\frac{1}{8\mu^2} < \gamma < 0$
- $Re(\gamma) = 0$: $\begin{cases} \gamma < -\frac{1}{8\mu^2} \vee \gamma > \frac{1}{8(4-\mu^2)} \ , & \mu < 2 \\ \frac{1}{8(4-\mu^2)} < \gamma < -\frac{1}{8\mu^2} \ , & \mu > 2 \end{cases}$

Again resorting to Eq. (3.16), we find that

$$z\left(\rho, \arccos\left(\pm\sqrt{\frac{1}{1-\frac{1}{32\gamma}-\frac{1}{4}\mu^2}}\right)\right) = \frac{\rho^2}{(\rho-1)^2[1+8\gamma\left(\mu^2-4\right)-32\gamma^2\rho^2]} \ , \tag{3.17}$$

in the limit $\rho = 1$ we have

$$z^2 = -\frac{1}{32\gamma^2} \ . \tag{3.18}$$

In the case of the critical point N, we see a divergence that is cancelled out by the value for θ; however, we find that it leads to an unphysical, imaginary expansion rate.

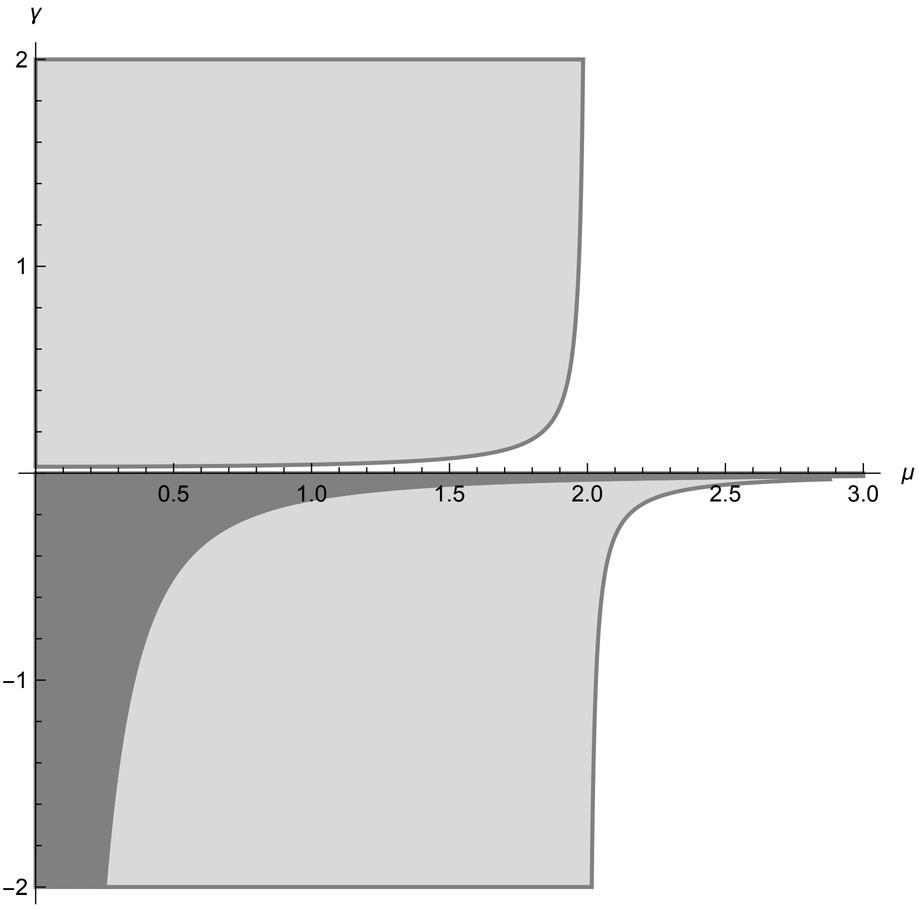

FIGURE 3.4: Degenerate eigenvalues of $S_{\pm}$: real and negative (dark gray), pure imaginary (light gray).

Chapter 4

Conclusions

In this work, we have studied the dynamics of an $SO(3)$-invariant massive vector field [2] nonminimally coupled to the curvature.

A De Sitter, exponential inflationary phase, is admitted by the system, for a restricted region of the parameter space, $\gamma > 1/4$ (cf. Eq. (2.12)). We considered some specific regimes for exponential inflation; for the massless case, $\mu = 0$, we have obtained physical solutions. The strong coupling limit, $\mu^2\gamma \gg 1$ is only viable if the coupling to the Ricci scalar is stronger than to the Ricci tensor, $\alpha > \beta$. A weak coupling limit, $\mu^2\gamma \ll 1$ is not achievable, as it breaks the aforementioned constraint, $\gamma > 1/4$. A power law behavior for the scale factor and vector field was also studied: however, it is not possible to find a simple monomial solution for the dynamical system embodied in Eqs. (2.13)-(2.15).

The dynamical system arising from the equations of motion for this theory has been studied, leading to 9 finite critical points and 3 critical points at infinity. In the former case, the origin is a trivial critical point, with no impact arising from the nonminimal coupling between the vector field and curvature: the behavior of this fixed point is thus naturally equivalent to that obtained in Ref. [2]. The other 8 non-trivial points lead to a constant expansion rate and are saddle points, with only 2 of them have physical meaning, *i.e.* that obey the constraint $\gamma > 1/4$ (real expansion rate).

Regarding the 3 fixed points at infinity, we have, $N(1, \frac{\pi}{2})$, which, depending on the value of γ, can behave as a saddle point, an unstable point or a focus. If the critical point N is a focus, this may lead to a Big Rip scenario, in which the Universe evolving towards a infinite expansion rate. The other 2 critical points, $S_{\pm} = (1, \pm\frac{8\gamma\sqrt{-\gamma(8\gamma\mu^2+1)}}{1+8\gamma(\mu^2-4)})$, lead to an imaginary expansion rate, and can thus be identified with an oscillating scale factor. These possibilities, not studied in this theory, display the interesting dynamics that can arise from the model under consideration.

Thus, we conclude that a massive vector field can lead to inflationary solutions, provided the nonminimal coupling to gravity is non-vanishing and satisfy the condition $\alpha \neq \beta \to \gamma \neq 0$, *i.e.* that the couplings with the Ricci Scalar and Tensor do not cancel each other out. This is a particulary interesting and pleasing new feature of the presented model.

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